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ON A NEW INTEGRAL TRANSFORM AND SOLUTION OF SOME INTEGRAL EQUATIONS

Shaikh Sadik Ali[1 §], M.S. Chaudhary[2]

[1]Maulana Azad College
Aurangabad, M.S., 431001, INDIA
[2]Department of Mathematics
Shivaji University Kolhapur
M.S., INDIA

Abstract: In this paper, the convolution theorem and uniqueness theorem for the Elzaki transform is proved.Applicability of Elzaki transform is demonstrated for solving integral equations of convolution type and Abels integral equations.

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1. Introduction

In the literature there are many integral transforms used in engineering and applied sciences. Among the all transforms, the Laplace transform plays versatile role in its applications. Integral transforms method is efficient tool to solve differential equations, integral equations. Integral transform is very effective method due to the fact that it converts system of differential equations and integral equations into algebraic equations. Subsequently Tarig Elzaki [1, 2, 3, 4, 5] derived the Elzaki transform of ordinary and partial derivatives. The main objective of this paper is to demonstrate the applicability of the Elzaki transform to solve integral equations of convolution type and Abel's integral equations.

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§Correspondence author

2. Preliminaries

Definition 1. The convolution type Voltera integral equation is defined as

$$f(x) = h(x) + \lambda \int_0^x k(s-t)f(t)dt \qquad (\lambda \ isconstant) \tag{1}$$

where $f(.)$ is to be determined. We call (2.1) is homogenious Voltera integral equations of convolution type if $h(x) = 0$

Definition 2. Abel integral equation is expressed as

$$f(t) = \int_0^t \frac{g(x)}{(t-x)^\alpha} dt \quad 0 < x < t \tag{2}$$

Definition 3. The Elzaki transform of the functions belonging to a class A, where

$$A = \{f(t)|\exists M, k_1, k_2, |f(t)| \leq Me^{\frac{t}{k_j}} where j \in (-1)^j \times [0, \infty)\} \tag{3}$$

f(t) is denoted by E[f(t)]=F(v) and defined as

$$E[f(t)] = v \int_0^\infty e^{\frac{-t}{v}} f(t)dt \qquad v \in (k_1; k_2) \tag{4}$$

Theorem 4. *If F(v) is the Elzaki transform of f(t); then*

$$i) E(f'(t)) = \frac{F(v)}{v} - vf(0) \tag{5}$$

$$ii) E(f''(t)) = \frac{F(v)}{v^2} - f(0) - vf'(0) \tag{6}$$

$$iii) E(t^n) = n! v^{n+2} \tag{7}$$

$$iv) E(e^{at}) = \frac{v^2}{1-av} \tag{8}$$

$$v) E(sin(at)) = \frac{av^3}{1+a^2v^2} \tag{9}$$

$$vi) E(cos(at)) = \frac{v^2}{1+a^2v^2} \tag{10}$$

$$vii) If f(t) = \sum a_n t^n \quad then \quad E[f(t)] = \sum n! a_n v^{n+2}. \tag{11}$$

Theorem 5. *If $F(v)$ and $G(v)$ are Elzaki transform of $f(t)$ and $g(t)$ respectively, then*

$$E[f(t) * g(t)] = \frac{1}{v} E[f(t)] E[g(t)] \tag{12}$$

Where $*$ denotes convolution product.

Proof. Consider functions $f(x)$ and $g(x)$, which are Elzaki transformable functions and for $t \geq 0$ they can be written as infinite convergent series

$$f(x) = \sum_{n=0}^{\infty} a_n t^n \text{ and } g(x) = \sum_{m=0}^{\infty} a_m t^m \tag{13}$$

By definition of convolution

$$f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau \tag{14}$$

from (2.13)

$$\begin{aligned} f(t) * g(t) &= \int_0^t \sum_{n=0}^{\infty} a_n \tau^n \sum_{m=0}^{\infty} a_m (t-\tau)^m d\tau \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m \int_0^t \tau^n (t-\tau)^m d\tau \end{aligned}$$

Expanding$(t-\tau)^m$ by the binomial theorem, we obtain

$$\begin{aligned} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_n b_m \binom{m}{k} (-1)^k \int_0^t \tau^n \tau^k t^{(m-k)} d\tau \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_n b_m \binom{m}{k} (-1)^k t^{(m-k)} \int_0^t \tau^{(n+k)} d\tau \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_n b_m \binom{m}{k} (-1)^k \frac{t^{(m+n+1)}}{n+k+1} \end{aligned}$$

But, for any positive integers m and n, the beta function is connected with gamma function by the relation :

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{1}{n+k+1} = \frac{m!n!}{(m+n+1)!} \tag{15}$$

Therefore(2.14) becomes

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m} a_n b_m \frac{m!n!}{(m+n+1)!} t^{(m+n+1)} \tag{16}$$

Applying Elzaki transform, we obtain

$$E[f(t)*g(t)] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m} a_n b_m \frac{m!n!}{(m+n+1)!}(m+n+1)! \quad v^{(m+n+1)}$$
$$= \frac{1}{v} \quad E[f(t)] \cdot E[g(t)]$$

□

T.M. Elzaki and S.M. Elzaki [1] discussed relation between Laplace Transform and Elzaki Transform.

Corollary 6. *If $F(s)$ and $G(v)$ are Laplace transform and Elzaki transform of $f(t)$ respectively, then*

$$F(s) = sG(\frac{1}{s}) \tag{17}$$

Theorem 7. *(Uniqueness Theorem)*
If $F(v)$ and $G(v)$ are Elzaki transforms of $f(t)$ and $g(t)$ respectively. Then $F(v) = G(v)$ implies $f(t) = g(t)$.

Proof. Consider

$$F(v) = G(v)$$

$$E[f(t); v] = E[g(t); v]$$

$$vL[f(t); \frac{1}{v}] = vL[g(t); \frac{1}{v}]$$

$$L[f(t); \frac{1}{v}] = L[g(t); \frac{1}{v}]$$

By uniqueness of Laplace transform. We obtain

$$f(t) = g(t)$$

□

3. Main Result

Consider convolution type Voltera integral equation of first kind

$$f(x) = \int_0^x K(x-t)h(t)dt \tag{18}$$

We apply Elzaki transform on (3.1) and using convolution theorem. We obtain

$$F(v) = vK(v)H(v)$$

So

$$H(v) = \frac{vF(v)}{K(v)}$$

By taking inverse Elzaki transform, we obtain.

$$h(x) = E^{-1}[\frac{vF(v)}{K(v)}]$$

Simillarly, consider convolution type Voltera integral equation of second kind

$$h(x) = f(x) + \lambda \int_0^x k(x-t)h(t)dt \tag{19}$$

Again applying Elzaki transform on above equation. We obtain

$$H(v) = \frac{vF(v)}{v - \lambda K(v)}$$

Another problem of frequent interest in connection with (3.2) is the resolvent kernel, that is, the determination of function$\lceil(t)$ such that

$$h(x) = f(x) + \lambda \int_0^x \lceil(x-t)f(t)dt \tag{20}$$

Now above equation may be written as

$$H(v) = F(v) + \frac{\lambda K(v)}{v - \lambda K(v)} F(v)$$

So that $\lceil(t)$ is the inverse Elzaki transform of $\frac{\lambda K(v)}{v-\lambda K(v)}$. Now, we find solution of Abel's integral equation using Elzaki transform. We write Abel's integral equation in the form

$$f(t) = \int_0^x \frac{g(x)}{(t-x)^\alpha} dx, \quad 0 < \alpha < 1. \tag{21}$$

Above equation can be written as

$$f(t) = g(t) * t^{-\alpha} H(t) \tag{22}$$

Where $H(t)$ is Heaviside's unit step function.

Applying Elzaki transform on (3.5). We obtain

$$\begin{aligned}
F(v) &= \frac{1}{v} G(v) \lceil (1-\alpha) v^{-\alpha+2} \\
G(v) &= \frac{F(v)}{\lceil (1-\alpha)} v^{\alpha-1} \\
G(v) &= \frac{F(v) \lceil \alpha}{\lceil (1-\alpha) \lceil (\alpha)} v^{\alpha-1} \\
vG(v) &= \frac{sin(\pi\alpha) \lceil \alpha}{\alpha} \frac{1}{v} F(v) v^{\alpha+1} \\
vG(v) &= \frac{sin(\pi\alpha)}{\alpha} E[f(t) * t^{(\alpha-1)}]
\end{aligned}$$

$$\begin{aligned}
E[\int_0^t g(x)dx]) &= \frac{sin(\pi\alpha)}{\alpha} E[\int_0^t f(x)(t-x)^{(\alpha-1)} dx] \\
E[\int_0^t g(x)dx]) &= \frac{sin(\pi\alpha)}{\alpha} E[k(t)],
\end{aligned}$$

where $k(t) = \int_0^t f(x)(t-x)^{(\alpha-1)} dx, \quad k(0) = 0$. We have

$$E[k'(t)] = \frac{K(v)}{v}$$

Therefore

$$\begin{aligned}
vG(v) &= \frac{sin(\pi\alpha)}{\alpha} v E[k'(t)] \\
E[g(t)] &= E[\frac{sin(\pi\alpha)}{\alpha} k'(t)]
\end{aligned}$$

By uniqueness theorem. We obtain

$$g(t) = \frac{sin(\pi\alpha)}{\alpha} \frac{d}{dt} [\int_0^t f(x)(t-x)^{(\alpha-1)} dx]$$

This is the a solution of Abel's integral equation.

4. Application to Integral Equations

In this section, we illustrate some integral equations by using the Elzaki transform.

Example 4.1. Solve the integral equation

$$t = \int_0^t e^{t-x} g(x)dx \tag{23}$$

Solution. Applying Elzaki transform. We obtain

$$v^3 = \frac{1}{v} E[e^t] E[g(t)]$$

Therefore,

$$G(v) = v^2 - v^3. \tag{24}$$

Here,

$$E[g(t)] = G(v)$$

Taking innverse Elzaki transform of equation (25), we obtain

$$g(t) = E^{-1}[v^2] E^{-1}[v^3]$$
$$g(t) = 1 - t$$

Solution by The Laplace Transform. By applying the Laplace transform to (24), we obtain.

$$\frac{1}{s^2} = \frac{G(s)}{s-1},$$

where

$$L[g(t)] = G(s)$$

Therefore

$$G(s) = \frac{1}{s} - \frac{1}{s^2}$$

Applying inverse Laplace transform, we obtain

$$g(t) = 1 - t.$$

Example 4.2. Solve the integral equation:

$$sin(2t) = \int_0^t (\tau - t) h(t) d\tau \tag{25}$$

Solution by the Laplace transform. Applying the Laplace transform on (26), we obtain

$$\frac{2}{s^2+4} = \frac{-1}{s^2}H(s).$$

Here

$$L[h(t)] = H(s).$$

Therefore

$$H(s) = \frac{-2s^2}{s^2+4}$$
$$H(s) = \frac{-2s^2-8+8}{s^2+4}$$
$$H(s) = -2 + \frac{8}{s^2+4}$$

Applying inverse Laplace transform, we obtain

$$h(t) = -2\delta(t) + 4sin(2t) \tag{26}$$

Solution by Elzaki transform. Applying the Elzaki transform on equation (26), we obtain

$$\frac{2v^3}{1+4v^2} = \frac{1}{v}(-v^3)H(v).$$

Here

$$E[h(t)] = H(v).$$

Therefore

$$\begin{aligned} E[h(t)] &= \frac{-2v}{1+4v^2}. \\ &= \frac{8v^3}{1+4v^2} - \frac{2v(1+4v^2)}{1+4v^2} \\ &= 4\frac{2v^3}{1+4v^2} - 2v \end{aligned}$$

Applying the inverse Elzaki transform, we obtain

$$h(t) = 4sin(2t) - 2\delta(t)$$

Example 4.3. Find the function $g(t)$ which satisfies the equation

$$g(t) = t + \int_0^t g(\tau)sin(t-\tau)d\tau. \tag{27}$$

Solution. Applying Elzaki transform on (28), we get

$$G(v) = v^3 + \frac{1}{v}\frac{v^2 G(v)}{1+v^2}$$
$$G(v) = v^3 + v^5$$
$$G(v) = v^(1+2) + v^(3+2)$$

Applying inverse Elzaki transform, we obtain the required solution as,

$$g(t) = t + \frac{t^3}{3!}.$$

Example 4.4. Find the resolvent kernel$\lceil(t)$ of the following equation

$$g(t) = f(t) + \lambda \int_0^t e^(t-\tau) g(\tau) d\tau. \tag{28}$$

Solution. Let $\Omega(v)$ be the Elzaki transform of $\lceil(t)$ from equation(21),

$$\Omega(v) = \frac{\lambda E[e^t]}{v - \lambda E[e^t]} \tag{29}$$

$$\Omega(v) = \frac{\lambda \frac{v^2}{1-v}}{v - \lambda \frac{v^2}{1-v}} \tag{30}$$

$$\Omega(v) = \frac{\lambda v}{1-(1+\lambda)v} \tag{31}$$

Applying inverse Elzaki transform, we obtain

$$\lceil(t) = \lambda e^{-(1+\lambda)} t.$$

Therefore equation (28) leads in this case

$$g(t) = f(t) + \lambda \int_0^t e^{-(t-\tau)(1+\lambda)} f(\tau) d\tau.$$

5. Conclusion

Elzaki transform provides powerful method for solving integral equations of convolution type. It is heavily used in solving Abel's integral equations and convolution type integral equations.

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