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POSITIVE SOLUTIONS TO A SECOND ORDER m-POINT INTEGRAL BOUNDARY VALUE PROBLEM

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Abstract: In this paper, by using the fixed-point index theorems, we study the existence of at least one or two positive solutions to the nonlinear second order m-point integral boundary value problem

$$u''(t) + a(t)f(u(t)) = 0, \qquad 0 < t < 1,$$

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} u(s)ds,$$

where $0 = \eta_0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < \eta_{m-1} = 1$, $0 < \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2) < 2$, $\alpha_i \geq 0$ for $i \in \{1, \ldots, m-3\} \cup \{m-1\}$ and $\alpha_{m-2} > 0$. $a \in C([0,1],[0,\infty))$ and $f \in C([0,\infty),[0,\infty))$. As an application, we give some examples that illustrate our results.

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1. Introduction

We are interested in the existence of positive solutions of the following m-point boundary value problem (BVP):

$$u''(t) + a(t)f(u(t)) = 0, \qquad 0 < t < 1, \tag{1}$$

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} u(s)ds, \tag{2}$$

where $0 = \eta_0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < \eta_{m-1} = 1$, $0 < \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2) < 2$, $\alpha_i \geq 0$ for $i \in \{1, \ldots, m-3\} \cup \{m-1\}$, $\alpha_{m-2} > 0$, $f \in C([0,\infty),[0,\infty))$, $a \in C([0,1],[0,\infty))$, and there exists $t_0 \in [\eta_{m-2}, 1]$ such that $a(t_0) > 0$.

The study of multi-point boundary-value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [1-2]. Gupta [3] studied three-point boundary-value problems for nonlinear second order ordinary differential equations. Since then, the existence of solutions for nonlinear three-point, m-point, integral and nonlocal boundary value problems has been studied by several authors by using the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, coincidence degree theory and the fixed-point theorem in cones. We refer the reader to [4-21, 24-30] and the references therein. However, most of those papers have studied the existence and multiplicity of solutions (or positive solutions) to boundary value problems with boundary conditions specified as solution values or the slope of solution values at given points or as more general nonlocal integral boundary conditions. There are also a few papers that consider nonlinear second order ordinary differential equations with boundary conditions specified as an area under the curve of solutions; see [22-23].

For the existence problems of positive solution of the BVP (1)-(2), by defining

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u},$$

in [23], the authors used the Krasnoselskii theorem to prove the following result.

Theorem 1. (*See*[23]). *The BVP* (1)-(2) *has at least one positive solution in the case*

(D_1) $f_0 = 0$ *and* $f_\infty = \infty$ *(f is superlinear), or*

(D_2) $f_0 = \infty$ *and* $f_\infty = 0$ *(f is superlinear).*

From Theorem 1, the following two problems are natural.

Problem 1. Whether we can obtain a similar conclusion or not, if $f_0 = f_\infty = 0$, or $f_0 = f_\infty = \infty$.

Problem 2. Whether we can get a similar conclusion or not, if $f_0, f_\infty \notin \{0, \infty\}$.

Motivated by the results of [23], the aim of this paper is to establish some simple criteria for the existence of positive solutions of the BVP (1)-(2), which gives a positive answer to the questions stated above. The key tool in our approach is the following fixed-point index theorem [31].

Theorem 2. *Let E be a Banach space, and $K \subset E$ be a cone in E. Let $r > 0$, and define $\Omega_r = \{x \in K \mid \|x\| < r\}$. Assume $A : \overline{\Omega_r} \to K$ is a completely continuous operator such that $Ax \neq x$ for $x \in \partial\Omega_r$.*

(i) If $\|Au\| \leq \|u\|$ for $u \in \partial\Omega_r$, then

$$i(A, \Omega_r, K) = 1.$$

(ii) If $\|Au\| \geq \|u\|$ for $u \in \partial\Omega_r$, then

$$i(A, \Omega_r, K) = 0.$$

2. The Existence Results of the BVP (1)-(2) for the Case: $f_0 = f_\infty = \infty$ or $f_0 = f_\infty = 0$

In this section, we establish conditions for the existence of two positive solutions for the BVP (1)-(2) under $f_0 = f_\infty = \infty$ or $f_0 = f_\infty = 0$.

Lemma 3. (See[23]) *Let $\alpha_i \geq 0$ for $i = 1, 2, \ldots, m-1$, and $\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2) \neq 2$. If $y \in C[0, 1]$, then the problem*

$$u''(t) + y(t) = 0, \quad 0 < t < 1, \tag{3}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} u(s)ds, \tag{4}$$

has a unique solution

$$u(t) = -\int_0^t (t-s)y(s)ds + \frac{2t}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \int_0^1 (1-s)y(s)ds$$

$$- \frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-s)^2y(s)ds$$
$$+ \frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2y(s)ds. \qquad (5)$$

Lemma 4. *(See[23]) Let $\alpha_i \geq 0$ for $i = 1, 2, \ldots, m-1$, and $\sum_{i=1}^{m-1}\alpha_i(\eta_i^2 - \eta_{i-1}^2) < 2$. If $y \in C([0,1],[0,\infty))$, then the unique solution u of (3)-(4) satisfies $u(t) \geq 0$ for $t \in [0,1]$.*

Lemma 5. *(See[23]) Let $\alpha_i \geq 0$ for $i \in \{1,\ldots,m-3\}\cup\{m-1\}, \alpha_{m-2} > 0$ and $\sum_{i=1}^{m-1}\alpha_i(\eta_i^2 - \eta_{i-1}^2) > 2$. If $y \in C[0,1]$ and $y(t) \geq 0$ for $t \in (0,1)$, then (3)-(4) has no positive solution.*

Lemma 6. *(See[23]) Let $\alpha_i \geq 0$ for $i \in \{1,2,\ldots,m-3\}\cup\{m-1\}$, $\alpha_{m-2} > 0$ and $\sum_{i=1}^{m-1}\alpha_i(\eta_i^2 - \eta_{i-1})^2 < 2$. If $y \in C[0,1]$ and $y(t) \geq 0$ for $t \in (0,1)$, then the unique solution u of problem (3)-(4) satisfies*

$$\inf_{t\in[\eta_{m-2},1]} u(t) \geq \gamma\|u\|,$$

where

$$\gamma = \min\left\{\eta_{m-2}, \frac{1}{2}\sum_{i=1}^{m-2}\alpha_i(\eta_i^2-\eta_{i-1}^2), \frac{\alpha_{m-2}(\eta_{m-2}^2-\eta_{m-3}^2)(1-\eta_{m-2})}{\eta_{m-2}(2-\alpha_{m-2}(\eta_{m-2}^2-\eta_{m-3}^2))},\right.$$
$$\left.\frac{1}{2}\sum_{i=1}^{m-2}\frac{\alpha_i}{\eta_i}(\eta_i^2-\eta_{i-1}^2)(1-\eta_i)\right\}. \qquad (6)$$

Let $E = C[0,1]$, and only the sup norm is used. It is easy to see that BVP (1)-(2) has a solution $u = u(t)$ if and only if u is a solution of the operator equation

$$u = Au,$$

where

$$(Au)(t) = -\int_0^t (t-s)a(s)f(u(s))ds$$
$$+ \frac{2t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1(1-s)a(s)f(u(s))ds$$
$$- \frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-s)^2a(s)f(u(s))ds$$

$$+\frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds. \quad (7)$$

Denote

$$K=\{u\in E: u\geq 0, \min_{\eta_{m-2}\leq t\leq 1}u(t)\geq\gamma\|u\|\}. \quad (8)$$

It is obvious that K is a cone in E. Moreover, by Lemma 6, $A(K)\subset K$. It is also easy to see that $A:K\to K$ is completely continuous.

In the following, for the sake of convenience, set

$$\Lambda_1=\frac{2\int_0^1(1-s)a(s)ds+\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)},$$

$$\Lambda_2=\gamma\frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^1(1-s)a(s)ds.$$

Theorem 7. *Suppose that the following assumptions are satisfied.*
$(H1)$ $f_0=f_\infty=\infty$.
$(H2)$ *There exist constants* $\rho_1>0$ *and* $M_1\in(0,\Lambda_1^{-1})$ *such that*

$$f(u)\leq M_1\rho_1, \qquad u\in[0,\rho_1].$$

Then, the BVP (1)-(2) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0<\|u_1\|<\rho_1<\|u_2\|.$$

Proof. At first, in view of $f_0=\lim_{u\to 0^+}(f(u)/u)=\infty$, then for any $M_*\in(\Lambda_2^{-1},\infty)$, there exist $\rho_*\in(0,\rho_1)$ such that

$$f(u)\geq M_*u, \qquad 0\leq u\leq\rho_*. \quad (9)$$

Set $\Omega_{\rho_*}=\{u\in K:\|u\|<\rho_*\}$. Since $u\in\partial\Omega_{\rho_*}\subset K$, we have $\min_{\eta_{m-2}\leq t\leq 1}u(t)\geq\gamma\|u\|$. Thus, from (7), (9), for any $u\in\partial\Omega_{\rho_*}$, we have

$$u(1)=\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}u(s)ds$$
$$=\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\Bigg(-\int_0^s(s-r)a(r)f(u(r))dr$$
$$+\frac{2s}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1(1-r)a(r)f(u(r))dr$$

$$
\begin{aligned}
&- \frac{s}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-r)^2a(r)f(u(r))dr \\
&+ \frac{s}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-r)^2a(r)f(u(r))dr\Bigg)ds \\
&= -\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\int_0^s(s-r)a(r)f(u(r))drds \\
&+ \frac{2\int_0^1(1-r)a(r)f(u(r))dr}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\left(\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}sds\right) \\
&- \frac{\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-r)^2a(r)f(u(r))dr}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\left(\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}sds\right) \\
&+ \frac{\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-r)^2a(r)f(u(r))dr}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\left(\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}sds\right) \\
&= \frac{1}{2}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds \\
&- \frac{1}{2}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-s)^2a(s)f(u(s))ds \\
&+ \frac{\int_0^1(1-s)a(s)f(u(s))ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\left(\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)\right) \\
&- \frac{\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-s)^2a(s)f(u(s))ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\left(\sum_{i=1}^{m-1}\frac{\alpha_i}{2}(\eta_i^2-\eta_{i-1}^2)\right) \\
&+ \frac{\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\left(\sum_{i=1}^{m-1}\frac{\alpha_i}{2}(\eta_i^2-\eta_{i-1}^2)\right) \\
&= \frac{\int_0^1(1-s)a(s)f(u(s))ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2) \\
&- \frac{\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-s)^2a(s)f(u(s))ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)} \\
&+ \frac{\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\times \\
&\sum_{i=1}^{m-1}\alpha_i\Bigg[\int_0^{\eta_{i-1}}(\eta_i-\eta_{i-1})(2-(\eta_i+\eta_{i-1}))sa(s)f(u(s))ds \\
&+\int_{\eta_{i-1}}^{\eta_i}\big[(\eta_i^2-\eta_{i-1}^2)(1-s)-(\eta_i-s)^2\big]a(s)f(u(s))ds \\
&+\int_{\eta_i}^{1}(\eta_i^2-\eta_{i-1}^2)(1-s)a(s)f(u(s))ds\Bigg] \\
&\geq \frac{1}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\times \\
&\sum_{i=1}^{m-1}\alpha_i\Bigg[\int_{\eta_{i-1}}^{\eta_i}(\eta_i-\eta_{i-1})\big[\eta_{i-1}(1-s)+(1-\eta_i)s\big]a(s)f(u(s))ds \\
&+(\eta_i^2-\eta_{i-1}^2)\int_{\eta_i}^{1}(1-s)a(s)f(u(s))ds\Bigg] \\
&\geq \frac{1}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)\int_{\eta_i}^{1}(1-s)a(s)f(u(s))ds \\
&\geq \ \frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^{1}(1-s)a(s)f(u(s))ds \\
&\geq \ \frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^{1}(1-s)a(s)M_*u(s)ds \\
&\geq \ M_*\gamma\frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^{1}(1-s)a(s)ds\|u\| \\
&= M_*\Lambda_2\|u\| > \|u\|, \qquad (10)
\end{aligned}
$$

which yields

$$\|Au\| \geq |u(1)| > \|u\|, \quad u \in \partial\Omega_{\rho_*}.$$

Hence, Theorem 2 implies

$$i(A,\Omega_{\rho_*},K)=0. \tag{11}$$

Next, since $f_\infty = \lim_{u\to\infty}(f(u)/u) = \infty$, then for any $M^* \in (\Lambda_2^{-1}, \infty)$, there exist $\rho^* > \rho_1$ such that

$$f(u) \geq M^*u, \qquad u \geq \gamma\rho^*. \tag{12}$$

Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$, for $u \in \partial\Omega_{\rho^*}$, since $u \in K$, $\min_{\eta_{m-2} \le t \le 1} u(t) \ge \gamma\|u\| = \gamma\rho^*$, and hence, for any $u \in \partial\Omega_{\rho^*}$, from (7), (12), by using the method to get (10), we have

$$\begin{aligned}
u(1) &\ge \frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^1 (1-s)a(s)f(u(s))ds \\
&\ge \frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^1 (1-s)a(s)M^*u(s)ds \\
&\ge M^*\gamma\frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^1 (1-s)a(s)ds\|u\| \\
&= M^*\Lambda_2\|u\| > \|u\|,
\end{aligned}$$

which implies

$$\|Au\| \ge |u(1)| > \|u\|, \quad u \in \partial\Omega_{\rho^*}.$$

Thus, Theorem 2 yields

$$i(A, \Omega_{\rho^*}, K) = 0. \tag{13}$$

Finally, set $\Omega_{\rho_1} = \{u \in K : \|u\| < \rho_1\}$. For any $u \in \partial\Omega_{\rho_1}$, from (7) and $(H2)$ we obtain

$$\begin{aligned}
Au(t) &\le \frac{2t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1 (1-s)a(s)f(u(s))ds \\
&\quad + \frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds \\
&\le \frac{2M_1\rho_1}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1 (1-s)a(s)ds \\
&\quad + \frac{M_1\rho_1}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds \\
&\le \left(\frac{2\int_0^1(1-s)a(s)ds+\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\right)M_1\rho_1 \\
&= \Lambda_1M_1\rho_1 < \rho_1 = \|u\|,
\end{aligned}$$

which yields

$$\|Au\| < \|u\|, \qquad u \in \partial\Omega_{\rho_1}.$$

Thus, an application of Theorem 2 again shows that

$$i(A, \Omega_{\rho_1}, K) = 1. \tag{14}$$

Hence, since $\rho_* < \rho_1 < \rho_*$ and (11), (13), (14), it follows from the additivity of the fixed-point index that

$$i(A, \Omega_{\rho_1} \setminus \overline{\Omega}_{\rho_*}, K) = 1,$$
$$i(A, \Omega_{\rho^*} \setminus \overline{\Omega}_{\rho_1}, K) = -1.$$

Therefore, A has a fixed point u_1 in $\Omega_{\rho_1} \setminus \overline{\Omega}_{\rho_*}$ and a fixed point u_2 in $\Omega_{\rho^*} \setminus \overline{\Omega}_{\rho_1}$. Both are positive solutions of the BVP (1)-(2) and

$$0 < \|u_1\| < \rho_1 < \|u_2\|.$$

The proof is complete. □

Theorem 8. *Assume that the following assumptions are satisfied.*
$(H3)$ $f_0 = f_\infty = 0$.
$(H4)$ *There exist constants $\rho_2 > 0$ and $M_2 \in (\Lambda_2^{-1}, \infty)$ such that*

$$f(u) \geq M_2\rho_2, \quad u \in [\gamma\rho_2, \rho_2].$$

Then, the BVP (1)-(2) has at least two positive solutions u_1 and u_2 such that

$$0 < \|u_1\| < \rho_2 < \|u_2\|.$$

Proof. Firstly, since $f_0 = \lim_{u\to 0^+}(f(u)/u) = 0$, for any $\varepsilon \in (0, \Lambda_1^{-1})$, there exist $\rho_* \in (0, \rho_2)$ such that

$$f(u) \leq \varepsilon u, \quad u \in [0, \rho_*]. \tag{15}$$

Setting $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for any $u \in \partial\Omega_{\rho_*}$, from (7) and (15) we get

$$\begin{aligned} Au(t) &\leq \frac{2t}{2 - \sum_{i=1}^{m-1}\alpha_i(\eta_i^2 - \eta_{i-1}^2)}\int_0^1 (1-s)a(s)f(u(s))ds \\ &+ \frac{t}{2 - \sum_{i=1}^{m-1}\alpha_i(\eta_i^2 - \eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds \\ &\leq \frac{2\varepsilon\rho_*}{2 - \sum_{i=1}^{m-1}\alpha_i(\eta_i^2 - \eta_{i-1}^2)}\int_0^1 (1-s)a(s)ds \end{aligned}$$

$$+\frac{\varepsilon\rho_*}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds$$
$$\leq\left(\frac{2\int_0^1(1-s)a(s)ds+\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\right)\varepsilon\rho_*$$
$$=\Lambda_1\varepsilon\rho_*<\rho_*=\|u\|,$$

which yields

$$\|Au\|<\|u\|,\quad u\in\partial\Omega_{\rho_*}.$$

Hence, Theorem 2 implies

$$i(A,\Omega_{\rho_*},K)=1. \tag{16}$$

Secondly, in view of $f_\infty=\lim_{u\to\infty}(f(u)/u)=0$, then for any $\varepsilon\in(0,\Lambda_1^{-1})$, there exist $\rho_0>\rho_2$ such that

$$f(u)\leq\varepsilon u,\qquad u\in[\rho_0,\infty), \tag{17}$$

and we consider two cases.

Case (i). Suppose that $f(u)$ is unbounded; then from $f\in C([0,\infty),[0,\infty))$, we know that there is $\rho^*>\rho_0$ such that

$$f(u)\leq f(\rho^*),\qquad u\in[0,\rho^*]. \tag{18}$$

Since $\rho^*>\rho_0$, then from (17), (18), we have

$$f(u)\leq f(\rho^*)\leq\varepsilon\rho^*,\qquad u\in[0,\rho^*]; \tag{19}$$

for $u\in K$, $\|u\|=\rho^*$, (7) and (19), we can get

$$Au(t)\leq\frac{2t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1(1-s)a(s)f(u(s))ds$$
$$+\frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds$$
$$\leq\frac{2\varepsilon\rho^*}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1(1-s)a(s)ds$$
$$+\frac{\varepsilon\rho^*}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds$$

$$\leq \left(\frac{2\int_0^1 (1-s)a(s)ds + \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_{i-1}} (\eta_{i-1}-s)^2 a(s)ds}{2-\sum_{i=1}^{m-1} \alpha_i(\eta_i^2-\eta_{i-1}^2)} \right) \varepsilon\rho^*$$
$$= \Lambda_1 \varepsilon \rho^* < \rho^* = \|u\|,$$

Case (ii). Suppose that $f(u)$ is bounded, that is $f(u) \leq L$. Taking $\rho^* \geq \max\{L/\varepsilon, \rho_2\}$, for $u \in K$, $\|u\| = \rho^*$, from (7), we have

$$Au(t) \leq \frac{2t}{2-\sum_{i=1}^{m-1} \alpha_i(\eta_i^2-\eta_{i-1}^2)} \int_0^1 (1-s)a(s)f(u(s))ds$$
$$+ \frac{t}{2-\sum_{i=1}^{m-1} \alpha_i(\eta_i^2-\eta_{i-1}^2)} \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_{i-1}} (\eta_{i-1}-s)^2 a(s)f(u(s))ds$$
$$\leq \frac{2L}{2-\sum_{i=1}^{m-1} \alpha_i(\eta_i^2-\eta_{i-1}^2)} \int_0^1 (1-s)a(s)f(u(s))ds$$
$$+ \frac{L}{2-\sum_{i=1}^{m-1} \alpha_i(\eta_i^2-\eta_{i-1}^2)} \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_{i-1}} (\eta_{i-1}-s)^2 a(s)f(u(s))ds$$
$$\leq \frac{2\varepsilon\rho^*}{2-\sum_{i=1}^{m-1} \alpha_i(\eta_i^2-\eta_{i-1}^2)} \int_0^1 (1-s)a(s)ds$$
$$+ \frac{\varepsilon\rho^*}{2-\sum_{i=1}^{m-1} \alpha_i(\eta_i^2-\eta_{i-1}^2)} \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_{i-1}} (\eta_{i-1}-s)^2 a(s)ds$$
$$\leq \left(\frac{2\int_0^1 (1-s)a(s)ds + \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_{i-1}} (\eta_{i-1}-s)^2 a(s)ds}{2-\sum_{i=1}^{m-1} \alpha_i(\eta_i^2-\eta_{i-1}^2)} \right) \varepsilon\rho^*$$
$$= \Lambda_1 \varepsilon \rho^* < \rho^* = \|u\|,$$

Hence, in either case, we always may say $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$ such that

$$\|Au\| < \|u\|, \qquad u \in \partial\Omega_{\rho^*}.$$

Therefore, Theorem 2 yields

$$i(A, \Omega_{\rho^*}, K) = 1. \tag{20}$$

Finally, set $\Omega_{\rho_2} = \{u \in K : \|u\| < \rho_2\}$. For any $u \in \partial\Omega_{\rho_2}$, since $u \in K$, $\min_{\eta_{m-2} \leq t \leq 1} u(t) \geq \gamma\|u\| = \gamma\rho_2$, and hence, for any $u \in \partial\Omega_{\rho_2}$, from (7) and $(H4)$, by using the method to get (10), we obtain

$$u(1) \geq \frac{\sum_{i=1}^{m-1} \alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1} \alpha_i(\eta_i^2-\eta_{i-1}^2)} \int_{\eta_{m-2}}^1 (1-s)a(s)f(u(s))ds$$

$$\geq \frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^{1}(1-s)a(s)M_2\rho_2 ds$$
$$\geq M_2\rho_2\gamma\frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^{1}(1-s)a(s)ds$$
$$= M_2\rho_2\Lambda_2 > \rho_2 = \|u\|,$$

which implies

$$\|Au\| \geq |u(1)| > \|u\|, \qquad u \in \partial\Omega_{\rho_2}.$$

Thus, an application of Theorem 2 again shows that

$$i(A, \Omega_{\rho_2}, K) = 0. \tag{21}$$

Hence, since $\rho_* < \rho_2 < \rho_*$ and (16), (20), (21), it follows from the additivity of the fixed-point index that

$$i(A, \Omega_{\rho_2} \setminus \overline{\Omega}_{\rho_*}, K) = -1,$$
$$i(A, \Omega_{\rho^*} \setminus \overline{\Omega}_{\rho_2}, K) = 1.$$

Therefore, A has a fixed point u_1 in $\Omega_{\rho_2} \setminus \overline{\Omega}_{\rho_*}$ and a fixed point u_2 in $\Omega_{\rho^*} \setminus \overline{\Omega}_{\rho_2}$. Both are positive solutions of the BVP (1)-(2) and

$$0 < \|u_1\| < \rho_2 < \|u_2\|.$$

This completes the proof. □

3. The Existence Results of the BVP (1)-(2) for the Case: f_0, $f_\infty \notin \{0, \infty\}$

In this section we discuss the existence for positive solution of the BVP (1)-(2) assuming f_0, $f_\infty \notin \{0, \infty\}$.

Theorem 9. *Suppose* $(H2)$ *and* $(H4)$ *hold and that* $\rho_1 \neq \rho_2$. *Then the BVP* (1)-(2) *has at least one positive solution* u *satisfying* $\rho_1 < \|u\| < \rho_2$ *or* $\rho_2 < \|u\| < \rho_1$.

Proof. Without loss of generality, we may assume that $\rho_1 < \rho_2$.

Set $\Omega_{\rho_1} = \{u \in K : \|u\| < \rho_1\}$. For any $u \in \partial\Omega_{\rho_1}$, from (7) and $(H2)$, we have

$$Au(t) \leq \frac{2t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1(1-s)a(s)f(u(s))ds$$

$$+\frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds$$
$$\leq \frac{2M_1\rho_1}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1(1-s)a(s)ds$$
$$+\frac{M_1\rho_1}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds$$
$$\leq \left(\frac{2\int_0^1(1-s)a(s)ds+\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\right)M_1\rho_1$$
$$=\Lambda_1M_1\rho_1<\rho_1=\|u\|,$$

which yields

$$\|Au\|<\|u\|,\qquad u\in\partial\Omega_{\rho_1}.$$

Hence, Theorem 2 implies

$$i(A,\Omega_{\rho_1},K)=1. \tag{22}$$

Next, set $\Omega_{\rho_2}=\{u\in K:\|u\|<\rho_2\}$. For any $u\in\partial\Omega_{\rho_2}$, since $u\in K$, $\min_{\eta_{m-2}\leq t\leq 1}u(t)\geq\gamma\|u\|=\gamma\rho_2$, and hence, for any $u\in\partial\Omega_{\rho_2}$, from (7) and $(H4)$, by using the method to get (10), we get

$$u(1)\geq\frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^1(1-s)a(s)f(u(s))ds$$
$$\geq\frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^1(1-s)a(s)M_2\rho_2ds$$
$$\geq M_2\rho_2\gamma\frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^1(1-s)a(s)ds$$
$$=M_2\rho_2\Lambda_2>\rho_2=\|u\|,$$

which implies

$$\|Au\|\geq|u(1)|>\|u\|,\qquad u\in\partial\Omega_{\rho_2}.$$

Thus, an application of Theorem 2 again shows that

$$i(A,\Omega_{\rho_2},K)=0. \tag{23}$$

Hence, since $\rho_1<\rho_2$ and (22), (23), it follows from the additivity of the fixed-point index that

$$i(A,\Omega_{\rho_2}\setminus\overline{\Omega}_{\rho_1},K)=-1.$$

Therefore, A has a fixed point u in $\Omega_{\rho_2} \setminus \overline{\Omega}_{\rho_1}$. Moreover, it is a positive solution of the BVP (1)-(2) and

$$\rho_1 < \|u\| < \rho_2.$$

The proof is complete. □

Corollary 10. *Assume that the following assumptions hold.*
$(H5)$ $f_0 = \xi_1 \in [0, \theta_1 \Lambda_1^{-1})$, *where the constant* $\theta_1 \in [0, 1)$.
$(H6)$ $f_\infty = \beta_1 \in ((\theta_2/\gamma)\Lambda_2^{-1}, \infty)$, *where the constant* $\theta_2 > 1$.
Then, the BVP (1)-(2) *has at least one positive solution.*

Proof. In view of $f_0 = \xi_1 \in [0, \theta_1 \Lambda_1^{-1})$, for $\varepsilon = \theta_1 \Lambda_1^{-1} - \xi_1 > 0$, there exists a sufficiently small $\rho_1 > 0$ such that

$$f(u) \leq (\xi_1 + \varepsilon)u = \theta_1 \Lambda_1^{-1} u \leq \theta_1 \Lambda_1^{-1} \rho_1, \qquad u \in [0, \rho_1].$$

Since $\theta_1 \in [0, 1)$, then $\theta_1 \Lambda_1^{-1} \in [0, \Lambda_1^{-1})$. By the inequality above, $(H2)$ is satisfied.

Since $f_\infty = \beta_1 \in ((\theta_2/\gamma)\Lambda_2^{-1}, \infty)$, for $\varepsilon = \beta_1 - (\theta_2/\gamma)\Lambda_2^{-1} > 0$, there exists a sufficiently large $\rho_2 (> \rho_1)$ such that

$$\frac{f(u)}{u} \geq \beta_1 - \varepsilon = \frac{\theta_2}{\gamma} \Lambda_2^{-1}, \qquad u \in [\gamma \rho_2, \infty),$$

thus, when $u \in [\gamma \rho_2, \rho_2]$, we have

$$f(u) \geq \frac{\theta_2}{\gamma} \Lambda_2^{-1} u \geq \theta_2 \Lambda_2^{-1} \rho_2.$$

Since $\theta_2 > 1$, $\theta_2 \Lambda_2^{-1} \in (\Lambda_2^{-1}, \infty)$, then from the above inequality, condition $(H4)$ is satisfied. Therefore, from Theorem 9, the desired result holds. □

Corollary 11. *Suppose that the following assumptions hold.*
$(H7)$ $f_0 = \xi_2 \in ((\theta_2/\gamma)\Lambda_2^{-1}, \infty)$, *where the constant* $\theta_2 > 1$.
$(H8)$ $f_\infty = \beta_2 \in [0, \theta_1 \Lambda_1^{-1})$, *where the constant* $\theta_1 \in [0, 1)$.
Then, the BVP (1)-(2) *has at least one positive solution.*

Proof. Since $f_0 = \xi_2 \in ((\theta_2/\gamma)\Lambda_2^{-1}, \infty)$, for $\varepsilon = \xi_2 - (\theta_2/\gamma)\Lambda_2^{-1} > 0$, there exists a sufficiently small $\rho_2 > 0$ such that

$$\frac{f(u)}{u} \geq \xi_2 - \varepsilon = \frac{\theta_2}{\gamma} \Lambda_2^{-1}, \qquad u \in (0, \rho_2),$$

thus, when $u \in [\gamma\rho_2, \rho_2]$, we have

$$f(u) \geq \frac{\theta_2}{\gamma}\Lambda_2^{-1}u \geq \theta_2\Lambda_2^{-1}\rho_2,$$

which implies $(H4)$ holds.

In view of $f_\infty = \beta_2 \in [0, \theta_1\Lambda_1^{-1})$, for $\varepsilon = \theta_1\Lambda_1^{-1} - \beta_2 > 0$, there exists a sufficiently large $\rho_0(> \rho_2)$ such that

$$\frac{f(u)}{u} \leq \beta_2 + \varepsilon = \theta_1\Lambda_1^{-1}, \qquad u \in [\rho_0, \infty). \tag{24}$$

We consider the following two cases.

Case (i). Suppose that $f(u)$ is unbounded. Because $f \in C([0,\infty), [0,\infty))$, we know that there is a $\rho_1 > \rho_0$ such that

$$f(u) \leq f(\rho_1), \qquad u \in [0, \rho_1]. \tag{25}$$

Since $\rho_1 > \rho_0$, then from (24), (25), we can get

$$f(u) \leq f(\rho_1) \leq \theta_1\Lambda_1^{-1}\rho_1, \qquad u \in [0, \rho_1].$$

Since $\theta_1 \in [0, 1)$, then $\theta_1\Lambda_1^{-1} \in [0, \Lambda_1^{-1})$. By the inequality above, $(H2)$ is satisfied.

Case (ii). Suppose that $f(u)$ is bounded, that is

$$f(u) \leq M, \qquad u \in [0, \infty). \tag{26}$$

In this case, taking sufficiently large $\rho_1 > M/\theta_1\Lambda_1^{-1}$, then from (26), we have

$$f(u) \leq M \leq \theta_1\Lambda_1^{-1}\rho_1, \qquad u \in [0, \rho_1].$$

Since $\theta_1 \in [0, 1)$, then $\theta_1\Lambda_1^{-1} \in [0, \Lambda_1^{-1})$. By the inequality above, $(H2)$ is satisfied.

Therefore, from Theorem 9, we get the conclusion of Corollary 11 □

Corollary 12. *Assume Conditions $(H2)$, $(H6)$ and $(H7)$ hold. Then, the BVP (1)-(2) has at least two positive solutions u_1 and u_2 such that*

$$0 < \|u_1\| < \rho_1 < \|u_2\|.$$

Proof. From $(H6)$ and the proof of Corollary 10, we get that there exists a sufficiently large $\rho_2 > \rho_1$ such that

$$f(u) \geq \theta_2 \Lambda_2^{-1} \rho_2 = M_2 \rho_2, \qquad u \in [\gamma \rho_2, \rho_2],$$

where $M_2 = \theta_2 \Lambda_2^{-1} \in (\Lambda_2^{-1}, \infty)$.

In view of $(H7)$ and the proof of Corollary 11, we have that there exists a sufficiently small $\rho_2^* \in (0, \rho_1)$ such that

$$f(u) \geq \theta_2 \Lambda_2^{-1} \rho_2^* = M_2 \rho_2^*, \qquad u \in [\gamma \rho_2^*, \rho_2^*],$$

where $M_2 = \theta_2 \Lambda_2^{-1} \in (\Lambda_2^{-1}, \infty)$.

Using this and $(H2)$, we know by Theorem 9 that the BVP (1)-(2) has two positive solutions u_1 and u_2 such that

$$\rho_2^* < \|u_1\| < \rho_1 < \|u_2\| < \rho_2.$$

The proof is complete. □

Corollary 13. *Assume Conditions $(H4)$, $(H5)$ and $(H8)$ hold. Then, the BVP (1)-(2) has at least two positive solutions u_1 and u_2 such that*

$$0 < \|u_1\| < \rho_2 < \|u_2\|.$$

Proof. By $(H5)$ and the proof of Corollary 10, we obtain that there exists a sufficiently small $\rho_1 \in (0, \rho_2)$ such that

$$f(u) \leq \theta_1 \Lambda_1^{-1} \rho_1 = M_1 \rho_1, \qquad u \in [0, \rho_1],$$

where $M_1 = \theta_1 \Lambda_1^{-1} \in (0, \Lambda_1^{-1})$.

In view of $(H8)$ and the proof of Corollary 11, we see that there exists a sufficiently large $\rho_1^* > \rho_2$ such that

$$f(u) \leq \theta_1 \Lambda_1^{-1} \rho_1^* = M_1 \rho_1^*, \qquad u \in [0, \rho_1^*],$$

where $M_1 = \theta_1 \Lambda_1^{-1} \in (0, \Lambda_1^{-1})$.

Using this and $(H4)$, we know by Theorem 9 that the BVP (1)-(2) has two positive solutions u_1 and u_2 such that

$$\rho_1 < \|u_1\| < \rho_2 < \|u_2\| < \rho_1^*.$$

The proof is complete. □

4. Some Examples

In this section, in order to illustrate our result, we consider some examples.

Example 14. Consider the three-point boundary value problem

$$u''(t) + (\frac{1}{2} - t)^2(u^{\frac{1}{2}} + u^2) = 0, \qquad 0 < t < 1, \tag{27}$$

$$u(0) = 0, \qquad u(1) = \int_0^{\frac{1}{2}} u(s)ds + \frac{1}{2}\int_{\frac{1}{2}}^1 u(s)ds \tag{28}$$

Set $\eta_0 = 0, \eta_1 = \frac{1}{2}, \eta_2 = 1$, $\alpha_1 = 1, \alpha_2 = \frac{1}{2}$, $a(t) = (\frac{1}{2} - t)^2$ and $f(u) = u^{\frac{1}{2}} + u^2$. We can show that $\sum_{i=1}^{2} \alpha_i(\eta_i^2 - \eta_{i-1}^2) = 5/8 < 2$. Through a simple calculation we can get $f_0 = f_\infty = \infty$, then $(H1)$ holds. Again $\gamma = \min\{\eta_{m-2}, \sum_{i=1}^{m-2} \alpha_i(\eta_i^2 - \eta_{i-1}^2)/2, \alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2)(1 - \eta_{m-2})/\eta_{m-2}(2 - \alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2)), \sum_{i=1}^{m-2} \alpha_i(\eta_i^2 - \eta_{i-1}^2)(1 - \eta_i)/2\eta_i\} = 1/8$, $\Lambda_1 = (2\int_0^1 (1 - s)a(s)ds + \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^2 a(s)ds)/(2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)) = 83/1320$, because $f(u)$ is monotone increasing function for $u \geq 0$, taking $\rho_1 = 9, M_1 = 28/3 \in (0, \Lambda_1^{-1})$, then when $u \in [0, \rho_1]$, we get

$$f(u) \leq f(9) = 84 = M_1\rho_1,$$

which implies $(H2)$ holds. Hence, by Theorem 7, the BVP (27)-(28) has at least two positive solutions u_1 and u_2 such that

$$0 < \|u_1\| < 9 < \|u_2\|.$$

Example 15. Consider the three-point boundary value problem

$$u''(t) + e^4(1 - t)^2u^2e^{-u} = 0, \qquad 0 < t < 1, \tag{29}$$

$$u(0) = 0, \qquad u(1) = 16\int_0^{\frac{1}{4}} u(s)ds + \int_{\frac{1}{4}}^1 u(s)ds \tag{30}$$

Set $\eta_0 = 0, \eta_1 = \frac{1}{4}, \eta_2 = 1$, $\alpha_1 = 16, \alpha_2 = 1$, $a(t) = e^4(1-t)^2$ and $f(u) = u^2e^{-u}$. We can show that $\sum_{i=1}^{2} \alpha_i(\eta_i^2 - \eta_{i-1}^2) = 31/16 < 2$. Since $f_0 = f_\infty = 0$, then $(H3)$ holds. Again $\gamma = \min\{\eta_{m-2}, \sum_{i=1}^{m-2} \alpha_i(\eta_i^2 - \eta_{i-1}^2)/2, \alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2)(1 - \eta_{m-2})/\eta_{m-2}\ (2 - \alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2))$, $\sum_{i=1}^{m-2} \alpha_i(\eta_i^2 - \eta_{i-1}^2)(1 -$

$\eta_i)/2\eta_i\} = 1/4$, $\Lambda_2 = \gamma\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)\int_{\eta_{m-2}}^1(1-s)a(s)ds/(2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)) = 7533e^4/16384$, because $f(u)$ is monotone decreasing function for $u \geq 1$, taking $\rho_2 = 4, M_2 = 4e^{-4} \in (\Lambda_2^{-1}, \infty)$, then when $u \in [\gamma\rho_2, \rho_2]$, we get

$$f(u) \geq f(4) = 16e^{-4} = M_2\rho_2,$$

which implies $(H4)$ holds. Hence, by Theorem 8, the BVP (29)-(30) has at least two positive solutions u_1 and u_2 such that

$$0 < \|u_1\| < 4 < \|u_2\|.$$

Example 16. Consider the three-point boundary value problem

$$u''(t) + \frac{1}{40}\frac{aue^{2u}}{b + e^u + e^{2u}} = 0, \qquad 0 < t < 1, \tag{31}$$

$$u(0) = 0, \qquad u(1) = 30\int_0^{\frac{1}{4}} u(s)ds + \frac{1}{15}\int_{\frac{1}{4}}^1 u(s)ds, \tag{32}$$

where $a = 80$, $b = 38$. Set $\eta_0 = 0, \eta_1 = 1/4, \eta_2 = 1$, $\alpha_1 = 30, \alpha_2 = 1/15$, $a(t) = 1/40$ and $f(u) = aue^{2u}/(b + e^u + e^{2u})$. We can show that $\sum_{i=1}^{2}\alpha_i(\eta_i^2 - \eta_{i-1}^2) = 31/16 < 2$. Since

$$\begin{aligned}\gamma = \min\Big\{&\eta_{m-2}, \sum_{i=1}^{m-2}\alpha_i(\eta_i^2 - \eta_{i-1}^2)/2,\\ &\alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2)(1 - \eta_{m-2})/\eta_{m-2}(2 - \alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2)),\\ &\sum_{i=1}^{m-2}\alpha_i(\eta_i^2 - \eta_{i-1}^2)(1 - \eta_i)/2\eta_i\Big\} = 1/4,\end{aligned}$$

$$\begin{aligned}\Lambda_1 = \Big(2\int_0^1(1-s)a(s)ds+\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds\Big)/\Big(2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)\Big)\\ = 2881/7200,\end{aligned}$$

$$\Lambda_2 = \gamma\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)\int_{\eta_{m-2}}^1(1-s)a(s)ds/\Big(2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)\Big) = 279/5120,$$

and $f_0 = \frac{a}{2+b} = 2$, $f_\infty = a = 80$. Taking $\theta_1 \in (5762/7200, 1)$, $\theta_2 \in (1, 5580/5120)$, thus $f_0 \in (0, \theta_1\Lambda_1^{-1})$, $f_\infty \in ((\theta_2/\gamma)\Lambda_2^{-1}, \infty)$, which imply $(H5)$ and $(H6)$ hold. Therefore, by Corollary 10, the BVP (31)-(32) has at least one positive solution.

Example 17. Consider the three-point boundary value problem

$$u''(t) + tu\left(1 + \frac{c}{1+u^2}\right) = 0, \qquad 0 < t < 1, \tag{33}$$

$$u(0) = 0, \qquad u(1) = 2\int_0^{\frac{1}{2}} u(s)ds + \frac{1}{3}\int_{\frac{1}{2}}^1 u(s)ds, \tag{34}$$

where $c = 959$. Set $\eta_0 = 0, \eta_1 = 1/2, \eta_2 = 1$, $\alpha_1 = 2, \alpha_2 = 1/3$, $a(t) = t$ and $f(u) = u(1+c/(1+u^2))$. We can show that $\sum_{i=1}^{2} \alpha_i(\eta_i^2 - \eta_{i-1}^2) = 3/4 < 2$. Since $\gamma = \min\{\eta_{m-2}, \sum_{i=1}^{m-2} \alpha_i(\eta_i^2 - \eta_{i-1}^2)/2, \alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2)(1-\eta_{m-2})/\eta_{m-2}$ $(2-\alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2)), \sum_{i=1}^{m-2} \alpha_i(\eta_i^2 - \eta_{i-1}^2)(1-\eta_i)/2\eta_i\} = 1/4$, $\Lambda_1 = (2\int_0^1(1-s)a(s)ds + \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2 a(s)ds)/(2-\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)) = 193/720$, $\Lambda_2 = \gamma \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2) \int_{\eta_{m-2}}^1 (1-s)a(s)ds/(2-\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)) = 1/80$ and $f_0 = 1 + c = 960$, $f_\infty = 1$. Taking $\theta_1 \in (193/720, 1)$, $\theta_2 \in (1, 3)$, thus $f_0 \in ((\theta_2/\gamma)\Lambda_2^{-1}, \infty)$, $f_\infty \in (0, \theta_1\Lambda_1^{-1})$, which imply $(H7)$ and $(H8)$ hold. Therefore, by Corollary 11, the BVP (33), (34) has at least one positive solution.

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