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STABILITY FOR MIXED TYPE OF ADDITIVE AND QUADRATIC FUNCTIONAL EQUATION IN RANDOM NORMED SPACES VIA FIXED POINT METHOD

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Abstract: In this paper, we prove the stability in random normed space via fixed point method for the following *additive and quadratic type functional equation*

$$f(x+y) - f(-x-y) - f(x) + f(-x) - 4f(y) + f(2y) = 0. \qquad (0.1)$$

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1. Introduction and Preliminaries

The stability problem of functional equations has originally been formulated by Ulam [20] : *under what condition does there exists a homomorphism near an approximate homomorphism?* Hyers [5] answered the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [1], and for approximately linear mappings was presented by Rassias [16] by

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considering an unbounded Cauchy difference. The paper work of Rassias [16] has had a lot of influence in the development of what call the *generalized Hyers-Ulam stability* of functional equations. Since then, more generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings have been investigated (see [2]-[4], [6]-[12]).

Recall, almost all subsequent proofs in this very active area used the Hyers' method, called by a direct method. Namely, the function, which is the solution of a functional equation, is explicitly constructed, starting from the given function. Recently, Radu [15] observed that the existence of the solution of a functional equation and the estimation of the difference with the given function can be obtained from the fixed point alternative. Quite recently, Mihet and Radu [14] applied this method to prove the stability theorems of additive functional equation

$$f(x+y) - f(x) - f(y) = 0 \tag{1.1}$$

in the random normed space. We call solutions of the functional equation (1.1) as *additive maps*. It is easy to see that the mappings $f(x) = ax^2 + bx$ is a solution of the functional equation (0.1), where a, b are real constants. Every solution of the additive and quadratic type functional equation is said to be a *quadratic-additive mapping*.

In this paper, using the fixed point method, we prove the stability for the functional equation (0.1) in random normed space.

We now state the usual terminology, notations and conventions of the theory of random normed spaces, as in [18, 19] : Firstly, the space of all probability distribution functions is denoted by

$$\Delta^+ := \{F \big| \mathbb{R} \cup \{-\infty, \infty\} \to [0,1] \big| \, F \text{ is left-continuous and nondecreasing} \\ \text{on } \mathbb{R}, \text{ where } F(0) = 0 \text{ and } F(+\infty) = 1\}.$$

And let the subset $D^+ \subseteq \Delta^+$ be the set $D^+ := \{F \in \Delta^+ | l^- F(+\infty) = 1\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function $\varepsilon_0 : \mathbb{R} \cup \{0\} \to [0, \infty)$ given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t \geq 0 \\ 1 & \text{if } t \leq 0. \end{cases}$$

Definition 1.1. (see [18]) A mapping $\tau : [0,1] \times [0,1] \to [0,1]$ is a *continuous triangular norm* (briefly, a *continuous t-norm*) if τ satisfies the following conditions :

(a) τ is commutative and associative;

(b) τ is continuous;

(c) $\tau(a, 1) = a$ for all $a \in [0, 1]$;

(d) $\tau(a, b) \leq \tau(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t-norms are $\tau_P(a, b) = ab$, $\tau_M(a, b) = \min(a, b)$ and $\tau_L(a, b) = \max(a + b - 1, 0)$.

Definition 1.2. (see [19]) A *random normed space* (briefly, *RN-space*) is a triple (X, Λ, τ), where X is a vector space, τ is a continuous t-norm, and Λ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\Lambda_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,

(RN2) $\Lambda_{\alpha x}(t) = \Lambda_x(t/|\alpha|)$ for all x in X, $\alpha \neq 0$ and all $t \geq 0$,

(RN3) $\Lambda_{x+y}(t + s) \geq \tau(\Lambda_x(t), \Lambda_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

If $(X, \|\cdot\|)$ is a normed space, we can define a mapping $\Lambda : X \to D^+$ by

$$\Lambda_x(t) = \frac{t}{t + \|x\|}$$

for all $x \in X$ and $t > 0$. Then (X, Λ, τ_M) is a random normed space. This space is called the *induced random normed space.*

Definition 1.3. Let (X, Λ, τ) be an RN-space.

(i) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\Lambda_{x_n - x}(t) > 1 - \varepsilon$ whenever $n \geq N$.

(ii) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\Lambda_{x_n - x_m}(t) > 1 - \varepsilon$ whenever $n \geq m \geq N$.

(iii) An RN-space (X, Λ, τ) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4. (see [18]) *If* (X, Λ, τ) *is an RN-space and* $\{x_n\}$ *is a sequence such that* $x_n \to x$, *then* $\lim_{n\to\infty} \Lambda_{x_n}(t) = \Lambda_x(t)$.

2. On Stability in Random Normed Space

We recall the fundamental result in the fixed point theory.

Theorem 2.1. (see [13] or [17]) *Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

or there exists a nonnegative integer k such that:

(1) $d(J^n x, J^{n+1} x) < +\infty$ for all $n \geq k$;

(2) the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;

(4) *$d(y, y^*) \leq (1/(1-L)) d(y, Jy)$ for all $y \in Y$.*

We use the following abbreviation for a given mapping $f : X \to Y$:

$$Df(x,y) := f(x+y) - f(-x-y) - f(x) + f(-x) - 4f(y) + f(2y)$$

for all $x, y \in X$.

Lemma 2.2. *If $f : X \to Y$ is a mapping such that $Df(x,y) = 0$ for all $x, y \in X \backslash \{0\}$ with $f(0) = 0$, then f is a quadratic-additive mapping.*

Proof. The result follows from the fact

$$Df(0,x) = \frac{2}{3} Df(x,x) + \frac{1}{3} Df(-x,-x) = 0$$

and $Df(x,0) = 0$ for all $x \in X \backslash \{0\}$. □

Now we will establish the stability for the functional equations (0.1) in random normed spaces via fixed point method.

Theorem 2.3. *Let X be a linear space, (Z, Λ', τ_M) be an RN-space, (Y, Λ, τ_M) be a complete RN-space and $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\varphi : (X \backslash \{0\})^2 \to Z$ such that*

$$\Lambda_{Df(x,y)}(t) \geq \Lambda'_{\varphi(x,y)}(t) \tag{2.1}$$

for all $x, y \in X\backslash\{0\}$ and $t > 0$. If for all $x, y \in X\backslash\{0\}$ and $t > 0$ φ satisfies one of the following conditions:

(i) $\Lambda'_{\alpha\varphi(x,y)}(t) \leq \Lambda'_{\varphi(2x,2y)}(t)$ *for some* $0 < \alpha < 2$,

(ii) $\Lambda'_{\varphi(2x,2y)}(t) \leq \Lambda'_{\alpha\varphi(x,y)}(t)$ *for some* $4 < \alpha$,

then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\Lambda_{f(x)-F(x)}(t) \geq= \begin{cases} M(x, 2(2-\alpha)t) & \text{if } \varphi \text{ satisfies (i)}, \\ M(x, (\alpha-4)t) & \text{if } \varphi \text{ satisfies (ii)} \end{cases} \tag{2.2}$$

for all $x \in X\backslash\{0\}$ and $t > 0$, where

$$M(x,t) := \tau_M\left\{\Lambda'_{\varphi(x,x)}(t), \Lambda'_{\varphi(-x,-x)}(t)\right\}.$$

Proof. We will prove the theorem in three cases, φ satisfies one of the conditions (i) or (ii).

Case 1. Assume that φ satisfies the condition (i). Let S be the set of all functions $g : X \to Y$ with $g(0) = 0$ and introduce a generalized metric on S by

$$d(g,h) = \inf\left\{u \in \mathbb{R}^+ : \Lambda_{g(x)-h(x)}(ut) \geq M(x,t) \ \text{ for all } \ x \in X\backslash\{0\}\right\}.$$

Consider the mapping $J : S \to S$ defined by

$$Jf(x) := \frac{f(2x) - f(-2x)}{4} + \frac{f(2x) + f(-2x)}{8},$$

then we have

$$J^n f(x) = \frac{1}{2}\left(4^{-n}\left(f(2^n x) + f(-2^n x)\right) + 2^{-n}\left(f(2^n x) - f(-2^n x)\right)\right)$$

for all $x \in X$. Let $f, g \in S$ and let $u \in [0, \infty]$ be an arbitrary constant with $d(g, f) \leq u$. From the definition of d, (RN2), and (RN3), for the given $0 < \alpha < 2$ we have

$$\begin{aligned}
\Lambda_{Jg(x)-Jf(x)}\left(\frac{\alpha u}{2}t\right) =&\Lambda_{\frac{3(g(2x)-f(2x))}{8} - \frac{g(-2x)-f(-2x)}{8}}\left(\frac{\alpha u}{2}t\right) \\
\geq&\tau_M\left\{\Lambda_{\frac{3(g(2x)-f(2x))}{8}}\left(\frac{3\alpha ut}{8}\right), \Lambda_{\frac{g(-2x)-f(-2x)}{8}}\left(\frac{\alpha ut}{8}\right)\right\} \\
\geq&\tau_M\{\Lambda_{g(2x)-f(2x)}(\alpha ut), \Lambda_{g(-2x)-f(-2x)}(\alpha ut)\} \\
\geq&\tau_M\{\Lambda'_{\varphi(2x,2x)}(\alpha t), \Lambda'_{\varphi(-2x,-2x)}(\alpha t)\} \\
\geq&M(x,t)
\end{aligned}$$

for all $x \in X\backslash\{0\}$, which follows that

$$d(Jf, Jg) \leq \frac{\alpha}{2} d(f, g).$$

That is, J is a strictly contractive self-mapping of S with the Lipschitz constant $\frac{\alpha}{2}$. Moreover, by (2.1), we see that

$$\begin{aligned}
\Lambda_{f(x)-Jf(x)}\left(\frac{t}{4}\right) =& \Lambda_{\frac{-5Df(x,x)}{24}+\frac{-Df(-x,-x)}{24}}\left(\frac{t}{2}\right) \\
\geq& \tau_M\left\{\Lambda_{\frac{5Df(x,x)}{24}}\left(\frac{t}{24}\right), \Lambda_{\frac{Df(-x,-x)}{24}}\left(\frac{t}{24}\right)\right\} \\
\geq& \tau_M\{\Lambda_{Df(x,x)}(t), \Lambda_{Df(-x,-x)}(t)\} \\
\geq& \tau_M\{\Lambda'_{\varphi(x,x)}(t), \Lambda'_{\varphi(-x,-x)}(t)\}
\end{aligned}$$

for all $x \in X\backslash\{0\}$. It means that $d(f, Jf) \leq \frac{1}{4} < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : X \to Y$ of J in the set $T = \{g \in S | d(f, g) < \infty\}$, which is represented by

$$F(x) := \lim_{n\to\infty}\left(\frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}\right)$$

for all $x \in X$. Since

$$d(f, F) \leq \frac{1}{1 - \frac{\alpha}{2}} d(f, Jf) \leq \frac{1}{2(2-\alpha)},$$

the inequality (2.2) holds. Next we will show that F is a quadratic-additive map. Let $x, y \in X\backslash\{0\}$. Then by (RN3) we have

$$\begin{aligned}
\Lambda_{DF(x,y)}(t) \geq \tau_M\Bigg\{&\Lambda_{(F-J^n f)(x+y)}\left(\frac{t}{12}\right), \Lambda_{(J^n f-F)(-x-y)}\left(\frac{t}{12}\right), \\
&\Lambda_{(F-J^n f)(-x)}\left(\frac{t}{12}\right), \Lambda_{2(J^n f-F)(x)}\left(\frac{t}{12}\right), \Lambda_{4(J^n f-F)(y)}\left(\frac{t}{12}\right), \\
&\Lambda_{(F-J^n f)(2y)}\left(\frac{t}{12}\right), \Lambda_{DJ^n f(x,y)}\left(\frac{t}{2}\right)\Bigg\} \qquad (2.3)
\end{aligned}$$

for all $n \in \mathbb{N}$. The first six terms on the right hand side of the above inequality tend to 1 as $n \to \infty$ by the definition of F. Now consider that

$$\Lambda_{DJ^n f(x,y)}\left(\frac{t}{2}\right) \geq \tau_M\Bigg\{\Lambda_{\frac{Df(2^n x, 2^n y)}{2\cdot 4^n}}\left(\frac{t}{8}\right), \Lambda_{\frac{Df(-2^n x, -2^n y)}{2\cdot 4^n}}\left(\frac{t}{8}\right),$$

$$\Lambda_{\frac{Df(2^nx,2^ny)}{2\cdot 2^n}}\left(\frac{t}{8}\right), \Lambda_{\frac{Df(-2^nx,-2^ny)}{2\cdot 2^n}}\left(\frac{t}{8}\right)\Big\}$$
$$\geq \tau_M\Big\{\Lambda_{Df(2^nx,2^ny)}\left(\frac{4^nt}{4}\right), \Lambda_{Df(-2^nx,-2^ny)}\left(\frac{4^nt}{4}\right),$$
$$\Lambda_{Df(2^nx,2^ny)}\left(\frac{2^nt}{4}\right), \Lambda_{Df(-2^nx,-2^ny)}\left(\frac{2^nt}{4}\right)\Big\}$$
$$\geq \tau_M\Big\{\Lambda'_{\varphi(x,y)}\left(\frac{4^nt}{4\alpha^n}\right), \Lambda'_{\varphi(-x,-y)}\left(\frac{4^nt}{4\alpha^n}\right),$$
$$\Lambda'_{\varphi(x,y)}\left(\frac{2^nt}{4\alpha^n}\right), \Lambda'_{\varphi(-x,-y)}\left(\frac{2^nt}{4\alpha^n}\right)\Big\}$$

which tends to 1 as $n \to \infty$ by (RN3) and $\frac{2}{\alpha} > 1$. Therefore it follows from (2.3) that

$$\Lambda_{DF(x,y)}(t) = 1$$

for each $x, y \in X\backslash\{0\}$ and $t > 0$. By (RN1) and Lemma 2.2, this means that $DF(x,y) = 0$ for all $x, y \in X$.

Case 2. Assume that φ satisfies the condition (ii). Let the set (S,d) be as in the proof of the case 1. Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := g\left(\frac{x}{2}\right) - g\left(-\frac{x}{2}\right) + 2\left(g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right)\right)$$

for all $g \in S$ and $x \in X$. Notice that

$$J^ng(x) = 2^{n-1}\left(g\left(\frac{x}{2^n}\right) - g\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2}\left(g\left(\frac{x}{2^n}\right) + g\left(-\frac{x}{2^n}\right)\right)$$

and $J^0g(x) = g(x)$ for all $x \in X$. Let $f, g \in S$ and let $u \in [0,\infty]$ be an arbitrary constant with $d(g,f) \leq u$. From the definition of d, (RN2), and (RN3), we have

$$\begin{aligned}
\Lambda_{Jg(x)-Jf(x)}\left(\frac{4u}{\alpha}t\right) =&\Lambda_{3(g(\frac{x}{2})-f(\frac{x}{2}))+g(-\frac{x}{2})-f(-\frac{x}{2})}\left(\frac{4u}{\alpha}t\right)\\
\geq&\tau_M\left\{\Lambda_{3(g(\frac{x}{2})-f(\frac{x}{2}))}(\frac{3u}{\alpha}t), \Lambda_{g(-\frac{x}{2})-f(-\frac{x}{2})}\left(\frac{u}{\alpha}t\right)\right\}\\
\geq&\tau_M\left\{\Lambda_{g(\frac{x}{2})-f(\frac{x}{2})}\left(\frac{u}{\alpha}t\right), \Lambda_{g(-\frac{x}{2})-f(-\frac{x}{2})}\left(\frac{u}{\alpha}t\right)\right\}\\
\geq&\tau_M\left\{\Lambda'_{\varphi(\frac{x}{2},\frac{x}{2})}\left(\frac{t}{\alpha}\right), \Lambda'_{\varphi(-\frac{x}{2},-\frac{x}{2})}\left(\frac{t}{\alpha}\right)\right\}\\
\geq&M(x,t)
\end{aligned}$$

for all $x \in X\backslash\{0\}$, which follows that

$$d(Jf, Jg) \leq \frac{4}{\alpha} d(f, g)$$

That is, J is a strictly contractive self-mapping of S with the Lipschitz constant $0 < \frac{4}{\alpha} < 1$. Moreover, by (2.1), we see that

$$\begin{aligned}
\Lambda_{f(x)-Jf(x)}\left(\frac{t}{\alpha}\right) &= \Lambda_{\frac{2}{3}Df(\frac{x}{2},\frac{x}{2})+\frac{1}{3}Df(-\frac{x}{2},-\frac{x}{2})}\left(\frac{t}{\alpha}\right) \\
&\geq \tau_M\left\{\Lambda'_{\varphi(\frac{x}{2},\frac{x}{2})}\left(\frac{t}{\alpha}\right), \Lambda'_{\varphi(-\frac{x}{2},-\frac{x}{2})}\left(\frac{t}{\alpha}\right)\right\} \\
&\geq M(x, t)
\end{aligned}$$

for all $x \in X\backslash\{0\}$. It means that $d(f, Jf) \leq \frac{1}{\alpha} < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : X \to Y$ of J in the set $T = \{g \in S | d(f, g) < \infty\}$, which is represented by

$$F(x) := \lim_{n\to\infty}\left(2^{n-1}\left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2}\left(f\left(\frac{x}{2^n}\right) + f\left(-\frac{x}{2^n}\right)\right)\right)$$

for all $x \in X$. Since

$$d(f, F) \leq \frac{1}{1 - \frac{4}{\alpha}} d(f, Jf) \leq \frac{1}{\alpha - 4},$$

the inequality (2.2) holds.

Next we will show that F is quadratic-additive. Let $x, y \in X\backslash\{0\}$. Then by (RN3) we have the inequality (2.3) for all $n \in \mathbb{N}$. The first six terms on the right hand side of the inequality (2.3) tend to 1 as $n \to \infty$ by the definition of F. Now consider that

$$\begin{aligned}
\Lambda_{DJ^n f(x,y)}\left(\frac{t}{2}\right) \geq& \tau_M\Big\{\Lambda_{2^{2n-1}Df(\frac{x}{2^n},\frac{y}{2^n})}\left(\frac{t}{8}\right), \Lambda_{2^{2n-1}Df(\frac{-x}{2^n},\frac{-y}{2^n})}\left(\frac{t}{8}\right), \\
&\Lambda_{2^{n-1}Df(\frac{x}{2^n},\frac{y}{2^n})}\left(\frac{t}{8}\right), \Lambda_{-2^{n-1}Df(\frac{-x}{2^n},\frac{-y}{2^n})}\left(\frac{t}{8}\right)\Big\} \\
\geq& \tau_M\Big\{\Lambda'_{\varphi(x,y)}\left(\frac{\alpha^n t}{4^{n+1}}\right), \Lambda'_{\varphi(-x,-y)}\left(\frac{\alpha^n t}{4^{n+1}}\right), \\
&\Lambda'_{\varphi(x,y)}\left(\frac{\alpha^n t}{2^{n+2}}\right), \Lambda'_{\varphi(-x,-y)}\left(\frac{\alpha^n t}{2^{n+2}}\right)\Big\},
\end{aligned}$$

which tends to 1 as $n \to \infty$ by (RN3) for all $x, y \in X\backslash\{0\}$. Therefore it follows from (2.3) that

$$\Lambda_{DF(x,y)}(t) = 1$$

for each $x, y \in X\backslash\{0\}$ and $t > 0$. By (RN1) and Lemma 2.2, this means that $DF(x, y) = 0$ for all $x, y \in X$. It completes the proof of Theorem 2.3. □

Now we have the generalized Hyers-Ulam stability of the quadratic-additive functional equation (0.1) in the framework of normed spaces. Let $\Lambda_x(t) = \frac{t}{t+\|x\|}$. Then (X, Λ, τ_M) is an induced random normed space, which leads us to get the following result.

Corollary 2.4. *Let X be a linear space, $\mathbb{R}$ be the set of real numbers, and Y be a complete normed space and $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\varphi : (X\backslash\{0\})^2 \to Z$ such that*

$$\|Df(x, y)\| \le \varphi(x, y)$$

for all $x, y \in X\backslash\{0\}$. If for all $x, y \in X\backslash\{0\}$ φ satisfies one of the following conditions:

(i) $\alpha\varphi(x, y) \ge \varphi(2x, 2y)$ *for some* $0 < \alpha < 2$,

(ii) $\varphi(2x, 2y) \ge \alpha\varphi(x, y)$ *for some* $4 < \alpha$,

then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le = \begin{cases} \frac{\max(\varphi(x,x),\varphi(-x,-x))}{2(2-\alpha)} & \text{if } \varphi \text{ satisfies (i)}, \\ \frac{\max(\varphi(x,x),\varphi(-x,-x))}{\alpha-4} & \text{if } \varphi \text{ satisfies (ii)} \end{cases}$$

for all $x \in X\backslash\{0\}$.

Now we have the generalized Hyers-Ulam stability of the quadratic-additive type functional equation (0.1).

Corollary 2.5. *Let X be a normed space, $p, q \in (-\infty, 1) \cup (2, \infty)$ and Y a complete normed-space. If $f : X \to Y$ is a mapping with $f(0) = 0$ such that*

$$\|Df(x, y)\| \le \|x\|^p + \|y\|^q$$

for all $x, y \in X\backslash\{0\}$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le = \begin{cases} \frac{\|x\|^p+\|x\|^q}{2-2^{\max\{p,q\}}} & \text{if } p, q < 1, \\ \frac{\|x\|^p+\|x\|^q}{2^{\min\{p,q\}}-4} & \text{if } 2 < p, q \end{cases}$$

for all $x \in X\backslash\{0\}$ and f is itself a quadratic-additive map if $p, q < 0$.

Proof. If we denote by $\varphi(x,y) = \|x\|^p + \|y\|^q$, then the induced random normed space (X, Λ_x, τ_M) holds the conditions of Corollary 2.4 with $\alpha = 2^{\max\{p,q\}}$ if $p,q \in (-\infty,0)\cup(0,1)$ and $\alpha = 2^{\min\{p,q\}}$ if $p,q > 2$. □

Corollary 2.6. *Let X be a normed space and Y a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \le \theta\|x\|^p\|y\|^q$$

for all $x,y \in X\backslash\{0\}$ with $f(0) = 0$, where $\theta \ge 0$ and $p+q \in (-\infty,1)\cup(2,\infty)$. Then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le = \begin{cases} \frac{\|x\|^{p+q}}{2-2^{p+q}} & \text{if } p+q < 1, \\ \frac{\|x\|^{p+q}}{2^{p+q}-4} & \text{if } 2 < p+q \end{cases}$$

for all $x \in X\backslash\{0\}$.

Proof. If we denote by $\varphi(x,y) = \|x\|^p\|y\|^q$, then the induced random normed space (X, Λ_x, τ_M) holds the conditions of Corollary 2.4 with $\alpha = 2^{p+q}$. □

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