

## REMARKS ON THE STABILISATION OF STOCHASTIC DELAY DIFFERENTIAL EQUATIONS BY NOISE

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**Abstract:** This paper is concerned with the stabilisation of finite dimensional stochastic delay differential equation of the form

$$x'(t) = f_1(t, x(t), x(t - \tau_1(t)), \dots, x(t - \tau_n(t))) + \int_{t - \tau_0(t)}^t f_2(s, t, x(s)) ds, \quad t \geq 0,$$

where  $n \in \mathbf{N}$ ,  $f_1 \in C(\mathbf{R}^+ \times (\mathbf{R}^d)^{n+1}; \mathbf{R}^d)$ ,  $f_2 \in C(\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^d; \mathbf{R}^d)$ . Under more general conditions, we show that a linear multiplicative noise can always stabilise a general finite-dimensional functional differential equation whenever the delay is sufficiently small. Our results complement and improve existing results, and are also in line with the results of Appleby [1] which in itself generalised some well known results in the literature.

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noise

### 1. Introduction

Stochastic differential equations have received the attention of many authors, see for instance [1-6]. In an interesting paper [3], Mao proved that general finite dimensional stochastic differential equations can be stabilised or destabilised by Brownian motion. An equally interesting question arose from this: given an unstable functional differential equation (1.1), can it be stabilised by stochastic perturbation of the form  $\sigma x(t)dB$ ? Here  $B(t)$  is a Brownian motion and  $\sigma$  is a real number. Appleby [1] answered this question affirmatively when he considered stochastic perturbations of the deterministic functional differential equation

$$x'(t) = f_1(t, x(t), x(-\tau_1(t)), \dots, x(t - \tau_n(t))) + \int_{t-\tau_0(t)}^t f_2(s, t, x(s))ds, \tag{1.1}$$

$t \geq 0$

where  $n \in \mathbf{N}$  and functions  $f_1 \in C(\mathbf{R}^+ \times (\mathbf{R}^d)^{n+1}; \mathbf{R}^d)$ ,  $f_2 \in C(\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^d; \mathbf{R}^d)$ . Moreover it was assumed that  $f_1$  and  $f_2$  satisfy the following global Lipschitz conditions in the space parameters  $x$ , so that for all  $t \geq 0$ , there exists  $L > 0$  such that

$$\| f_1(t, x_1, \dots, x_{n+1}) - f_1(t, y_1, \dots, y_{n+1}) \| \leq L \sum_{j=1}^{n+1} \| x_j - y_j \|, \quad x_j, y_j \in \mathbf{R}^d, \tag{1.2}$$

$$\| f_2(s, t, x) - f_2(s, t, y) \| \leq L \| x - y \|, \quad 0 \leq s \leq t, \quad x, y \in \mathbf{R}^d. \tag{1.3}$$

Furthermore, Appleby [1] showed that if both conditions (1.2) and (1.3) are satisfied and, moreover, if the time delays  $\tau_i$ ,  $0 \leq i \leq n$  are all sufficiently small, then it is possible to choose  $\sigma$  (a real constant) sufficiently large so that the equation

$$dX(t) = \left( f_1(t, X(t), X(t - \tau_1(t)), X(t - \tau_2(t)), \dots, X(t - \tau_n(t))) + \int_{t-\tau_0(t)}^t f_2(s, t, X(s))ds \right) dt + \sigma X(t)dB(t), \quad t \geq 0 \tag{1.4}$$

$$X(t) = \psi(t), \quad -\bar{\tau} \leq t \leq 0, \tag{1.5}$$

will be almost surely exponentially stable.

In this paper, we seek to use the approach of Appleby [1] to weaken Lipschitz conditions (1.2) and (1.3) so that

$$\| f_1(t, x_1, \dots, x_{n+1}) - f_1(t, y_1, \dots, y_{n+1}) \| \leq P(t) \sum_{j=1}^{n+1} \| x_j - y_j \|, \quad x_j, y_j \in \mathbf{R}^d, \quad (1.6)$$

$$\| f_2(s, t, x) - f_2(s, t, y) \| \leq P(t) \| x - y \|, \quad 0 \leq s \leq t, \quad x, y \in \mathbf{R}^d, \quad (1.7)$$

where  $P(t)$  is a non negative process. Our results complement those of Appleby [1].

### 2. Preliminaries

Our approach in this work shall follow the work of Appleby [1]. Throughout this paper, unless otherwise specified, we use the following notation. Let  $C(I; J)$  be the space of continuous functions taking the fin

$$\| f_1(t, x_1, x_2, \dots, x_{n+1}) \| \leq \sum_{j=1}^{n+1} K_j \| x_j \| \quad \forall \quad t > 0 \quad x_j \in \mathbf{R}^d, \quad (2.1)$$

$$\| f_2(s, t, x) \| \leq K_0 \| x \| \quad \forall \quad 0 \leq s \leq t, \quad x \in \mathbf{R}^d, \quad (2.2)$$

where  $k_j, = 0, \dots, n + 1$  are non-negative constants.

In what follows, we now place some assumptions on  $\tau_j$  ,  $j = 0, 1, \dots, n$ .  $\tau_j \in C(\mathbf{R}^+; \mathbf{R}^+)$  and  $\bar{\tau}(t) = \max_{j=0,1,\dots,n} \tau_j(t)$  is a continuous function on  $\mathbf{R}^+$ . Furthermore, we suppose that  $\sup_{t \geq 0} \bar{\tau}(t) = \bar{\tau}$ , i.e, there exists a finite  $\bar{\tau}$  satisfying

$$\bar{\tau} = \sup_{t \geq 0} \max_{j=0,1,\dots,n} \tau_j(t). \quad (2.3)$$

Let  $\phi \in C([-\bar{\tau}, 0], \mathbf{R}_d)$ , where  $\bar{\tau}$  satisfies (2.3). Now let

$$x(t) = \psi(t), \quad -\bar{\tau} \leq t \leq 0, \quad (2.4)$$

then, according to Appleby [1], if  $f_1, f_2$  satisfy (1.2), (1.3), (2.1), (2.2) and  $\tau_j$  satisfies the above continuity hypotheses, it follows that there is a unique solution of the equation (1.1). Note moreover that if  $\psi(t) = 0, \forall -\bar{\tau} \leq t \leq 0$ , then  $x \equiv 0$  is the unique solution of the equation (1.1), which we call the zero solution of the equation (1.1). We refer the reader to Appleby [1] where the

following almost sure asymptotic stability of solution to the equation (1.1) was established. Let  $(B(t))_{t \geq 0}$  be a one dimensional standard Brownian motion on a complete filtered probability space  $(\Omega, \mathbf{F}, \mathbf{P})$  with natural filtration  $(\mathbf{F})_{t \geq 0}$ , where  $\mathbf{F}_t = \sigma(B(s) : 0 \leq s \leq t)$ . Now, let  $\sigma$  be real constant and consider the stochastic functional differential equation.

$$dX(t) = \left( f_1(t, X(t), X(t - \tau_1(t)), X(t - \tau_2(t)), \dots, X(t - \tau_n(t))) + \int_{t-\tau_0(t)}^t f_2(s, t, X(s)) ds \right) dt + \sigma X(t) dB(t), \quad t \geq 0 \tag{2.5}$$

$$X(t) = \psi(t), \quad -\bar{\tau} \leq t \leq 0. \tag{2.6}$$

If the above assumptions on  $\psi, f_1, f_2$  and  $\tau_j, j = 0, \dots, n$  hold, then equation (2.5) has a unique, strong continuous solution on  $\mathbf{R}^+$ . Again, if  $\psi \equiv 0$ , then  $X(t) = 0, \forall t \geq 0$ , almost surely. Observe that when  $\sigma = 0$ , the problem reduces to the deterministic functional differential equation (1.1). Hence the question of whether these are values of  $\sigma = 0$  such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \| X(t) \| \leq -\lambda \quad a.s \tag{2.7}$$

was answered in [1].

### 3. Main Result

Suppose that assumptions (1.6) and (1.7) are satisfied. Suppose further that

$$\sigma^2 > 2 \sum_{j=2}^{n+1} K_j$$

and let  $\tau(\sigma^2, K)$  be given by

$$\tau(\sigma^2, K) = \frac{2}{\sigma^2} \log \left( \frac{K_0 + \sqrt{K_0^2 + \sigma^2(\frac{1}{2}\sigma^2 - K_1)(2K_0 + \sigma^2 \sum_{j=2}^{n+1} K_j)}}{(2K_0 + \sigma^2 \sum_{j=2}^{n+1} K_j)} \right).$$

If  $\bar{\tau}$  defined by

$$\bar{\tau} = \sup_{t \geq 0} \max_{j=0, \dots, n} \tau_j(t)$$

satisfies  $\bar{\tau} < \tau(\sigma^2, K)$ , then the solution of the equation

$$dX(t) = \left( f_1(t, X(t), X(t - \tau_1(t)), X(t - \tau_2(t)), \dots, X(t - \tau_n(t))) + \int_{t-\tau_0(t)}^t f_2(s, t, X(s)) ds \right) dt + \sigma X(t) dB(t), \quad t \geq 0, \tag{3.1}$$

$$X(t) = \psi(t), \quad -\bar{\tau} \leq t \leq 0, \tag{3.2}$$

satisfies

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \| X(t) \| \leq -\lambda_1(\sigma^2, \bar{\tau}) < 0, \quad a.s., \tag{3.3}$$

where  $\lambda_1$  is given by

$$\lambda_1(\sigma^2, \bar{\tau}) = \frac{\sigma^2}{2} - K_1 - \sum_{j=2}^{n+1} K_j e^{\sigma^2 \bar{\tau}/2} - K_0 \frac{1}{\frac{\sigma^2}{2}} (e^{\sigma^2 \bar{\tau}} - e^{\sigma^2 \bar{\tau}/2}).$$

#### 4. Proof of the Main Result

We shall need the following auxiliary functions in the proof of our main result. Suppose  $\Phi(t) = I_d$ , for  $t \in [-\bar{\tau}, 0]$  and consider the matrix stochastic differential equation  $d\Phi(t) = \sigma\Phi(t)dB(t)$ . This has solution  $\Phi(t) = \varphi(t)I_d$ , where  $(\varphi(t))_{t>0}$  is the scalar geometric Brownian motion which satisfies  $d\varphi(t) = \sigma\varphi(t)dB(t)$  with  $\varphi(t) = 1, t \in [\bar{\tau}, 0]$  and hence is given by

$$\begin{aligned} \varphi(t) &= e^{\int -\frac{\sigma^2}{2} dt + \int \sigma dB(t)} \quad (\text{by Ito's formula}) \\ &= e^{\int -\frac{\sigma^2}{2} dt + \sigma B(t)}. \end{aligned} \tag{4.1}$$

Thus,  $\Phi(t)^{-1}$  exists for all  $t \geq -\bar{\tau}$  a.s., and in particular  $\Phi(t)^{-1} = \varphi(t)^{-1}I_d$ , so  $\| \Phi(t) \| = \varphi(t), \| \Phi(t)^{-1} \| = \varphi(t)^{-1}$ . Now for  $t \geq 0$ , let

$$R(t) = f_1(t, X(t), \dots, X(t - \tau_n(t))) + \int_{t-\tau_0(t)}^t f(s, t, x(s)) ds$$

therefore,

$$dX = R(t)dt + \sigma X(t)dB(t).$$

On employing Ito's product rule, let

$$dX_1 = \sigma X_1(t)dB(t),$$

$$\begin{aligned}
 dX_2 &= A(t)d(t) + C(t)dB(t), \\
 dX &= [\sigma X_1(t)dB(t)]X_2(t) + [A(t)d(t) + C(t)dB(t)]X_1(t) + \sigma X_1(t)d(t) \\
 &= \sigma(X_1(t)X_2(t))dB(t) + A(t)X_1dt + C(t)X_1(t)dB(t) + \sigma C(t)X_1(t)dt \\
 &= \sigma(X(t))dB(t) + A(t)X_1dt + C(t)X_1(t)dB(t) + \sigma C(t)X_1(t)dt,
 \end{aligned}$$

then,

$$R(t)dt = A(t)X_1dt + \sigma C(t)X_1(t)dt + C(t)X_1(t)dB(t).$$

On choosing  $C = 0$  and  $A = R(t)X_1^{-1}$ , we have

$$\begin{aligned}
 dX_1 &= \sigma X_1(t)dB(t), \\
 dX_2 &= R(t)X_1^{-1}dt.
 \end{aligned}$$

On employing Ito’s lemma (see e.g. [1]), it follows that

$$\begin{aligned}
 X_1 &= \exp\left(\int_0^t -\frac{\sigma^2}{2}ds + \int_0^t \sigma dB(t)\right) \\
 &= e^{-\frac{\sigma^2}{2}t + \sigma B(t)}, \\
 X_2 &= \psi(0) + \int_0^t R(s)X_1^{-1}ds \\
 X &= X_1(\psi_0 + \int_0^t R(s)X_1^{-1}ds).
 \end{aligned}$$

But  $X_1 = \Phi(t)$ , hence

$$X(t) = \Phi(t)(\psi(0) + \int_0^t \Phi(s)^{-1}R(s)ds), t \geq 0. \tag{4.2}$$

Now, let  $y(t) = \Phi(t)^{-1}X(t)$  for  $t \geq -\bar{\tau}$ . As  $t \mapsto \Phi(t), R(t)$  is continuous, equation (4.2) implies  $y \in C'(\mathbf{R}^+; \mathbf{R}^d)$ . Thus for  $t \geq 0, t - \tau_j(t) \geq -\bar{\tau}$ , we have

$$\begin{aligned}
 y'(t) &= \Phi(t)(f_1(t, \Phi(t)y(t), \dots, \Phi(t - \tau_n(t))y(t - \tau_n(t))) \\
 &\quad + \int_{t-\tau_0(t)}^t f_2(s, t, \Phi(s)y(s))ds). \tag{4.3}
 \end{aligned}$$

Thus,

$$\| y'(t) \| \leq \| K_1(t) \| \| y(t) \| + \sum_{j=2}^{n+1} \varphi(t)^{-1} \varphi(t - \tau_{j-1}(t)) \| K_j(t) \| \| y(t - \tau_{j-1}(t)) \|$$

$$+ \int_{t-\tau_0(t)}^t \varphi(t)^{-1} \varphi(s) \| K_0(t) \| \| y(s) \| ds, \tag{4.4}$$

and if we define  $p(t)$  for  $t \geq \bar{\tau}$  by

$$p(t) = \| K_1(t) \| + \sum_{j=2}^{n+1} \varphi(t)^{-1} \varphi(t - \tau_{j-1}(t)) \| K_j(t) \| + \int_{t-\tau_0(t)}^t \varphi(t)^{-1} \varphi(s) \| K_0(t) \| \| y(s) \| ds, \tag{4.5}$$

we have that for  $t \geq \bar{\tau}$ ,

$$\| y(t) \| \geq \| y(\bar{\tau}) \| + \int_{\bar{\tau}}^t p(s) \max_{s-\bar{\tau} \leq u \leq s} \| y(u) \| ds. \tag{4.6}$$

On letting

$$y^*(t) = \max_{t-\bar{\tau} \leq s \leq t} \| y(s) \|, \quad \text{for } t \geq \bar{\tau},$$

we have

$$y^*(t) \leq \| y(\bar{\tau}) \| + \int_{\bar{\tau}}^t p(s) y^*(s) ds.$$

On using Grownwall's inequality, it follows that

$$\| y(t) \| \leq y^*(t) \leq \| y(\bar{\tau}) \| e^{\int_{\bar{\tau}}^t p(s) ds}, \quad t \geq \bar{\tau}.$$

If there exists a finite deterministic constant  $C$ , such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_{2\bar{\tau}}^t p(s) ds \leq C \tag{4.7}$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \| \phi(t) \| = -\frac{\sigma^2}{2}, \quad \text{a.s.},$$

then the definition of  $y$  gives

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \| X(t) \| = -\frac{\sigma^2}{2} + C, \quad \text{a.s.} \tag{4.8}$$

We shall need the following results in the proof of our main results.

**Lemma 1.** *Suppose  $\tau \in C(R^+; [0, \bar{\tau}])$ , and  $\varphi(t)$  satisfies*

$$\varphi(t) = e^{\int -\frac{\sigma^2}{2} dt + \sigma B(t)}, \tag{4.9}$$

then,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_{2\bar{\tau}}^t \|K_j\| \varphi(s)^{-1} \varphi(s - \tau(s)) ds \leq K_j e^{\sigma^2 \bar{\tau}}, \quad a.s. \quad (4.10)$$

**Lemma 2.** Let  $\tau \in C(\mathbf{R}^+; [0, \bar{\tau}])$  and

$$q_1(t) = \int_{t-\bar{\tau}(t)}^t \varphi(t)^{-1} \varphi(s) ds,$$

where  $\varphi(t)$  satisfies equation (4.9). Then,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_{2\bar{\tau}}^t q_1(s) ds \leq K_0 e^{\sigma^2 \bar{\tau}/2} \int_0^{\bar{\tau}} e^{\sigma^2 u/2} du, \quad a.s. \quad (4.11)$$

*Proof Lemma 1.* From the equation (4.9),

$$\varphi(s)^{-1} \varphi(s - \tau(s)) \leq e^{\sigma^2 \bar{\tau}/2} e^{-\sigma(B(s) - B(s - \tau(s)))}. \quad (4.12)$$

For each fixed  $t \geq \bar{\tau}$ , as  $0 \leq \tau(t) \leq \bar{\tau}$ , a.s., we have

$$B(t) - B(t - \tau(t)) \leq \max_{t-\bar{\tau} \leq s \leq t} B(t) - B(s) = \max_{0 \leq s \leq \bar{\tau}} W_s(t),$$

where  $s \mapsto W_s(t)$  is a standard Brownian motion. As  $\max_{0 \leq s \leq \bar{\tau}} W_s(t)$  has the same distribution as  $|W_{\bar{\tau}}(t)|$  and

$$E[e^{\lambda|X|}] = e^{\frac{1}{2}\lambda^2\alpha^2}, \quad \text{for } X \sim \mathcal{N}(0, \alpha^2),$$

we get,

$$\begin{aligned} E[e^{B(t) - B(t - \tau(t))}] &\leq E[e^{\max_{0 \leq s \leq \bar{\tau}} W_s(t)}] = E[e^{|W_{\bar{\tau}}(t)|}] = e^{\frac{1}{2}\bar{\tau}} \\ E[e^{4(B(t) - B(t - \tau(t)))}] &\leq e^{8\bar{\tau}}. \end{aligned} \quad (4.13)$$

Since  $K_j(t) \sim \mathcal{N}(K_j, D_j)$ ,  $j = 1, \dots, n$  are independent random variables,

$$\begin{aligned} E[\|K_j(t)\| e^{\sigma|W_{\bar{\tau}}(t)|}] &= E[\|K_j(t)\|] E[e^{\sigma|W_{\bar{\tau}}(t)|}] \leq \|E[K_j(t)]\| e^{\frac{\sigma^2}{2}\bar{\tau}} \\ &= \|K_j\| e^{\frac{\sigma^2}{2}\bar{\tau}} = |K_j| e^{\frac{\sigma^2}{2}\bar{\tau}}, \quad j = 1, \dots, n. \end{aligned} \quad (4.14)$$

Since  $K_j$  are constants,

$$E[\|K_j(t)\| e^{4\sigma(B(t) - B(t - \tau(t)))}] \leq |K_j| e^{8\sigma\bar{\tau}}. \quad (4.15)$$



Define for  $n \in \mathbf{N}$ , the two sequence of random variables

$$X_n = \int_{(2n+1)\bar{\tau}}^{(2n+2)\bar{\tau}} \| K_j(t) \| e^{\sigma(B(s)-B(t-\tau(s)))} ds \quad j = 1, \dots, n,$$

$$Y_n = \int_{(2n)\bar{\tau}}^{(2n+1)\bar{\tau}} \| K_j(t) \| e^{\sigma(B(s)-B(t-\tau(s)))} ds \quad j = 1, \dots, n.$$

By Holders inequality and (4.19), we get

$$\begin{aligned} E[X_n] &\leq |K_j|\bar{\tau}e^{\frac{\sigma}{2}\bar{\tau}}, \\ E[Y_n] &\leq |K_j|\bar{\tau}e^{\frac{\sigma}{2}\bar{\tau}}, \\ E[X_n^4] &\leq |K_j|\bar{\tau}e^{8\sigma\bar{\tau}}, \\ E[Y_n^4] &\leq |K_j|\bar{\tau}e^{8\sigma\bar{\tau}}. \end{aligned}$$

Finally, as  $0 \leq \tau(t) \leq \bar{\tau}$ , we see that  $\{X_n\}$  is a sequence of independent random variables, as is  $\{Y_n\}$ . By, Strong Law of Large Numbers (see [8]),

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n X_j \leq |K_j|\bar{\tau}e^{\frac{\sigma}{2}\bar{\tau}}, \tag{4.16}$$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n Y_j \leq |K_j|\bar{\tau}e^{\frac{\sigma}{2}\bar{\tau}}. \tag{4.17}$$

With  $t \geq 4\bar{\tau}$ , there is  $n = n(t) \in \mathbf{N}$  such that  $(2n + 2)\bar{\tau} \leq t < (2n + 4)\bar{\tau}$ , and

$$\begin{aligned} \frac{1}{t} \int_t^{(2)\bar{\tau}} \| K_j(t) \| e^{\sigma(B(s)-B(t-\tau(s)))} ds &= \frac{n}{t} \left( \frac{1}{n} \sum_{j=1}^n X_j + \frac{1}{n} \sum_{j=1}^n Y_j \right) \\ &+ \frac{1}{t} \int_{(2n+2)\bar{\tau}}^t \| K_j(t) \| e^{\sigma(B(s)-B(t-\tau(s)))} ds \quad j = 1, \dots, n. \end{aligned} \tag{4.18}$$

Since  $\lim_{n \rightarrow \infty} \frac{n(t)}{t} = \frac{1}{2\bar{\tau}}$ , we will show that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{(2n(t)+2)\bar{\tau}}^t \| K_j(t) \| e^{\sigma(B(s)-B(t-\tau(s)))} ds = 0. \tag{4.19}$$

By Markov’s inequality and Borel Cantelli lemma,

$$E[U_n(4)] \leq C(n + 1)^{-1} \quad \text{for some } C > 0,$$

where  $U_n = \frac{X_n+Y_n}{n+1}$ . Thus,

$$\lim_{n \rightarrow \infty} U_n = 0.$$

This implies that (4.19) is true.

On using inequality (4.12), we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_{2\bar{\tau}}^t \|K_j\| \varphi(s)^{-1} \varphi(s - \tau(s)) ds \leq K_j e^{\sigma^2 \bar{\tau}}, \quad a.s.$$

*Proof Lemma 2.* From equation (4.9),

$$\varphi(t) = e^{\int -\frac{\sigma^2}{2} dt + \sigma B(t)},$$

and thus

$$\varphi(s)^{-1} \varphi(s - \tau(s)) \leq e^{\sigma^2 \bar{\tau}/2} e^{-\sigma(B(s) - B(s - \tau(s)))}. \quad (4.20)$$

For each fixed  $t - \bar{\tau}$ , as  $0 \leq \tau(t) \leq \bar{\tau}$ , a.s, we have

$$B(t) - B(t - \tau(t)) \leq \max_{t - \bar{\tau} \leq s \leq t} B(t) - B(s) = \max_{0 \leq s \leq \bar{\tau}} W_s(t),$$

where  $s \mapsto W_s(t)$  is a standard Brownian motion (see [7]). As  $\max_{0 \leq s \leq \bar{\tau}}$  has the same distribution as  $|W_{\bar{\tau}}(t)|$ , and  $E[e^{\lambda|x|}] = 2e^{\frac{1}{2}\lambda^2\alpha^2}$  for  $X \sim N(0, \alpha^2)$ ,

$$E[e^{B(t) - B(t - \tau(t))}] \leq E[e^{\max_{0 \leq s \leq \bar{\tau}} W_s(t)}] = E[e^{|W_{\bar{\tau}}(t)|}] = e^{\frac{1}{2}\bar{\tau}} \quad (4.21)$$

$$E[e^{4(B(t) - B(t - \tau(t)))}] = e^{8\bar{\tau}}, \quad (4.22)$$

therefore,

$$E[K_0(t)e^{B(t) - B(t - \tau(t))}] = E[K_0(t)]E[B(t) - B(t - \tau(t))].$$

Since  $K_0(t)$  is independent of  $B(t) - B(t - \tau(t))$ , then,

$$E[K_0(t)e^{B(t) - B(t - \tau(t))}] = K_0 e^{\frac{1}{2}\bar{\tau}}$$

$$E[e^{4(B(t) - B(t - \tau(t)))}] = K_0 e^{8\bar{\tau}}.$$

Set  $q(t) = \int_{t - \bar{\tau}}^t K_0(t) e^{-\sigma(B(t) - B(s))} ds$ . From (4.20),

$$K_0(t) \phi^{-1} \phi(s - \tau(s)) \leq e^{\sigma^2 \bar{\tau}/2} K_0(t) e^{-\sigma(B(s) - B(s - \tau(s)))}. \quad (4.23)$$

Since  $q(t) = \int_{t - \tau(t)}^t K_0(t) \phi^{-1}(t) \phi(s) ds$ , then  $q(t) \leq e^{\sigma^2 \bar{\tau}/2} q(t)$ . Thus we would be done if we can prove that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{2\bar{\tau}}^t q(s) ds = K_0 \int_0^{\bar{\tau}} e^{\frac{1}{2}\sigma^2 u} du \quad a.s. \quad (4.24)$$

Now, let

$$X_n = \int_{(2n+1)\bar{\tau}}^{(2n+2)\bar{\tau}} q(s) ds$$

and

$$Y_n = \int_{(2n+1)\bar{\tau}}^{(2n+2)\bar{\tau}} q(s) ds \quad n \geq 1,$$

where  $X_n$  and  $Y_n$  are sequences of independent random variables. Then,

$$E(X_n) = K_0 \bar{\tau} \int_0^{\bar{\tau}} e^{\frac{1}{2}\sigma^2 u} du$$

and

$$E(Y_n) = K_0 \bar{\tau} \int_0^{\bar{\tau}} e^{\frac{1}{2}\sigma^2 u} du.$$

Then by Strong Law of Large Numbers (see [8]),

$$\lim_{n \rightarrow \infty} \sup \frac{1}{n} \sum_{j=1}^n X_j \leq K_0 \bar{\tau} \int_0^{\bar{\tau}} e^{\frac{1}{2}\sigma^2 u} du, \quad (4.25)$$

$$\lim_{n \rightarrow \infty} \sup \frac{1}{n} \sum_{j=1}^n Y_j \leq K_0 \bar{\tau} \int_0^{\bar{\tau}} e^{\frac{1}{2}\sigma^2 u} du, \quad (4.26)$$

$$\frac{1}{t} \int_{2\bar{\tau}}^t q(s) ds = \frac{n}{t} \left( \frac{1}{n} \sum_{j=1}^n X_j + \frac{1}{n} \sum_{j=1}^n Y_j \right) + \frac{1}{t} \int_{(2n+2)\bar{\tau}}^t q(s) ds. \quad (4.27)$$

Since  $\lim_{t \rightarrow \infty} \frac{n(t)}{t} = \frac{1}{2\bar{\tau}}$ , if we can show that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{(2n(t)+2)\bar{\tau}}^t q(s) ds = 0 \quad a.s., \quad (4.28)$$

then, (4.24) is true. Let us observe that since  $\lim_{n \rightarrow \infty} U_n = 0$  a.s., where  $U_n = \frac{X_n + Y_n}{n+1}$ , then equation (4.28) holds. But this is true by Markov's inequality and the Borel Cantelli Lemma as

$$E[U_n^4] \leq C(n+1)^{-4},$$

for some  $C > 0$ . Thus the completion of the proof.

Let us now apply Lemma 1 to each of the terms in the sum on the right hand side of (4.5), and also applying Lemma 2 to the integral term yields equation

(4.5) with  $C = K_1 + \sum_{j=2}^{n+1} K_j e^{\sigma^2 \bar{\tau}/2} + K_0 e^{\sigma^2 \bar{\tau}/2} \int_{\bar{\tau}}^0 e^{\sigma^2 \bar{u}/2} du$ . Now, for  $\bar{\tau} > 0$ , define

$$\lambda_1(\sigma^2, \bar{\tau}) = \frac{\sigma^2}{2} - K_1 - \sum_{j=2}^{n+1} K_j e^{\sigma^2 \bar{\tau}/2} - K_0 \frac{1}{\sigma^2/2} (e^{\sigma^2 \bar{\tau}} - e^{\sigma^2 \bar{\tau}/2}).$$

Define also  $K = (K_0, K_1, \dots, K_{n+1})^T$ . Then there exists a unique  $\tau = \tau(\sigma^2, K) > 0$  such that  $\lambda_1(\sigma^2, \tau(\sigma^2, K)) = 0$  and  $\lambda_1(\sigma^2, \tau) > 0$  for all  $\tau < \tau(\sigma^2, K)$ . In fact, we can explicitly compute

$$\tau(\sigma^2, K) = \frac{2}{\sigma^2} \log \left( \frac{K_0 + \sqrt{K_0^2 + \sigma^2(\frac{1}{2}\sigma^2 - K_1)(2K_0 + \sigma^2 \sum_{j=2}^{n+1} K_j)}}{(2K_0 + \sigma^2 \sum_{j=2}^{n+1} K_j)} \right)$$

which agrees with Appleby [1].

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