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EXTENSION OF THE EULER-LAGRANGE IDENTITY BY SUPERQUADRATIC POWER FUNCTIONS

S. Abramovich[1], S. Ivelić[2 §], J. Pečarić[3]

[1]Depertment of Mathmatich
University of Haifa
Haifa, ISRAEL

[2]Faculty of Civil Engineering
Architecture and Geodesy
University of Split
15, Matice Hrvatske, 21000, Split, CROATIA

[3]Faculty of Textile Technology
University of Zagreb
30, Prilaz Baruna Filipovića 30, 10000, Zagreb, CROATIA

Abstract: Using convexity, superquadracity and superterzacity we extend in this paper the Euler-Lagrange identity, Bohr's inequality and the triangle inequality.

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1. Generalization of the Triangle Inequality via Convexity

In [5, Theorem 1.1], inequalities related to the Euler-Lagrange identity are proved on Banach space. Using the convexity of $f(x) = x^p$, $p \geq 1$, $x \geq 0$ we prove in this section a generalization of this theorem for complex numbers, for

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[§]Correspondence author

which Bohr's inequality is a special case. This gives us the tools to achieve the main results of Section 2 and 3. In Section 2, we extend the result to the superquadratic functions $f(x) = x^p$, $p \geq 2$, $x \geq 0$ and obtain the Euler-Lagrange identity as a special case. In Section 3, using so called superterzatic functions, which are closely related to superquadratic functions, we obtain a new extension of the Euler-Lagrange identity.

Theorem 1. *Let* $x, \ y, \ a, \ b$ *be complex numbers and let* $\mu, \ \nu, \ \lambda \in \mathbb{R} \backslash \{0\}$. *Then*

$$\frac{|x|^p}{\mu} + \frac{|y|^p}{\nu} \geq \frac{|ax+by|^p}{\lambda}$$

holds if:

(i) $\mu > 0, \ \nu > 0, \ \lambda > 0$ *and*

$$|\lambda|^{1/(p-1)} \geq |\mu|^{1/(p-1)} |a|^q + |\nu|^{1/(p-1)} |b|^q ,$$

(ii) $\mu < 0, \ \nu > 0, \ \lambda < 0$ *and*

$$|\lambda|^{1/(p-1)} \leq |\mu|^{1/(p-1)} |a|^q - |\nu|^{1/(p-1)} |b|^q ,$$

(iii) $\mu > 0, \ \nu < 0, \ \lambda < 0$ *and*

$$|\lambda|^{1/(p-1)} \leq - |\mu|^{1/(p-1)} |a|^q + |\nu|^{1/(p-1)} |b|^q ,$$

where $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1. Bohr's inequality

$$sx^p + ty^p \geq \frac{1}{(s-1)\, s^{p-2}} \left((s-1)\, x + y \right)^p \geq \frac{1}{2^{p-2}} \left((s-1)\, x + y \right)^p ,$$

when $1 < s \leq 2$, $\frac{1}{s} + \frac{1}{t} = 1$ and $p > 1$, is a special case of Theorem 1 for $a = s-1$, $b = 1$, $\mu = \frac{1}{s}$, $\nu = \frac{1}{t}$ and $\lambda = (s-1)\, s^{p-2}$ (see also [1]).

We first prove a theorem similar [5, Theorem 1.1] but by dealing with a general integer n instead of $n = 2$. Our proof is completely different than the proof in [5]. It relies on the convexity of $f(x) = x^p$, $p > 1$, $x \geq 0$.

Theorem 2. *Let* $x_i, \ a_i, \ i = 1, ..., n$ *be complex numbers. Let* $\lambda, \ \mu_i \in \mathbb{R} \backslash \{0\}$, $i = 1, ..., n$, *and* $p, q \in \mathbb{R}$ *with* $p \neq 0, 1$ *and* $\frac{1}{q} + \frac{1}{p} = 1$.

Case (i): If $\mu_i > 0$, $i = 1, ..., n$, $\lambda > 0$ *and* $p > 1$, *then*

$$\sum_{i=1}^{n} \frac{|x_i|^p}{\mu_i} \geq \frac{\left|\sum_{i=1}^{n} a_i x_i\right|^p}{\lambda} \tag{1.1}$$

where

$$|\lambda|^{\frac{1}{p-1}} \geq \left(\overline{\lambda}\right)^{\frac{1}{p-1}} = \sum_{i=1}^{n} |\mu_i|^{\frac{1}{p-1}} |a_i|^q . \tag{1.2}$$

Case (ii): If $\mu_1 > 0$, $\mu_i < 0$, $i = 2, ..., n$, $\lambda > 0$ *and* $p > 1$, *then*

$$\sum_{i=1}^{n} \frac{|x_i|^p}{\mu_i} \leq \frac{|\sum_{i=1}^{n} a_i x_i|^p}{\lambda} \tag{1.3}$$

where

$$|\lambda|^{\frac{1}{p-1}} \leq |\mu_1|^{\frac{1}{p-1}} |a_1|^q - \sum_{i=2}^{n} |\mu_i|^{\frac{1}{p-1}} |a_i|^q . \tag{1.4}$$

Case (iii): If $\mu_1 < 0$, $\mu_i > 0$, $i = 2, ..., n$, $\lambda < 0$ *and* $p > 1$, *then*

$$\sum_{i=1}^{n} \frac{|x_i|^p}{\mu_i} \geq \frac{|\sum_{i=1}^{n} a_i x_i|^p}{\lambda}$$

where λ *satisfies (1.4)*

Case (iv): If x_i , $a_i \in \mathbb{R}^+$, $i = 1, ..., n$, $\mu_i > 0$, $i = 1, ..., n$, *and* $\lambda > 0$, *then for* $p < 0$, *inequalities (1.1) and (1.2) hold for* $\lambda = \overline{\lambda}$.

Case (v): If x_i , $a_i \in \mathbb{R}^+$, $i = 1, ..., n$, $\mu_i > 0$, $i = 1, ..., n$, *and* $\lambda > 0$, *then for* $0 < p < 1$, *the reverse of inequality (1.1) holds when (1.2) holds for* $\lambda = \overline{\lambda}$.

Proof. Case (i): It is obvious that it is enough to prove this case of the theorem for a_i, $x_i \geq 0$, $i = 1, ..., n$ and show that here

$$\sum_{i=1}^{n} \frac{x_i^p}{\mu_i} \geq \frac{(\sum_{i=1}^{n} a_i x_i)^p}{\lambda} \tag{1.5}$$

holds if

$$\lambda^{\frac{1}{p-1}} \geq \left(\overline{\lambda}\right)^{\frac{1}{p-1}} = \sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q. \tag{1.6}$$

Let us consider first a more general inequality than (1.5) where instead of the function $f(x) = x^p$, $p > 1$, $x \geq 0$, we deal with a positive strictly increasing convex function f on $(0, \infty)$ which satisfies $f^{-1}(AB) \geq f^{-1}(A) f^{-1}(B)$ for A, $B > 0$. In this case we write

$$\sum_{i=1}^{n} \frac{f(x_i)}{\mu_i} = \sum_{i=1}^{n} Q_i f\left(f^{-1}\left(\frac{f(x_i)}{\mu_i Q_i}\right)\right), \tag{1.7}$$

and then by the convexity of f we get

$$\sum_{i=1}^{n} Q_i f\left(f^{-1}\left(\frac{f(x_i)}{\mu_i Q_i}\right)\right) \geq \left(\sum_{j=1}^{n} Q_j\right) f\left(\frac{\sum_{i=1}^{n} Q_i f^{-1}\left(\frac{f(x_i)}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j}\right). \tag{1.8}$$

As $f^{-1}(AB) \geq f^{-1}(A) f^{-1}(B)$ and f is increasing we get that

$$\left(\sum_{j=1}^{n} Q_j\right) f\left(\frac{\sum_{i=1}^{n} Q_i f^{-1}\left(\frac{f(x_i)}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j}\right) \geq \left(\sum_{j=1}^{n} Q_j\right) f\left(\frac{\sum_{i=1}^{n} x_i Q_i f^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j}\right). \tag{1.9}$$

Therefore, from (1.7), (1.8) and (1.9) it is enough to solve the equality

$$\left(\sum_{j=1}^{n} Q_j\right) f\left(\frac{\sum_{i=1}^{n} x_i Q_i f^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j}\right) = \frac{f\left(\sum_{i=1}^{n} a_i x_i\right)}{\overline{\lambda}},$$

in other words to solve

$$\frac{Q_i f^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j} = a_i, \qquad i = 1, ..., n \tag{1.10}$$

and then insert

$$\overline{\lambda} = \left(\sum_{j=1}^{n} Q_j\right)^{-1} \tag{1.11}$$

in order for $\overline{\lambda}$ to satisfy for given $\mu_i > 0$ and $a_i \geq 0$, $i = 1, ..., n$ the inequality

$$\sum_{i=1}^{n} \frac{f(x_i)}{\mu_i} \geq \frac{f\left(\sum_{i=1}^{n} a_i x_i\right)}{\overline{\lambda}}. \tag{1.12}$$

Replacing $\overline{\lambda}$ by

$$\lambda \geq \overline{\lambda} = \left(\sum_{j=1}^{n} Q_j\right)^{-1} \tag{1.13}$$

inequality (1.12) holds too.

Now we return to deal with our function $f(x) = x^p$, $p > 1$, $x \geq 0$. This is a nonnegative increasing convex function for $x \geq 0$ and it satisfies $f^{-1}(AB) = f^{-1}(A) f^{-1}(B)$ for $A, B > 0$.

Going back to the proof of (1.5) under the condition (1.6) we obtain from (1.10) that

$$Q_i (\mu_i Q_i)^{-\frac{1}{p}} \left(\sum_{j=1}^{n} Q_j \right)^{-1} = a_i, \qquad i = 1, ..., n. \tag{1.14}$$

Solving (1.14) we get that

$$Q_i = \frac{\mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q \right)^p}, \quad i = 1, ..., n, \tag{1.15}$$

and from (1.11) that

$$\overline{\lambda} = \left(\sum_{i=1}^{n} Q_i \right)^{-1} = \left(\sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q \right)^{p-1}. \tag{1.16}$$

Hence, from (1.13), (1.5) and (1.6) are proved when $a_i, x_i \geq 0$, $i = 1, ..., n$ and therefore, (1.1) and (1.2) are proved for the complex numbers x_i, a_i, $i = 1, ...n$.

Case (ii): If $\mu_1 > 0$, $\mu_i < 0$, $i = 2, ..., n$ and $\lambda > 0$, we rewrite (1.3) as

$$\frac{\left| \sum_{i=1}^{n} a_i x_i \right|^p}{\lambda} + \sum_{i=2}^{n} \frac{|x_i|^p}{|\mu_i|} \geq \frac{|x_1|^p}{\mu_1}. \tag{1.17}$$

Let us make the substitutions

$$|\mu_i| = \nu_i, \qquad i = 2, ..., n, \qquad \mu_1 = \Lambda, \qquad \lambda = \nu_1,$$

$$z_1 = \sum_{i=1}^{n} a_i x_i, \qquad z_i = x_i, \qquad i = 2, ..., n,$$

and

$$x_1 = \frac{1}{a_1} z_1 + \sum_{i=2}^{n} \left(\frac{-a_i}{a_1} \right) z_i = \sum_{i=1}^{n} C_i z_i.$$

Inequality (1.17) becomes

$$\sum_{i=1}^{n} \frac{|z_i|^p}{\nu_i} \geq \frac{\left| \sum_{i=1}^{n} C_i z_i \right|^p}{\Lambda}.$$

Therefore, from Case (i) we get that

$$\Lambda^{\frac{1}{p-1}} \geq \sum_{i=1}^{n} \nu_i^{\frac{1}{p-1}} \left|C_i\right|^q .$$

In other words, (1.3) holds when

$$\left|\mu_1\right|^{\frac{1}{p-1}} \geq \frac{\left|\lambda\right|^{\frac{1}{p-1}}}{\left|a_1\right|^q} + \sum_{i=2}^{n} \left|\mu_i\right|^{\frac{1}{p-1}} \left|\frac{a_i}{a_1}\right|^q ,$$

which is the same as (1.4).

Case (iii): This case follows immediately from Case (ii).

Case (iv): The fact that f is strictly increasing, and not just strictly monotone, is used only in (1.9). But in the case of power functions the inequality (1.9) reduces to equality. Therefore, this fact is not needed here. This is the reason that for $p<0$ we get that (1.1) when (1.2) holds.

Case (v): The function $f(x) = x^p$, $0<p<1$, $x>0$, is positive strictly increasing concave function. Therefore, we get the reverse of inequality (1.8), and as $f^{-1}(AB) = f^{-1}(A) f^{-1}(B)$ for $A, B > 0$, we have an equality in (1.9). Hence, the reverse of (1.1) holds when the reverse of inequality (1.2) holds.

This completes the proof of the theorem. □

Remark 2. It is obviously that for $n=2$, as a special case of our Theorem 2 for complex numbers x_i, a_i, $i=1,...n$, we get Theorem 1, i.e. [5, Theorem 1.1].

The Jensen-Steffensen inequality, see [4, Theorem 2.19], leads to the following result.

Theorem 3. *Let* μ_i, $a_i \in \mathbb{R}\backslash\{0\}$, $\mu_i a_i > 0$ *and* $x_i > 0$, $i=1,...,n$. *Let* $p, q \in \mathbb{R}$ *with* $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *Let also* $\frac{x_i^p}{(\mu_i a_i)^q} < \frac{x_{i+1}^p}{(\mu_{i+1} a_{i+1})^q}$, $i=1,...,n-1$, *and*

$$0 < \sum_{i=1}^{k} \frac{(\mu_i a_i)^q}{\mu_i} \leq \sum_{i=1}^{n} \frac{(\mu_i a_i)^q}{\mu_i}, \quad k=1,...,n.$$

Then

$$\sum_{i=1}^{n} \frac{x_i^p}{\mu_i} \geq \frac{\left(\sum_{i=1}^{n} a_i x_i\right)^p}{\overline{\lambda}},$$

where

$$\overline{\lambda} = \left(\sum_{i=1}^{n} \frac{(\mu_i a_i)^q}{\mu_i}\right)^{p-1} .$$

2. Extension of the Euler-Lagrange Type Identity

Now we extend the Euler-Lagrange type inequalities by introducing the set of superquadratic functions and its basic properties. The Euler-Lagrange identity is a special case of this extension.

A function $f:[0,b)\to\mathbb{R}$ is superquadratic provided that for all $x\in[0,b)$ there exists a constant $C_f(x)\in\mathbb{R}$ such that the inequality

$$f(y)\geq f(x)+C_f(x)(y-x)+f(|y-x|) \tag{2.1}$$

holds for all $y\in[0,b)$ (see [3, Definition 2.1]). The function $f:[0,b)\to\mathbb{R}$ is subquadratic if $-f$ is superquadratic.

According to [3, Theorem 2.2] the inequality

$$f\left(\int h(s)d\mu(s)\right)\leq\int f(h(s))-f\left(\left|h(s)-\int h(s)d\mu(s)\right|\right)d\mu(s) \tag{2.2}$$

holds for all probability measures μ and all nonnegative μ-integrable h, if and only if f is superquadratic.

The discrete version of (2.2) is

$$f\left(\sum_{i=1}^{n}\alpha_i x_i\right)\leq\sum_{i=1}^{n}\alpha_i\left(f(x_i)-f\left(\left|x_i-\sum_{j=1}^{n}\alpha_j x_j\right|\right)\right), \tag{2.3}$$

$x_i\in[0,b),\quad \alpha_i\geq 0,\quad 1=i,...,n,\quad \sum_{i=1}^{n}\alpha_i=1.$

The power functions $f(x)=x^p$, $x\geq 0$, are convex and superquadratic for $p\geq 2$, and convex and subquadratic for $1<p\leq 2$. Inequalities (2.1), (2.2) and (2.3) reduce to equality for the function $f(x)=x^2$.

Now we use (2.3) in order to get the Euler-Lagrange type inequality.

Theorem 4. *Let $x_i\geq 0$, $a_i\geq 0$ and $\mu_i>0$, $i=1,...,n$. Let $p,q\in\mathbb{R}$ with $p\geq 2$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\sum_{i=1}^{n}\frac{x_i^p}{\mu_i}\geq\frac{\left(\sum_{i=1}^{n}a_i x_i\right)^p}{\left(\sum_{j=1}^{n}\mu_j^{\frac{1}{p-1}}a_j^q\right)^{p-1}} \tag{2.4}$$

$$+\frac{\sum_{i=1}^{n}\mu_i^{\frac{1}{p-1}}a_i^q}{\left(\sum_{j=1}^{n}\mu_j^{\frac{1}{p-1}}a_j^q\right)^p}\left(\left|\left(\frac{1}{a_i\mu_i}\right)^{\frac{1}{p-1}}\left(\sum_{j=1}^{n}\mu_j^{\frac{1}{p-1}}a_j^q\right)x_i-\sum_{j=1}^{n}a_j x_j\right|\right)^p.$$

If $1<p\leq 2$, the inverse of (2.4) holds.

Proof. In Theorem 2 we showed that for $x_i \geq 0$, $a_i \geq 0$, $\mu_i > 0$, $i = 1, ..., n$. inequalities (1.5) and (1.6) hold. There

$$\sum_{i=1}^{n} Q_i (A_i)^p = \sum_{i=1}^{n} \frac{(x_i)^p}{\mu_i} \tag{2.5}$$

where

$$Q_i = \frac{\mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right)^p}, \qquad i = 1, ..., n, \tag{2.6}$$

$$A_i = \frac{1}{(a_i \mu_i)^{\frac{1}{p-1}}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i, \qquad i = 1, , ..., n \tag{2.7}$$

and

$$\frac{\sum_{i=1}^n Q_i A_i}{\sum_{j=1}^n Q_j} = \sum_{i=1}^{n} a_i x_i. \tag{2.8}$$

Therefore, as $f(x) = x^p$, $p \geq 2$, $x \geq 0$, is superquadratic, by inserting (2.6)-(2.8), the inequality (2.3) becomes

$$\sum_{i=1}^{n} Q_i (A_i)^p = \frac{\sum_{i=1}^n \mu_i^{\frac{1}{p-1}} a_i^q \left(\left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i\right)^p}{\left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right)^p} \tag{2.9}$$

$$\geq \frac{\left(\sum_{i=1}^n a_i x_i\right)^p}{\left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p-1}}$$

$$+ \frac{\sum_{i=1}^n \mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right)^p} \left(\left|\left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i - \sum_{j=1}^{n} a_j x_j\right|\right)^p.$$

Hence, from (2.5) and (2.9) we get that (2.4) holds.

If $1 < p \leq 2$, then $f(x) = x^p$, $x \geq 0$, is a subquadratic function. Therefore, the reverse of (2.4) holds. □

Corollary 1. Let x, y, a, b ≥ 0 and μ, $\nu > 0$. Let $p, q \in \mathbb{R}$ with $p \geq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\frac{x^p}{\mu} + \frac{y^p}{\nu} \geq \frac{(ax + by)^p}{\left(\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q\right)^{p-1}} \tag{2.10}$$

$$+\mu^{\frac{1}{p-1}}a^q\left(\left|\left(\frac{1}{a\mu}\right)^{\frac{1}{p-1}}x-\frac{ax+by}{\mu^{\frac{1}{p-1}}a^q+\nu^{\frac{1}{p-1}}b^q}\right|\right)^p$$
$$+\nu^{\frac{1}{p-1}}b^q\left(\left|\left(\frac{1}{\nu b}\right)^{\frac{1}{p-1}}y-\frac{ax+by}{\mu^{\frac{1}{p-1}}a^q+\nu^{\frac{1}{p-1}}b^q}\right|\right)^p.$$

Specially, for $p=2$ we get

$$\frac{x^2}{\mu}+\frac{y^2}{\nu}=\frac{(ax+by)^2}{\mu a^2+\nu b^2}+\frac{(\nu bx-a\mu y)^2}{\mu\nu\left(\mu a^2+\nu b^2\right)}, \tag{2.11}$$

which is the Euler-Lagrange type identity.

Proof. The first part of statement follows directly from Theorem 4 by choosing $n=2$. Specially, if $f(x)=x^2$, as inequality (2.4) reduces to equality, from (2.10) we get (2.11). □

Corollary 2. Let $x_i\geq 0$, $a_i\geq 0$ and $\mu_i>0$, $i=1,...,n$. Let $p,q\in\mathbb{R}$ with $1<p\leq 2$ and $\frac{1}{p}+\frac{1}{q}=1$. Then

$$0\leq\sum_{i=1}^{n}\frac{x_i^p}{\mu_i}-\frac{\left(\sum_{i=1}^{n}a_ix_i\right)^p}{\left(\sum_{i=1}^{n}\mu_i^{\frac{1}{p-1}}a_i^q\right)^{p-1}} \tag{2.12}$$
$$\leq\frac{\sum_{i=1}^{n}\mu_i^{\frac{1}{p-1}}a_i^q}{\left(\sum_{j=1}^{n}\mu_j^{\frac{1}{p-1}}a_j^q\right)^p}\left(\left|\left(\frac{1}{a_i\mu_i}\right)^{\frac{1}{p-1}}\left(\sum_{j=1}^{n}\mu_j^{\frac{1}{p-1}}a_j^q\right)x_i-\sum_{j=1}^{n}a_jx_j\right|\right)^p.$$

Proof. From Theorem 4, as $f(x)=x^p$, $1<p\leq 2$, $x\geq 0$, is both subquadratic and convex, we get (2.12). □

3. The Euler-Lagrange Type Identity for Superterzatic Functions

A function $g:[0,b)\to\mathbb{R}$ is called superterzatic provided that for all $\bar{x}\in[0,b)$ there exists a constant $C(\bar{x})\in\mathbb{R}$ such that the inequality

$$\sum_{i=1}^{n}\alpha_i g(x_i)-g(\bar{x}) \tag{3.1}$$

$$\geq \sum_{i=1}^{n} \alpha_i x_i \left[(x_i - \bar{x}) C(\bar{x}) + (|x_i - \bar{x}|)^{-1} g(|x_i - \bar{x}|) \right]$$
$$= \sum_{i=1}^{n} \alpha_i (x_i - \bar{x})^2 C(\bar{x}) + \sum_{i=1}^{n} \alpha_i x_i (|x_i - \bar{x}|)^{-1} g(|x_i - \bar{x}|)$$

holds for all $x_i \in [0,b)$, $\alpha_i \geq 0$, $i = 1,...,n$, and $\sum_{i=1}^{n} \alpha_i = 1$, where $\bar{x} = \sum_{j=1}^{n} \alpha_j x_j$, (see [2]). A function g is called subterzatic if $-g$ is superterzatic.

Let $f : [0,b) \to \mathbb{R}$ and $g : [0,b) \to \mathbb{R}$ be defined by $g(x) = xf(x)$. A sufficient condition for a function g to be superterzatic, is presented in [2, Theorem 1]. It is shown there that if f is superquadratic then g is superterzatic and if f is subquadratic then the reverse of inequality (3.1) holds, that is, g is subterzatic. Inequality in (3.1) becomes equality for $g(x) = xf(x) = x^3$.

When $g(x) = xf(x)$, f is superquadratic, it is also shown in [2, Theorem 1] that $C(x)$ in (3.1) is equal to $C_f(x)$, where $C_f(x)$ is as in (2.1). Also, when $g(x) = x^p$, $x > 0$, $p > 2$, then $C(x) = C_f(x) = (p-1)x^{p-2}$ (see [2, Example 2]).

Using (3.1) we extend the Euler-Lagrange type inequalities.

Theorem 5. *Let $x_i \geq 0$, $a_i \geq 0$ and $\mu_i > 0$, $i = 1,...,n$. Let $p,q \in \mathbb{R}$ with $p \geq 3$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{i=1}^{n} \frac{x_i^p}{\mu_i} \geq \frac{\bar{x}^p}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p-1}} \tag{3.2}$$
$$+ \frac{\sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p}} \left(\left(\frac{1}{a_i\mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i - \bar{x} \right)^2 (p-1)\bar{x}^{p-2}$$
$$+ \frac{\sum_{i=1}^{n} a_i x_i}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p-1}} \left(\left| \left(\frac{1}{a_i\mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i - \bar{x} \right| \right)^{p-1},$$

where $\bar{x} = \sum_{j=1}^{n} a_j x_j$.

If $2 < p \leq 3$, then the reverse of (3.2) holds.

Proof. The function $g(x) = x^p$, $p \geq 3$, $x \geq 0$ is superterzatic and therefore, as explained before, the inequality (3.1) holds with $C(\bar{x}) = (p-1)\bar{x}^{p-2}$ (see [2, Example 2]). Then by inserting (2.6)-(2.8) in (3.1) we get (3.2).

If $2 < p \leq 3$, then $g(x) = x^p$, $x \geq 0$, is subterzatic and therefore, the reverse of inequality (3.2) holds. ☐

Remark 3. For $p = 3$ the inequality (3.2) becomes equality since for $g(x) = x^3$ the inequality (3.1) reduce to equality.

Corollary 3. Let $x,\ y,\ a,\ b \geq 0$ and $\mu,\ \nu > 0$. Let $p, q \in \mathbb{R}$ with $p \geq 3$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned}
&\frac{x^p}{\mu} + \frac{y^p}{\nu} \qquad (3.3)\\
&\geq \frac{\bar{x}^p}{\left(\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q\right)^{p-1}}\\
&+ \frac{\mu^{\frac{1}{p-1}} a^q}{\left(\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q\right)^{p}} \left(\frac{1}{(\mu a)^{\frac{1}{p-1}}} \left(\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q\right) x - \bar{x}\right)^2 (p-1)\bar{x}^{p-2}\\
&+ \frac{\nu^{\frac{1}{p-1}} b^q}{\left(\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q\right)^{p}} \left(\frac{1}{(\nu b)^{\frac{1}{p-1}}} \left(\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q\right) y - \bar{x}\right)^2 (p-1)\bar{x}^{p-2}\\
&+ ax \left(\left|\frac{x}{(\mu a)^{\frac{1}{p-1}}} - \frac{\bar{x}}{\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q}\right|\right)^{p-1}\\
&+ by \left(\left|\frac{y}{(\nu b)^{\frac{1}{p-1}}} - \frac{\bar{x}}{\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q}\right|\right)^{p-1},
\end{aligned}$$

where $\bar{x} = ax + by$.

Specially, for $p = 3$ we get the Euler-Lagrange type identity:

$$\begin{aligned}
\frac{x^3}{\mu} + \frac{y^3}{\nu} = \frac{(ax+by)^3}{\left(\mu^{\frac{1}{2}} a^{\frac{3}{2}} + \nu^{\frac{1}{2}} b^{\frac{3}{2}}\right)^2} &+ \left(\frac{ab}{\mu\nu}\right)^{\frac{1}{2}} \left(\frac{\nu^{\frac{1}{2}} b^{\frac{1}{2}} x - \mu^{\frac{1}{2}} a^{\frac{1}{2}} y}{\mu^{\frac{1}{2}} a^{\frac{3}{2}} + \nu^{\frac{1}{2}} b^{\frac{3}{2}}}\right)^2 2(ax+by)\\
&+ \frac{\nu b^2 x + \mu a^2 y}{\mu\nu} \left(\frac{\nu^{\frac{1}{2}} b^{\frac{1}{2}} x - \mu^{\frac{1}{2}} a^{\frac{1}{2}} y}{\mu^{\frac{1}{2}} a^{\frac{3}{2}} + \nu^{\frac{1}{2}} b^{\frac{3}{2}}}\right)^2. \qquad (3.4)
\end{aligned}$$

Proof. The first part of statement follows directly from Theorem 5 by choosing $n = 2$. Specially, since for $p = 3$ the inequality (3.2) reduces to equality, from (3.3) we get (3.4). ☐

Corollary 4. Let $x_i \geq 0$, $a_i \geq 0$ and $\mu_i > 0$, $i = 1, ..., n$. Let $p, q \in \mathbb{R}$ with $2 < p \leq 3$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} \frac{x_i^p}{\mu_i} - \frac{\bar{x}^p}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p-1}} \qquad (3.5)\\
&\leq \frac{\sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p}} \left(\left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i - \bar{x}\right)^2 (p-1)\bar{x}^{p-2}\\
&+ \frac{\sum_{i=1}^{n} a_i x_i}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p-1}} \left(\left|\left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i - \bar{x}\right|\right)^{p-1},
\end{aligned}
$$

where $\bar{x} = \sum_{j=1}^{n} a_j x_j$.

Proof. From Theorem 5, since the function $g(x) = x^p$, $2 < p \leq 3$, $x \geq 0$, is both convex and superterzatic, we get (3.5). ☐

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