International Journal of Pure and Applied Mathematics
Volume 74 No. 3 2012, 367-372
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

FREE STREAMLINE FOR A FLOW OVER A STEP

F. Guechi[1 §], M. Khermache[2]
[1,2]Department of Mathematics
Ferhat Abbas University
19000, Sétif, ALGERIA

Abstract: A free surface flow over a step is calculated analytically. The flow is assumed to be steady, two dimensional and irrotational. The liquid is treated as inviscid and incompressible. The effects of gravity and surface tension are not taken into account. We use the method of the free streamline theory based on the hodograph method and Schwartz-Christoffel transformation technique to obtain the exact solution.

AMS Subject Classification: 76B07, 76D45, 76M40, 76B07
Key Words: free surface, flow, potential function, the free surface streamline theory

1. Introduction

We study the steady two dimensional potential flow of an inviscid incompressible heavy fluid in a domain bounded below by an infinites walls rigid and above by a free surface. The fluid is assumed to be inviscid, incompressible and the flow is irrotationnal. Far upstream the flow is uniform with a constant velocity U and a constant depth L. The classical problem of a free streamline flow of an ideal fluid has been studies by many authors [1-3] and [7-8]. Toison et al. [6] used an iterative method and Bloor [4-5] used a series truncation. The

Received: November 13, 2011

§Correspondence author

first work in this type of problem is characterized by the use of the Schwartz-Christoffel formula. The latter can treat the flows of border which combines rectilinear wall and unknown a free surface.

2. Formulation of Problem

We consider a steady irrotational flow of an incompressible and inviscid fluid over a step (Fig.1).The effects of gravity and surface tension are not taken into account. A system of cartesian coordinates is defined with the x-axis along the horizontal wall and the y-axis going through the vertical wall. As and ; the flow approaches a uniform stream with a constant velocity U and depth H.

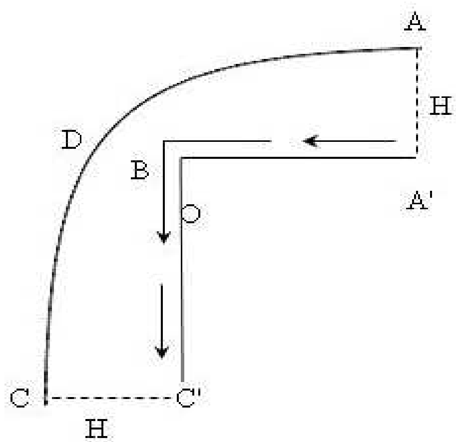

Figure 1: Sketch of the flow and of the system of coordinates

The flow is limited superiorly by the thread of current . One indicates by the complex velocity such that and are its components and as the complex function where and are the potential and the stream functions respectively. The function transforms the z-plan into an infinite band (see Fig.2).

The mathematical problem consists in determining the potential function

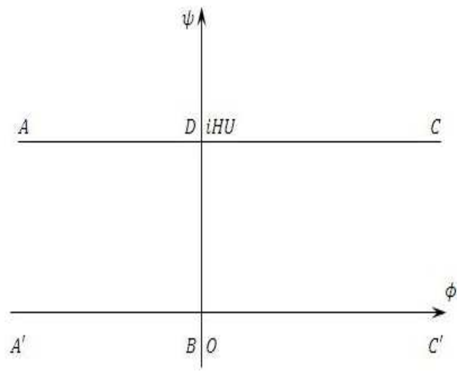

Figure 2: The flow configuration in the complex potential plane

which checks the following conditions:

$$\begin{cases} \Delta\phi = 0 & in\ the\ domain \\ \frac{1}{2}((\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2) + \frac{P}{\rho} = Cte & on\ the\ free\ surface\ of\ unknown\ form \\ \frac{\partial\phi}{\partial x} = 0 & on\ BC' \\ \frac{\partial\phi}{\partial y} = 0 & on\ A'B \\ \phi \to U_x & when\ x \to \infty,\ \phi \to -\infty \end{cases} \tag{1}$$

3. Resolution of the Problem

To solve the problem (1), we initially use the method of the free surface streamline theory introduced by Kirchhoff, based on the hodograph transformation to find the form of the free surface. The complex transformation is defined by:

$$\Omega = log(\frac{U}{\frac{df}{dz}}) = log(\frac{U}{u - iv}) = log(\frac{U}{q}) + i\theta \tag{2}$$

Where $z = x + iy$, q and θ are the module speed and the angle between the velocity vector and the horizontal one, respectively. By this last transformation, the field occupied by the fluid in the $z-$plan is transformed into an infinite band in the $\Omega-$plan (see Fig.3).

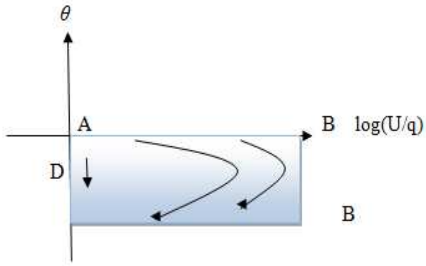

Figure 3: The flow domain in the Ω-plane.

The conforms transformation of an semi-infinite band in the plan to the lower half-plan of another complexes $\lambda-$plan, is given by the theorem of Schwartz-Christoffel, by respecting the direction and the orientation of the flow (see Fig.4).

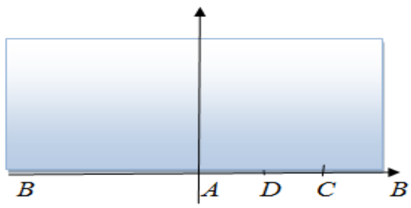

Figure 4: The flow domain in the $\lambda-$plane

This transformation is given by:

$$\lambda = \frac{1}{2}[1 - ch(-2\Omega)] = -sh^2\Omega \tag{3}$$

The transformation which transforms the interior of the infinite band of the $f-$plan towards the lower half-plan of the $\lambda-$plan is:

$$\lambda = \frac{1}{1 - e^{(\frac{\pi f}{HU})}} \tag{4}$$

After calculations, we find a relation between λ and f:

$$U\frac{dz}{d\lambda} = U\frac{dz}{df}\frac{df}{d\lambda} = \frac{-HU}{\pi}(\frac{1}{\lambda\sqrt{1-\lambda}} - i\frac{1}{(1-\lambda)\sqrt{\lambda}}) \tag{5}$$

By integrating (5) with the choice of $z = -L + iL$ into the point D where $\lambda = \frac{1}{2}$, the solution is as follows:

$$\begin{cases} x = -L + \frac{H}{\pi} log(\frac{1+\sqrt{1-\lambda}}{1-\sqrt{1-\lambda}}) - \frac{2H}{\pi} log(\sqrt{2}+1) \\ y = L - \frac{H}{\pi} log(\frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}}) + \frac{2H}{\pi} log(\sqrt{2}+1) \end{cases} \quad 0 \leq \lambda \leq 1 \tag{6}$$

The amplitude of the flow in the infinity is:

$$H = \lim_{\lambda \to 0} y(\lambda) = L + \frac{2H}{\pi} log(\sqrt{2}+1) \tag{7}$$

One represents C_e like the contraction degree of the flow:

$$C_e = \frac{H}{L} \tag{8}$$

We write $\overline{x} = \frac{x}{H}, \overline{y} = \frac{y}{H}$. The parametric equation of the free surface becomes:

$$\begin{cases} \overline{x} = -1 + \frac{1}{\pi} log(\frac{1+\sqrt{1-\lambda}}{1-\sqrt{1-\lambda}}) \\ \overline{y} = 1 - \frac{1}{\pi} log(\frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}}) \end{cases} \quad 0 \leq \lambda \leq 1 \tag{9}$$

The form of the free surface over a step is given in Figure .5.

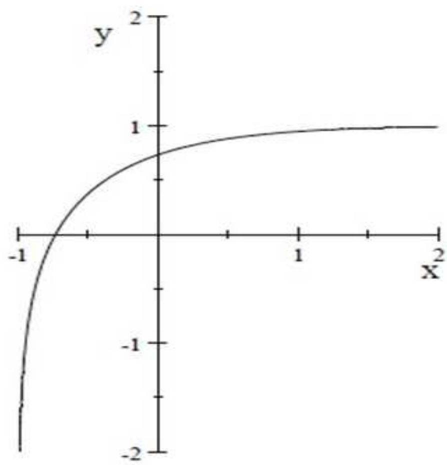

Figure 5: The form of the free surface

References

[1] D. Daboussi, F. Dias, J. M. Vanden Broeck, Gravity flows with a free surface of finite extent, *Eur. J. Mech. B – Fluids*, **17** (1998), 19-31.

[2] I. El bouihi, R. Sehaqui, Numerical stydy of two-dimensional incompressible Navier-Stokes equation in natural convection with nanofluids, *Int. J. Open Problems Compt. Math.*, **3** (2010), 72-82.

[3] L.K. Forbes, Two-Layer critical flow over a semi-circular obstruction, *J. Eng. Math.*, **23** (1989), 325-342.

[4] A.C. King, M.G.I. Bloor, Free surface flow over a step, *J. Fluid Mech.*, **182** (1987), 193-208.

[5] A.C. King, M.G.I. Bloor, Free streamline flow over curved topography, *Q. Appl. M.*, **48** (1990), 281-293.

[6] F. Toison, J. Hureau, Open Channel flow and waterfalls, *Eur. J. Mech. B-Fluids*, **19** (2000), 269-283.

[7] J.M. Vanden-Broeck, Flow under a gate, *Phys. Fluids*, **29** (1986), 3148-3151.

[8] J.M. Vanden-Broeck, Free streamline model for a rising bubble, *Phys. Fluids*, **29** (1986), 2798-2801.