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A CERTAIN p-ADIC SPECTRAL THEOREM

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Abstract: We extend a p-adic spectral theorem of M. M. Vishik to a certain class of p-adic Banach algebras. This class includes inductive limits of finite-dimensional p-adic Banach algebras of the form $B(\mathcal{X})$, where $\mathcal{X}$ is a p-adic Banach space of the form $\mathcal{X} \simeq \Omega_p(J)$, J being a finite nonempty set. In particular, we present a p-adic spectral theorem for p-adic UHF algebras and p-adic TUHF algebras (Triangular UHF Algebras).

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1. Introduction

In an germinal 1960 paper [6] J. Glimm introduced the class of uniformly hyperfinite (UHF) C^*-algebras. In that paper Glimm showed that UHF algebras are classified up to $*$-isomorphisms by their *supernatural numbers*. Glimm's classification of UHF C^*-algebras inspired the author of the present paper to introduce in [2] a class of non-selfadjoint operator algebras that are currently called *standard triangular UHF (TUHF)* algebras. Briefly: a *standard triangular UHF (TUHF)* operator algebra is any unital Banach algebra that is isometrically isomorphic to a Banach algebra inductive limit of the form $\mathcal{T} = \varinjlim(T_{p_n}; \sigma_{p_n\, p_m})$; where (p_n) is a sequence of positive integers such that $p_m \mid p_n$ whenever $m \leq n$.

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Here T_{p_n} is the algebra of $p_n \times p_n$ upper triangular complex matrices and for $m \leq n$, $\sigma_{p_n p_m} : T_{p_m} \to T_{p_n}$ is the mapping $x \hookrightarrow 1 \otimes x = \mathrm{diag}(x, \dots, x)$. The main result in [2] is that two standard triangular UHF operator algebras are isometrically isomorphic if, and only if they have the same supernatural number. This result can be extended in at least three different directions. One direction is to view standard triangular UHF operator algebras as special cases of *Banach algebra inductive limits* of upper triangular matrix algebras, and the the goal is to use *purely Banach-algebraic* methods to extend the main result in [2] to these inductive limits. Such an extension is presented in the paper [3], where it is proved that the supernatural number associated to an arbitrary *triangular UHF (TUHF) Banach algebra* is an isomorphism invariant of the algebra, provided that the algebra satisfies certain "local dimensionality conditions." The proof of main result of [3], although relying only on classical complex Banach algebra techniques, makes essential use of the classical spectral theorem (the Riesz functional calculus) for complex Banach algebras. A second direction in which the main result in [2] has been extended is to view standard TUHF algebras as *triangular subalgebras* of UHF C^*-algebras, such that the diagonal of the triangular subalgebra is a *canonical masa* in the ambient C^*-algebra. In this vein, the results in [2] have been substantially extended by J.R. Peters, Y.T. Poon and B.H. Wagner [11], and by S.C. Power [9], [10]. Finally, a third direction in which the principal results of [2] can be extended is to convert these results into a "pure piece" of *p-adic functional analysis* (to borrow a turn of phrase used by I. Kaplansky). For any prime number p, let Ω_p be the p-adic counterpart of the complex numbers $\mathbf{C}$ (see [7], page 13). Replacing $\mathbf{C}$ by Ω_p in the definition of complex (T)UHF Banach algebras, we obtain the definition of *p-adic (T)UHF Banach algebras*. The results in [2] for complex TUHF algebras can be duplicated for p-adic TUHF algebras, provided that a sufficiently general p-adic version of the Riesz functional calculus can be developed for p-adic Banach algebras ([3]). The purpose of the present paper it to prove a p-adic spectral theorem (Theorem 2.10) that is an extension of a classical p-adic spectral theorem of M. M. Vishik (see [7], page 149). We then use this extended p-adic spectral theorem to prove a *p-adic Spectral Theorem for p-adic TUHF and UHF* Banach algebras (Theorem 3.3). In the preprint [4] we use Theorem 3.3, Theorem 3.4, and Lemma 3.5 to transfer, *mutatis mutandis*, the results of [3] to p-adic TUHF Banach algebras.

In the remainder of the present section of the paper we put forth the preliminary material on p-adic analysis and p-adic Banach algebras that is necessary for proving Theorem 2.10. Then in Section 2 we prove Theorem 2.10. Finally, in Section 3, we use Theorem 2.10 to prove Theorem 3.3.

Let p be a prime number, and let $\mathbf{Q}$ be the field of rational numbers. Let $|\cdot|_p$ be the function defined on $\mathbf{Q}$ by

$$\left|\frac{a}{b}\right|_p = p^{\mathrm{ord}_p b - \mathrm{ord}_p a}, \quad |0|_p = 0.$$

Here ord_p of a non-zero integer is the highest power of p dividing the integer. Then $|\cdot|_p$ is a norm on $\mathbf{Q}$. The field $\mathbf{Q}_p$ is defined to be the completion of $\mathbf{Q}$ under the norm $|\cdot|_p$. Define $\overline{\mathbf{Q}}_p$ to be the algebraic closure of $\mathbf{Q}_p$ (see [7]: p. 11). Unlike the case of the real numbers $\mathbf{R}$, whose algebraic closure $\mathbf{C}$ is only a quadratic extension of $\mathbf{R}$, the algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ has infinite degree over $\mathbf{Q}_p$. However, the norm $|\cdot|_p$ on $\mathbf{Q}_p$ can be extended to a norm $|\cdot|_p$ on $\overline{\mathbf{Q}}_p$. But it turns out that $\overline{\mathbf{Q}}_p$ is not complete under this extended norm. Thus, in order to do analysis, we must take a larger field than $\overline{\mathbf{Q}}_p$. We denote the completion of $\overline{\mathbf{Q}}_p$ under the norm $|\cdot|_p$ by Ω_p, that is,

$$\Omega_p = \overset{\wedge}{\overline{\mathbf{Q}}}_p,$$

where $^\wedge$ means completion with respect to $|\cdot|_p$. The symbol "$\mathbf{C}$" is usually used to denote Ω_p (see [7], page 13 and [12], page 140).

Definition 1.1. Let $r \geq 0$ be an non-negative real number, and let $a \in \Omega_p$ and let $\sigma \subseteq \Omega_p$. We have the following definitions.

$$\begin{aligned} D_a(r) &= \{\, x \in \Omega_p \mid \ |x-a|_p \leq r \,\}; \\ D_a(r^-) &= \{\, x \in \Omega_p \mid \ |x-a|_p < r \,\}; \\ D_\sigma(r) &= \{\, x \in \Omega_p \mid \ \mathrm{dist}(x,\sigma) \leq r \,\}; \\ D_\sigma(r^-) &= \{\, x \in \Omega_p \mid \ \mathrm{dist}(x,\sigma) < r \,\}. \end{aligned}$$

If $b \in D_a(r)$, then $D_b(r) = D_a(r)$, and if $b \in D_a(r^-)$, then $D_b(r^-) = D_a(r^-)$. Thus, any point in a disc is its center. Hence $D_a(r), D_a(r^-)$ are both open and closed in the topological sense. It is conventional to take $\inf \emptyset = +\infty$, hence for $r > 0$ we have $D_\emptyset(r^-) = \emptyset$ and $D_\emptyset(r) = \emptyset$ (see [5], page 4).

Lemma 1.2. *Let $\emptyset \neq \sigma \subseteq \Omega_p$ be compact. Then for any $s > 0$ there exists $0 < r \in |\Omega_p|_p$ and $a_1, ..., a_N \in \sigma$ such that $r < s$ and*

$$D_\sigma(r) = \bigcup_{i=1}^{N} D_{a_i}(r);$$

$$\sigma \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-),$$

where the $D_{a_i}(r)$ are disjoint.

Proof. Since σ is compact, there exist $a_1, ..., a_N \in \sigma$ such that

$$\sigma \subseteq \bigcup_{i=1}^{N} D_{a_i}(s^-), \tag{1}$$

where the union is disjoint and for $1 \leq i \leq N$, $\sigma \cap D_{a_i}(s^-) \neq \emptyset$. Now, for each $1 \leq i \leq N$, $D_{a_i}(s^-)$ is closed, and hence $\sigma \cap D_{a_i}(s^-)$ is compact. Therefore there exists $b_1, ..., b_N \in \sigma$ such that for $1 \leq i \leq N$, $b_i \in \sigma \cap D_{a_i}(s^-)$ and

$$|b_i - a_i|_p = \sup\{\, |x - a_i|_p \,:\, x \in \sigma \cap D_{a_i}(s^-)\,\}.$$

Because each $b_i \in D_{a_i}(s^-)$, we have, for $1 \leq i \leq N$, $s > |b_i - a_i|_p$. Now select any $r \in |\Omega_p|_p$ such that $r > 0$ and

$$s > r > |b_1 - a_1|_p, \ldots, |b_N - a_N|_p.$$

We claim that

$$\sigma \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-), \tag{2}$$

where $D_{a_1}(r), \ldots, D_{a_N}(r)$ are disjoint. To prove the claim, let $a \in \sigma$; then by (1), $a \in D_{a_i}(s^-)$ for some $1 \leq i \leq N$, and hence $a \in \sigma \cap D_{a_i}(s^-)$, which implies that

$$|a - a_i|_p \leq \sup\{\, |x - a_i|_p \,:\, x \in \sigma \cap D_{a_i}(s^-)\,\} = |b_i - a_i|_p < r.$$

This shows that $a \in D_{a_i}(r^-)$, which proves (2). Now let $1 \leq i, j \leq N$, with $i \neq j$. Suppose that $D_{a_i}(r) \cap D_{a_j}(r) \neq \emptyset$, and let $a \in D_{a_i}(r) \cap D_{a_j}(r)$. Then we have

$$|a_i - a_j|_p \leq \max\{\, |a_i - a|_p, |a - a_j|_p\,\} \leq r < s,$$

hence $a_i \in D_{a_i}(s^-) \cap D_{a_j}(s^-) = \emptyset$. This contradiction proves the claim. We now claim that

$$D_\sigma(r) = \bigcup_{i=1}^{N} D_{a_i}(r). \tag{3}$$

It is clear that $\bigcup_{i=1}^{N} D_{a_i}(r) \subseteq D_\sigma(r)$. Let $x \in D_\sigma(r)$, then because σ is compact, there exists $a \in \sigma$ such that $r \geq \operatorname{dist}(x, \sigma) = |x - a|_p$. Hence, $x \in D_a(r)$. Now, $a \in \sigma$, so by (2), there exists $1 \leq j \leq N$ such that $a \in D_{a_j}(r^-) \subseteq D_{a_j}(r)$. Therefore, $D_a(r) = D_{a_j}(r)$, which shows that $x \in \bigcup_{i=1}^{N} D_{a_i}(r)$. This proves (3). □

Lemma 1.3. *Let $\emptyset \neq \sigma \subseteq \Omega_p$. Let $r > 0$ and let $b_1, \ldots, b_M$ be in Ω_p, with*

$$\sigma \subseteq \bigcup_{i=1}^{M} D_{b_i}(r^-), \tag{1}$$

where the $D_{b_i}(r)$ are disjoint. Then there exist $a_1, \ldots, a_N$ in σ and $\emptyset \neq I \subseteq \{1, \ldots, M\}$ such that the $D_{a_i}(r)$ are disjoint and

$$D_\sigma(r^-) = \bigcup_{i=1}^{N} D_{a_i}(r^-) = \bigcup_{i \in I} D_{b_i}(r^-); \tag{2}$$

$$D_\sigma(r) = \bigcup_{i=1}^{N} D_{a_i}(r) = \bigcup_{i \in I} D_{b_i}(r).$$

Proof. Let $\emptyset \neq I \subseteq \{1, \ldots, M\}$ be the subset of $\{1, \ldots, M\}$ consisting of those $1 \leq i \leq M$ for which $\sigma \cap D_{b_i}(r^-) \neq \emptyset$. Write

$$I = \{k_1, \ldots, k_N\}.$$

For each $1 \leq i \leq N$, select an $a_i \in \sigma \cap D_{b_{k_i}}(r^-)$. Then for $1 \leq i \leq N$, we have $D_{a_i}(r^-) = D_{b_{k_i}}(r^-)$, and hence, by (1),

$$\sigma \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-) = \bigcup_{i \in I} D_{b_i}(r^-). \tag{3}$$

The assumption that the $D_{b_i}(r)$ are disjoint easily implies that the $D_{a_i}(r)$ are disjoint. We claim that

$$D_\sigma(r^-) \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-); \tag{4}$$

$$D_\sigma(r) \subseteq \bigcup_{i=1}^{N} D_{a_i}(r).$$

To prove (4) let $x \in D_\sigma(r^-)$. Then, by definition, $\mathrm{dist}(x,\sigma) < r$, hence there exists $a \in \sigma$ such that $|x-a|_p < r$. Then (3) implies that there exist $1 \le j \le N$ such that $|a - a_j|_p < r$. Therefore, we have

$$|x - a_j|_p = |(x-a) + (a-a_j)|_p \le \max\{\, |x-a|_p, |a-a_j|_p \,\} < r.$$

Thus, $x \in D_{a_j}(r^-)$. This proves the first part (4). To prove the second part of (4), suppose that there exists $x \in D_\sigma(r)$ with $x \notin \bigcup_{i=1}^{N} D_{a_i}(r)$. Then $|x-a_i|_p > r$ for $1 \le i \le N$, hence we can find $0 < r < s$ such that $|x-a_i|_p > s$ for $1 \le i \le N$. Now let $a \in \sigma$ be arbitrary, then by (3), $|a - a_i|_p < r$ for some $1 \le i \le N$. We then have

$$|x-a|_p = |(x-a_i) + (a_i - a)|_p = |x-a_i|_p > s. \tag{5}$$

In (5) we have used the easily proved fact that for $y, z \in \Omega_p$, if $|y|_p > |z|_p$, then $|y+z|_p = |y|_p$ (see [**?**], Theorem 2, page 5). But $a \in \sigma$ is arbitrary, hence (5) implies that $\mathrm{dist}(x,\sigma) \ge s > r$, which contradicts $x \in D_\sigma(r^-)$. This proves the second part of (4). It is easy to see that

$$\bigcup_{i=1}^{N} D_{a_i}(r^-) \subseteq D_\sigma(r^-); \tag{6}$$

$$\bigcup_{i=1}^{N} D_{a_i}(r) \subseteq D_\sigma(r).$$

Hence by (4) and (6), we see that (2) holds. □

Lemma 1.4. *Let $a \in \Omega_p$ and $0 < r \in |\Omega_p|_p$. Suppose that $f : D_a(r) \to \Omega_p$ is the uniform limit of rational functions with poles outside $D_a(r)$. Then there exists a sequence (c_k) in Ω_p such that $\lim_{k\to\infty} r^k |c_k|_p = 0$ and for all $x \in D_a(r)$*

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k.$$

Proof. See [7], Lemma 3, page 130. □

Lemma 1.5. *Let $a \in \Omega_p$, and let $0 < r \in |\Omega_p|_p$. Let $f : D_a(r) \to \Omega_p$. Suppose that there exists a sequence (c_k) in Ω_p be such that $\lim_{k\to\infty} r^k |c_k|_p = 0$,* *and for all* $x \in D_a(r)$,

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k.$$

Then $\max_{x \in D_a(r)} |f(x)|_p$ *is attained when* $|x-a|_p = r$, *and we have*

$$\max_{x \in D_a(r)} |f(x)|_p = \max_k r^k |c_k|_p\,.$$

Proof. See [7], Lemma 3, page 130. □

Definition 1.6. Let $a \in \Omega_p$, and let $0 < r \in |\Omega_p|_p$. Let $\emptyset \neq \sigma \subseteq \Omega_p$. A function $f : D_a(r) \to \Omega_p$ is said to be *Krasner analytic* on $D_a(r)$ iff f can be represented by a power series on $D_a(r)$ of the form $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$, where $\lim_{k \to \infty} r^k |c_k|_p = 0$. Define $B_r(\sigma)$ to be the set of all functions

$$f : D_\sigma(r) \to \Omega_p$$

such that f is Krasner analytic on $D_a(r)$ whenever $a \in \Omega_p$ and $D_a(r) \subseteq D_\sigma(r)$. If σ is compact, we define $L(\sigma) = \bigcup \{\, B_r(\sigma) \,|\, 0 < r \in |\Omega_p|_p \,\}$, and we call $L(\sigma)$ the set of *locally analytic functions* on σ (see [7], page 136).

Definition 1.7. Let $\emptyset \neq \sigma \subseteq \Omega_p$. For $0 < r, s \in |\Omega_p|_p$ and $a_1, \ldots, a_N, b_1, \ldots, b_M \in \Omega_p$, define the order relation $(s, b_1, \ldots, b_M) \leq (r, a_1, \ldots, a_N)$ by

$$(s, b_1, \ldots, b_M) \leq (r, a_1, \ldots, a_N) \iff s \leq r \text{ and } \bigcup_{i=1}^{M} D_{b_i}(s) \subseteq \bigcup_{i=1}^{N} D_{a_i}(r).$$

For $0 < r \in |\Omega_p|_p$ and $a_1, \ldots, a_N \in \Omega_p$, define

$$B_{r,a_1,\ldots,a_N} = \left\{ f : \bigcup_{i=1}^{N} D_{a_i}(r) \to \Omega_p \;\middle|\; f \text{ is Krasner analytic on each } D_{a_i}(r) \right\}.$$

Let $I(\sigma)$ to be the set of all $(r, a_1, \ldots, a_N)$ such that:

(1) $0 < r \in |\Omega_p|_p$ and $a_1, \ldots, a_N \in \Omega_p$;
(2) $D_{a_i}(r) \cap D_{a_j}(r) = \emptyset$ for $i \neq j$.
(3) $\sigma \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-)$.

Finally, define

$$\mathcal{L}(\sigma) = \bigcup \{\, B_{r,a_1,\ldots,a_N} \,|\, (r, a_1, \ldots, a_N) \in I(\sigma) \,\}.$$

Lemma 1.8. *Let $\emptyset \neq \sigma \subseteq \Omega_p$. Then the following statements hold.*

(a) *The set $I(\sigma)$ is decreasingly filtered under the order relation $\leq$.*

(b) *Let $(r, a_1, \ldots, a_N) \in I(\sigma)$ and $f \in B_{r,a_1,\ldots,a_N}$. Then f is bounded on $U = \bigcup_{i=1}^{N} D_{a_i}(r)$, and we define the uniform norm $\|f\|_u$ of f by*

$$\|f\|_u = \max_{x \in U} |f(x)|_p.$$

(c) *Let $(s, b_1, \ldots, b_M), (r, a_1, \ldots, a_N) \in I(\sigma)$, with*

$$(s, b_1, \ldots, b_N) \leq (r, a_1, \ldots, a_N).$$

Define $V = \bigcup_{i=1}^{M} D_{b_i}(s)$, then $f|_V \in B_{s,b_1,\ldots,b_M}$ and the mapping $f \mapsto f|_V$ is continuous on $B_{r,a_1,\ldots,a_N}$ in the uniform norm.

(d) *$\mathcal{L}(\sigma)$ is an algebra over Ω_p.*

(e) *For $\alpha = (r, a_1, \ldots, a_N) \in I(\sigma)$, set $B_\alpha = B_{r,a_1,\ldots,a_N}$. For $\beta = (s, b_1, \ldots, b_M)$ in $I(\sigma)$, with $\beta \leq \alpha$, let $V = \bigcup_{i=1}^{M} D_{b_i}(s)$ and define $\varphi^\alpha_\beta : B_\alpha \to B_\beta$ by*

$$\varphi^\alpha_\beta(f) = f|_V, \text{ all } f \in B_\alpha.$$

Then for $\alpha \geq \beta \geq \gamma \in I(\sigma)$ following conditions are satisfied

$$\varphi^\alpha_\alpha(f) = f, \text{ for all } f \in B_\alpha,$$
$$\varphi^\alpha_\beta \varphi^\beta_\gamma = \varphi^\alpha_\gamma.$$

It is worth noting here that we can define the inverse limit $\varprojlim B_\alpha$ of the system $\{ B_\alpha \,|\, \alpha \in I(\sigma) \}$ and place the projective topology on $\varprojlim B_\alpha$. But we will not make use of this topology in the present paper.

(f) *If σ is compact, then $\mathcal{L}(\sigma) = L(\sigma)$, i.e.,*

$$\mathcal{L}(\sigma) = \bigcup \{ B_r(\sigma) \,|\, 0 < r \in |\Omega_p|_p \}. \tag{1}$$

(g) *For all $(r, a_1, \ldots, a_N) \in I(\sigma)$, if $a_1, \ldots, a_N \in \sigma$, then $B_{r,a_1,\ldots,a_N} = B_r(\sigma)$.*

Proof. To prove (a) let $(s, b_1, \ldots, b_M), (r, a_1, \ldots, a_N)$ be members of $I(\sigma)$ such that

$(s, b_1, \ldots, b_M) \leq (r, a_1, \ldots, a_N)$. Suppose that $1 \leq i \leq M$, $1 \leq j \leq N$, and

$$D_{b_i}(s^-) \cap D_{a_j}(r^-) \neq \emptyset.$$

Let $x \in D_{b_i}(s^-) \cap D_{a_j}(r^-)$, then $|b_i - x|_p < s$ and $|a_j - x|_p < r$. Because $s \leq r$, we see that $x \in D_{b_i}(r^-) \cap D_{a_j}(r^-)$, and consequently $D_{b_i}(r^-) \cap D_{a_j}(r^-) \neq \emptyset$. Therefore $D_{b_i}(r^-) = D_{a_j}(r^-)$. Hence

$$D_{b_i}(s^-) = D_{b_i}(s^-) \cap D_{b_i}(r^-) = D_{b_i}(s^-) \cap D_{a_j}(r^-).$$

It follows that if I is the set of all $1 \leq i \leq M$ such that there exists a $1 \leq j \leq N$ for which

$$D_{b_i}(s^-) \cap D_{a_j}(r^-) \neq \emptyset,$$

then we have

$$\sigma \subseteq \left[\bigcup_{i=1}^{M} D_{b_i}(s^-)\right] \cap \left[\bigcup_{j=1}^{N} D_{a_j}(r^-)\right] = \bigcup_{i \in I} D_{b_i}(s^-).$$

Hence if we write $(b_i)_{i \in I} = (c_1, \ldots, c_K)$, then we have $(s, c_1, \ldots, c_K) \in I(\sigma)$, with

$$(s, c_1, \ldots, c_K) \leq (s, b_1, \ldots, b_M) \text{ and } (s, c_1, \ldots, c_K) \leq (r, a_1, \ldots, a_N).$$

This proves (a).

To prove (b), let $1 \leq i \leq N$ be arbitrary. Because f is Krasner analytic on $D_{a_i}(r)$, Lemma 1.5 implies that $\max\limits_{x \in D_{a_i}(r)} |f(x)|_p$ is attained on the circle $|x - a_i|_p = r$. It follows that f is bounded on U.

For the proof of (c), define $U = \bigcup\limits_{i=1}^{N} D_{a_i}(r)$. Let $f \in B_{r,a_1,\ldots,a_N}$ and let $1 \leq i \leq M$. Because $(s, b_1, \ldots, b_N) \leq (r, a_1, \ldots, a_N)$, we have $0 < s < r$ and $V \subseteq U$. Therefore there exists a $1 \leq j \leq N$ such that $b_i \in D_{a_j}(r)$, and hence $D_{b_i}(r) = D_{a_j}(r)$. Because f is Krasner analytic on $D_{a_j}(r)$, it follows from Lemma 1.4 that f is Krasner analytic on $D_{b_i}(r)$. Hence, because $s \leq r$, f is Krasner analytic on $D_{b_i}(s)$. This shows that $f|_V \in B_{s,b_1,\ldots,b_M}$. We have $V \subseteq U$, therefore it is clear that the mapping $f \mapsto f|_V$ is continuous on $B_{r,a_1,\ldots,a_N}$ in the uniform norm.

The proof of (d) follows form (a); and the proof of (e) follows from (c). To prove (f), assume that σ is compact. Let $f \in \mathcal{L}(\sigma)$. Then by Definition 1.7 there exist $a_1, \ldots, a_N$ in Ω_p and $0 < r \in |\Omega_p|_p$ such that

$$\sigma \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-), \tag{2}$$

where the $D_{a_i}(r)$ are disjoint and f is Krasner analytic on each $D_{a_i}(r)$. Then by (2) and Lemma 1.3, there exists $\emptyset \neq I \subseteq \{1, \ldots, N\}$ such that

$$D_\sigma(r) = \bigcup_{i \in I} D_{a_i}(r). \tag{3}$$

Let $b \in \Omega_p$, with $D_b(r) \subseteq D_\sigma(r)$. Then by (3), there exists $i \in I$ such that $b \in D_{a_i}(r)$, and consequently, $D_b(r) = D_{a_i}(r)$. Because f is Krasner analytic on $D_{a_i}(r)$ we see that $f : D_b(r) \to \Omega_p$ is the uniform limit of rational functions with poles outside $D_b(r)$. It follows from Lemma 1.4 that f is Krasner analytic on $D_b(r)$. This shows that $f \in B_r(\sigma)$. Because $f \in \mathcal{L}(\sigma)$ is arbitrary, we get

$$\mathcal{L}(\sigma) \subseteq \bigcup \{ B_r(\sigma) \,|\, 0 < r \in |\Omega_p|_p \}. \tag{4}$$

Now let $f \in \bigcup \{ B_r(\sigma) \,|\, 0 < r \in |\Omega_p|_p \}$, say, $f \in B_r(\sigma)$, $0 < r \in |\Omega_p|_p$. By Lemma 1.2 there exist $b_1, \ldots, b_M$ in σ and $0 < s \in |\Omega_p|_p$ such that $s < r$, the $D_{b_i}(s)$ are disjoint, and

$$D_\sigma(s) = \bigcup_{i=1}^{M} D_{b_i}(s), \quad \sigma \subseteq \bigcup_{i=1}^{M} D_{b_i}(s^-). \tag{6}$$

Let $1 \leq i \leq M$ be arbitrary. Because $b_i \in \sigma$, we have $D_{b_i}(r) \subseteq D_\sigma(r)$. By assumption, $f \in B_r(\sigma)$, therefore f is Krasner analytic on $D_{b_i}(r)$. Hence, by Definition 1.6, f can be represented by a power series on $D_{b_i}(r)$ of the form $f(x) = \sum_{k=0}^{\infty} c_k (x - b_i)^k$, where $\lim_{k \to \infty} r^k |c_k|_p = 0$. Since $s < r$, we have $D_{b_i}(s) \subseteq D_{b_i}(r)$, and hence, for $x \in D_{b_i}(s)$, we have $f(x) = \sum_{k=0}^{\infty} c_k (x - b_i)^k$, with $\lim_{k \to \infty} s^k |c_k|_p = 0$. This shows that f is Krasner analytic on $D_{b_i}(s)$. This proves that $f \in B_{s, b_1, \ldots, b_M}$. It follows that

$$\bigcup \{ B_r(\sigma) \,|\, 0 < r \in |\Omega_p|_p \} \subseteq \mathcal{L}(\sigma). \tag{7}$$

Statements (4) and (7) together imply (1). Finally, statement (g) is an easy consequence of Lemma 1.3 and Lemma 1.4. □

Definition 1.9. (The Shnirelman Integral) Let $0 < r \in |\Omega_p|_p$. Let $a \in \Omega_p$, and let f be an Ω_p-valued function whose domain contains all $x \in \Omega_p$ such that $|\, x - a\,|_p = r$. Let $\Gamma \in \Omega_p$, with $|\Gamma|_p = r$. Then the *Shnirelman integral* of f over the circle

$$\{\, x \in \Omega_p \,:\, |x - a|_p = r \,\}$$

is defined to be the following limit, provided the limit exists.

$$\int\limits_{a,\Gamma} f(x)\, dx = \lim_{\substack{n\to\infty \\ p \nmid n}} \frac{1}{n} \sum_{\xi^n = 1} f(a + \xi\Gamma).$$

Definition 1.10. Let $\emptyset \neq \sigma \subseteq \Omega_p$ be compact, let $H_0(\overline{\sigma})$ denote the set of functions $\varphi : \overline{\sigma} \to \Omega_p$ which are *Krasner analytic* on the complement $\overline{\sigma}$ of σ, i.e.,

(1) φ is the limit of rational functions whose poles are contained in σ, the limit being uniform in any set of the form

$$\overline{D}_\sigma(r) = \{\, z \in \Omega_p \mid \operatorname{dist}(z, \sigma) \geq r \,\}, \quad \sigma \subseteq D_\sigma(r^-), r > 0.$$

(2) $\lim\limits_{|z|_p \to \infty} \varphi(z) = 0.$

Lemma 1.11. *Let $\emptyset \neq \sigma \subseteq \Omega_p$ be compact. Let $0 < r_1, r_2 \in \Omega_p$, with $r_1 \leq r_2$. Let Γ_1, Γ_2 be in Ω_p, with $|\Gamma_1|_p = r_1$, $|\Gamma_2|_p = r_2$. Assume that*

$$a_1, \ldots, a_M; b_1, \ldots, b_N \in \Omega_p$$

are given, with

$$D_\sigma(r_1) = \bigcup_{i=1}^{M} D_{a_i}(r_1), \quad \sigma \subseteq \bigcup_{i=1}^{M} D_{a_i}(r_1{}^-);$$

$$D_\sigma(r_2) = \bigcup_{i=1}^{N} D_{b_i}(r_2), \quad \sigma \subseteq \bigcup_{i=1}^{N} D_{b_i}(r_2{}^-),$$

where the $D_{a_i}(r_1)$ and the $D_{b_i}(r_2)$ are disjoint. Then for $\varphi \in H_0(\overline{\sigma})$ and $f \in B_{r_2}(\sigma)$, we have $f \in B_{r_1}(\sigma)$, and the following sums exist and are equal.

$$\sum_{i=1}^{M} \int\limits_{a_i,\Gamma_1} f(x)(x-a_i)\varphi(x)\,dx = \sum_{i=1}^{N} \int\limits_{b_i,\Gamma_2} f(x)(x-b_i)\varphi(x)\,dx.$$

Proof. See [7], Lemma 8, page 138. □

Lemma 1.12. *Let $\emptyset \neq \sigma \subseteq \Omega_p$ be compact. Let $0 < r \in |\Omega_p|_p$ and let $\Gamma \in \Omega_p$, with $|\Gamma|_p = r$. Assume that*

$$D_\sigma(r) = \bigcup_{i=1}^{M} D_{b_i}(r); \tag{1}$$

$$\sigma \subseteq \bigcup_{i=1}^{M} D_{b_i}(r^-),$$

where $b_1, \ldots, b_M \in \sigma$ and the $D_{b_i}(r)$ are disjoint. Let $\varphi \in H_0(\overline{\sigma})$. Suppose that $a_1, \ldots, a_N \in \Omega_p$, with

$$\sigma \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-), \tag{2}$$

where the $D_{a_i}(r)$ are disjoint. Finally, assume that f is Krasner analytic on each of the $D_{a_i}(r)$. Then $f \in B_r(\sigma)$ and

$$\sum_{i=1}^{M} \int\limits_{b_i,\Gamma} f(x)(x-b_i)\varphi(x)\,dx = \sum_{i=1}^{N} \int\limits_{a_i,\Gamma} f(x)(x-a_i)\varphi(x)\,dx.$$

Proof. By (2) and Lemma 1.3, there exists $\emptyset \neq I \subseteq \{1, \ldots, N\}$ such that

$$D_\sigma(r^-) = \bigcup_{i\in I} D_{a_i}(r^-), \quad D_\sigma(r) = \bigcup_{i\in I} D_{a_i}(r). \tag{3}$$

To show that $f \in B_r(\sigma)$, let $b \in \Omega_p$, with $D_b(r) \subseteq D_\sigma(r)$. Then (3) implies that there exists $i \in I$ such that $b \in D_{a_i}(r)$, and hence $D_b(r) = D_{a_i}(r)$; because f is Krasner analytic on $D_{a_i}(r)$, it follows from Lemma 1.4 that f is Krasner analytic on $D_b(r)$. This proves that $f \in B_r(\sigma)$. Now, by (1), (3) and Lemma 1.11 we have

$$\sum_{i=1}^{M} \int\limits_{b_i,\Gamma} f(x)(x-b_i)\varphi(x)\,dx = \sum_{i\in I} \int\limits_{a_i,\Gamma} f(x)(x-a_i)\varphi(x)\,dx. \tag{4}$$

Now let $1 \le j \le N$, with $j \notin I$. We claim that $D_{a_j}(r) \subseteq \overline{\sigma}$. For suppose that $b \in \sigma$, with $b \in D_{a_j}(r)$. Then because $\sigma \subseteq D_\sigma(r)$, (3) implies that $b \in D_{a_i}(r)$ for some $i \in I$. We have $i \neq j$ and $b \in D_{a_i}(r) \cap D_{a_j}(r)$. This contradicts the assumption that $D_{a_i}(r)$ and $D_{a_j}(r)$ are disjoint when $i \neq j$. This proves the claim. By assumption, $\varphi \in H_0(\overline{\sigma})$, and since $f \in B_r(\sigma)$, we see that $f(x)(x - a_j)\varphi(x)$ is Krasner analytic on $D_{a_j}(r)$ for each $1 \le j \le N$ such that $j \notin I$. It follows from the p-adic Cauchy integral formula (see [7], Lemma 4, page 131) that for $j \notin I$, $\int\limits_{a_j,\Gamma} f(x)(x - a_j)\varphi(x)\,dx = 0$. Therefore, (4) gives

$$\begin{aligned}\sum_{i=1}^{M} \int\limits_{b_i,\Gamma} f(x)(x - b_i)\varphi(x)\,dx &= \sum_{i\in I} \int\limits_{a_i,\Gamma} f(x)(x - a_i)\varphi(x)\,dx \\ &= \sum_{i\in I} \int\limits_{a_i,\Gamma} f(x)(x - a_i)\varphi(x)\,dx + \\ &\quad \sum_{i\notin I} \int\limits_{a_i,\Gamma} f(x)(x - a_i)\varphi(x)\,dx \\ &= \sum_{i=1}^{N} \int\limits_{a_i,\Gamma} f(x)(x - a_i)\varphi(x)\,dx.\end{aligned}$$

This completes the proof of the lemma. □

Definition 1.13. A *p-adic Banach space* over Ω_p is a vector space $\mathcal{X}$ over Ω_p together with a norm $\|\cdot\|_p$ from $\mathcal{X}$ to the nonnegative real numbers such that for all $x, y \in \mathcal{X}$: (a) $\|x\|_p = 0$ if and only if $x = 0$; (b) $\|x+y\|_p \le \max\{\,\|x\|_p, \|y\|_p\,\}$; (c) $\|ax\|_p = |a|_p\|x\|_p$; (d) $\mathcal{X}$ is complete under $\|\cdot\|_p$. We shall assume that $\|\mathcal{X}\|_p = |\Omega_p|_p$, i.e., for every $x \neq 0$ in $\mathcal{X}$ there exists $a \in \Omega_p$ such that $\|ax\|_p = 1$. The *dual* $\mathcal{X}^*$ of a p-adic Banach space over Ω_p is defined in the usual way. If $\mathcal{X}$ and $\mathcal{Y}$ are p-adic Banach spaces over Ω_p, we define $\mathcal{X} \simeq \mathcal{Y}$ to mean that $\mathcal{X}$ and $\mathcal{Y}$ are isometrically isomorphic as p-adic Banach spaces over Ω_p. A *p-adic Banach algebra* over Ω_p is a p-adic Banach space $\mathcal{A}$ over Ω_p such that for $x, y \in \mathcal{A}$, we have $\|xy\|_p \le \|x\|_p\|y\|_p$. We shall assume that $\mathcal{A}$ has a unit. For $x \in \mathcal{A}$, the *spectrum* σ_x of x and the *resolvent set* $\rho(x)$ of x have the usual meaning; $\rho(x)$ is open and hence σ_x is closed (see [8], Theorem 4, page 114). The *resolvent* of x is defined by $R(z; x) = (z - x)^{-1}$, $z \in \rho(x)$. If $\mathcal{X}$ and $\mathcal{Y}$ are p-adic Banach spaces over Ω_p, then $B(\mathcal{X}, \mathcal{Y})$ is the vector space of Ω_p-linear continuous maps form $\mathcal{X}$ to $\mathcal{Y}$. $B(\mathcal{X}, \mathcal{Y})$ is a p-adic Banach space under the usual operator norm, and $B(\mathcal{X}) = B(\mathcal{X}, \mathcal{X})$ is a p-adic Banach algebra under this operator norm.

Definition 1.14. Let $\mathcal{X}$ be a p-adic Banach space over Ω_p. An operator $A \in B(\mathcal{X})$, with compact spectrum $\sigma_A \neq \emptyset$, is called *analytic* if the resolvent $R(z; A)$ is *Krasner analytic* on $\overline{\sigma}_A$, in the sense that for all $x \in \mathcal{X}$ and $h \in \mathcal{X}^*$, the function $z \mapsto h(R(z; A)x)$ is in $H_0(\overline{\sigma}_A)$.

Lemma 1.15. *Let J be any nonempty indexing set, and define $\Omega_p(J)$ to be the set of all "sequences" $c = (c_j)_{j \in J}$ in Ω_p such that for every $\epsilon > 0$ only finitely many $|c_j|_p$ are $> \epsilon$. Define $||c||_p = \max\limits_j |c_j|_p$. Then $\Omega_p(J)$ is a p-adic Banach space over Ω_p. The notation $\Omega_p(J)$ is used in [7], page 143. However the usual notation for $\Omega_p(J)$ is $c_0(J; \Omega_p)$, which is used in [12], page 185.*

Proof. See [12], Corollary 1, page 185. □

Lemma 1.16. *Let $\mathcal{X} \simeq \Omega_p(J)$, with $J \neq \emptyset$. Let $\emptyset \neq \sigma \subseteq \Omega_p$ be compact. Let $F : \overline{\sigma} \to B(\mathcal{X})$ be an analytic operator-valued function, i.e., for all $y \in \mathcal{X}, h \in \mathcal{X}^*$ the Ω_p-valued function*

$$F_{h,y}(x) = h(F(x)y), \quad x \in \overline{\sigma}$$

belongs to $H_0(\overline{\sigma})$. Let $a \in \Omega_p$ and $0 < r \in |\Omega_p|_p$. Let $\Gamma \in \Omega_p$, with $|\Gamma|_p = r$. Assume that there is no $b \in \sigma$ such that $|b - a|_p = r$. Finally, let $f : D_a(r) \to \Omega_p$ be Krasner analytic on $D_a(r)$. Define

$$S_n = \frac{1}{n} \sum_{\xi^n = 1} f(a + \xi\Gamma) F(a + \xi\Gamma), \quad n = 1, 2, \ldots.$$

Then the limit

$$\lim_{\substack{n \to \infty \\ p \nmid n}} S_n$$

exists as an operator in $B(\mathcal{X})$. We define $\int\limits_{a,\Gamma} f(x)F(x)\,dx$ to be this operator. Moreover, for any $y \in \mathcal{X}$ and $h \in \mathcal{X}^*$, we have

$$h\Bigg(\bigg(\int\limits_{a,\Gamma} f(x)F(x)\,dx\bigg)(y)\Bigg) = \int\limits_{a,\Gamma} f(x)F_{h,y}(x)\,dx.$$

In particular, assume that an analytic operator $A \in B(\mathcal{X})$ has compact spectrum $\sigma_A \neq \emptyset$. Then by definition, the resolvent $R(x; A) = (x - A)^{-1}$ is Krasner analytic on $\overline{\sigma}_A$ and hence the integral

$$\int\limits_{a,\Gamma} f(x)(x - a)R(x; A)\,dx$$

exists. Moreover, for all $y \in \mathcal{X}$ and all $h \in \mathcal{X}^*$, if $R_{h,y}(x;A) = h(R(x;A)y)$, then

$$h\Big(\Big(\int_{a,\Gamma} f(x)(x-a)R(x;A)\,dx\Big)(y)\Big) = \int_{a,\Gamma} f(x)(x-a)R_{h,y}(x;A)\,dx.$$

Proof. See [7], Lemma 3, page 149. □

Definition 1.17. Let $\mathcal{A}$ be a p-adic Banach algebra over Ω_p. Let $A \in \mathcal{A}$ have spectrum $\sigma_A \neq \emptyset$. Define $\mathcal{F}(A) = \mathcal{L}(\sigma_A)$.

We now state a "classical" p-adic spectral of M. M. Vishik (see [7], page 149). □

Spectral 1.18. Theorem I (Vishik). *Let $\mathcal{X} \simeq \Omega_p(J)$, where J is a nonempty indexing set. Let $A \in B(\mathcal{X})$ have compact spectrum $\sigma_A \neq \emptyset$, and assume that A is analytic. Let $f \in L(\sigma_A)$. Let $0 < r \in |\Omega_p|_p$ be such that $f \in B_r(\sigma_A)$, and assume that*

$$D_{\sigma_A}(r) = \bigcup_{i=1}^{N} D_{a_i}(r), \quad \sigma_A \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-),$$

where $a_1, \ldots, a_N \in \sigma_A$, with the $D_{a_i}(r)$ disjoint. Let $\Gamma \in \Omega_p$, with $|\Gamma|_p = r$. Set

$$f(A) = \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x;A)\,dx.$$

Let $g \in \mathcal{F}(A)$, and let $\alpha, \beta \in \Omega_p$. Then:

(1) *$(\alpha f + \beta g)(A) = \alpha f(A) + \beta g(A)$;*

(2) *$(f \cdot g)(A) = f(A)g(A)$;*

(3) *The mapping $f \mapsto f(A)$ is continuous on $B_r(\sigma_A)$.*

Proof. See [7], Spectral Theorem, page 149. □

Definition 1.19. Let $\mathcal{A}$ be a p-adic Banach algebra and let $\mathcal{X}$ be a p-adic Banach space. Let $A \in \mathcal{A}$ have compact spectrum σ_A. Then we say that A is $\mathcal{X}$-*analytic* iff there exists a unital Ω_p-monomorphism $\theta : \mathcal{A} \to B(\mathcal{X})$ such that $\theta(\mathcal{A})$ is a p-adic Banach subalgebra of $B(\mathcal{X})$, $\theta^{-1} : \theta(\mathcal{A}) \to \mathcal{A}$ is bounded and $\theta(A)$ is analytic in $B(\mathcal{X})$. Such a monomorphism $\theta : \mathcal{A} \to B(\mathcal{X})$ is said to be an *embedding* of $\mathcal{A}$ into $B(\mathcal{X})$. Let $B \in \mathcal{A}$. Then B is $\mathcal{A}$-*analytic* iff the following two conditions hold:

(a) For every $r > 0$, $\|R(x;B)\|_p$ is bounded on the complement $\overline{D}_{\sigma_B}(r)$ of $D_{\sigma_B}(r^-)$.

(b) There exists a sequence (B_k) in $\mathcal{A}$ and a sequence $(\mathcal{X}_k)$ of p-adic Banach spaces $\mathcal{X}_k$ of the form $\mathcal{X}_k \simeq \Omega_p(J_k)$, with $J_k \neq \emptyset$, such that $\sigma_{B_k} \neq \emptyset$ is compact, B_k is $\mathcal{X}_k$-analytic and $\lim_{k\to\infty} \|B - B_k\|_p = 0$.

Lemma 1.20. *Let $\mathcal{A}$ and $\mathcal{X}$ be as in Definition 1.19, with $\mathcal{X} \simeq \Omega_p(J)$, where J is a nonempty indexing set. Let $A \in \mathcal{A}$ have compact spectrum $\sigma_A \neq \emptyset$, and assume that A is $\mathcal{X}$-analytic. Let $\theta : \mathcal{A} \to B(\mathcal{X})$ be an embedding of $\mathcal{A}$ into $B(\mathcal{X})$, with $\theta(A)$ analytic. Let $0 < r \in |\Omega_p|_p$ and $\Gamma \in \Omega_p$, with $|\Gamma|_p = r$. Assume that $f \in B_r(\sigma_A)$ and*

$$\sigma_A \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-), \tag{1}$$

where $a_1, \ldots, a_N \in \sigma_A$ and the $D_{a_i}(r)$ are disjoint. Then we have

$$\begin{aligned}\theta^{-1}[f(\theta(A))] &= \lim_{\substack{n\to\infty \\ p\nmid n}} \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i - a_i)R(x_\xi^i; A), \\ &= \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x;A)\,dx, \end{aligned} \tag{2}$$

where for $1 \le i \le N$ and $\xi \in \Omega_p$, we define $x_\xi^i = a_i + \xi\Gamma$. Moreover, the limit in (1) is independent of the $\Gamma, r, a_1, \ldots, a_N$. Hence we may define $f(A)$ by $f(A) = \theta^{-1}[f(\theta(A))]$, and this definition will not depend on θ.

Proof. We have $\sigma_{\theta(A)} \subseteq \sigma_A$, hence (1) implies that

$$\sigma_{\theta(A)} \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-). \tag{3}$$

By Lemma 1.3, (3) implies that there exist $b_1, \ldots, b_M$ in $\sigma_{\theta(A)}$ and $\emptyset \neq I \subseteq \{1, \ldots, N\}$ such that the $D_{b_i}(r)$ are disjoint and

$$D_{\sigma_{\theta(A)}}(r^-) = \bigcup_{i=1}^{M} D_{b_i}(r^-) = \bigcup_{i\in I} D_{a_i}(r^-); \tag{4}$$

$$D_{\sigma_{\theta(A)}}(r) = \bigcup_{i=1}^{M} D_{b_i}(r) = \bigcup_{i\in I} D_{a_i}(r).$$

Let $y \in \mathcal{X}$ and $h \in \mathcal{X}^*$ be arbitrary. Then by assumption, the function $\varphi_{h,y}(x) = h(R(x;\theta(A))y)$ is in $H_0(\overline{\sigma}_{\theta(A)})$. Because each $a_i \in \sigma_A$ and $f \in B_r(\sigma_A)$, we see that f is Krasner analytic on each $D_{a_i}(r)$. Hence, by Lemma 1.12, it follows from (3) and (4) that $f \in B_r(\sigma_{\theta(A)})$ and

$$\sum_{i=1}^{M} \int_{b_i,\Gamma} f(x)(x-b_i)\varphi_{h,y}(x)\,dx = \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)\varphi_{h,y}(x)\,dx. \tag{5}$$

Hence, by Lemma 1.16, (5) implies that

$$h\Bigg(\bigg(\sum_{i=1}^{M} \int_{b_i,\Gamma_1} f(x)(x-b_i)R(x;\theta(A))\,dx - \sum_{i=1}^{N} \int_{a_i,\Gamma_2} f(x)(x-a_i)R(x;\theta(A))\,dx\bigg)(y)\Bigg) = 0.$$

Because $h \in \mathcal{X}^*$ is arbitrary, we see that

$$\bigg(\sum_{i=1}^{M} \int_{b_i,\Gamma} f(x)(x-b_i)R(x;\theta(A))\,dx - \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x;\theta(A))\,dx\bigg)(y) = 0. \tag{6}$$

Here we have used the fact that if $z \in \mathcal{X}$ is such that $k(z) = 0$ for all $k \in \mathcal{X}^*$, then $z = 0$ (see [7], Lemma 2, page 144). Finally, because $y \in \mathcal{X}$ is arbitrary, (6) implies that

$$\sum_{i=1}^{M} \int_{b_i,\Gamma} f(x)(x-b_i)R(x;\theta(A))\,dx = \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x;\theta(A))\,dx. \tag{7}$$

By (4) and Theorem 1.19, we have

$$f(\theta(A)) = \sum_{i=1}^{M} \int_{b_i,\Gamma} f(x)(x-b_i)R(x;\theta(A))\,dx,$$

consequently, (7) gives

$$f(\theta(A)) = \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x;\theta(A))\,dx. \tag{8}$$

Now let $1 \le i \le N$ and $\xi \in \Omega_p$, with $|\xi|_p = 1$. Then $x_\xi^i = a_i + \xi\Gamma \in \overline{\sigma}_A$, and therefore $R(x_\xi^i;A) \in \mathcal{A}$, form which it follows that

$$R(x_\xi^i;\theta(A)) = \theta(R(x_\xi^i;A)) \in \theta(\mathcal{A}). \tag{9}$$

Because $\theta(\mathcal{A})$ is a p-adic Banach subalgebra of $B(\mathcal{X})$ and

$$f(\theta(A)) = \lim_{\substack{n\to\infty \\ p\nmid n}} \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i - a_i)R(x_\xi^i;\theta(A)),$$

(9) implies that $f(\theta(A)) \in \theta(\mathcal{A})$. Hence we may define $f(A) = \theta^{-1}[f(\theta(A))]$. Because θ^{-1} is bounded, we have

$$\begin{aligned}
\theta^{-1}[f(\theta(A))] &= \theta^{-1}\Big[\sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x;\theta(A))\,dx\Big] \\
&= \theta^{-1}\Big[\lim_{\substack{n\to\infty \\ p\nmid n}} \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i - a_i)R(x_\xi^i;\theta(A))\Big] \\
&= \lim_{\substack{n\to\infty \\ p\nmid n}} \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i - a_i)\theta^{-1}[R(x_\xi^i;\theta(A))] \\
&= \lim_{\substack{n\to\infty \\ p\nmid n}} \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i - a_i)R(x_\xi^i;A).
\end{aligned}$$

Finally, by Theorem 1.19, the representation

$$f(\theta(A)) = \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x;\theta(A))\,dx$$

does not depend on $\Gamma, r, a_1, \dots, a_N$. □

Let $\mathcal{A}$ be a p-adic Banach algebra. The following version of Theorem 1.18 applies directly to elements $A \in \mathcal{A}$ that are $\mathcal{X}$-analytic.

Spectral 1.21. Theorem II (Vishik). Let $\mathcal{X} \simeq \Omega_p(J)$, where J is a nonempty indexing set. Let $\mathcal{A}$ be a p-adic Banach algebra, and let $A \in \mathcal{A}$ have compact spectrum $\sigma_A \neq \emptyset$, and assume that A is $\mathcal{X}$-analytic. Let $f \in L(\sigma_A)$. Let $0 < r \in |\Omega_p|_p$ be such that $f \in B_r(\sigma_A)$, and assume that

$$D_{\sigma_A}(r) = \bigcup_{i=1}^{N} D_{a_i}(r), \quad \sigma_A \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-),$$

where $a_1, \ldots, a_N \in \sigma_A$, with the $D_{a_i}(r)$ disjoint. Let $\Gamma \in \Omega_p$, with $|\Gamma|_p = r$. Set

$$f(A) = \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x - a_i)R(x; A)\, dx.$$

Let $g \in \mathcal{F}(A)$, and let $\alpha, \beta \in \Omega_p$. Then:

(1) $(\alpha f + \beta g)(A) = \alpha f(A) + \beta g(A)$;

(2) $(f \cdot g)(A) = f(A)g(A)$;

(3) The mapping $f \mapsto f(A)$ is continuous on $B_r(\sigma_A)$.

Proof. Let $\theta : \mathcal{A} \to B(\mathcal{X})$ be an embedding of $\mathcal{A}$ into $B(\mathcal{X})$, where $\mathcal{X} \simeq \Omega_p(J)$, $J \neq \emptyset$. Assume that $\theta(A)$ is $\mathcal{X}$-analytic. Then Theorem 1.18 applies to $\theta(A)$, and hence Theorem 1.21 follows form Lemma 1.20 and the fact that for $f \in B_{\sigma_A}(r)$, $f(A) = \theta^{-1}[f(\theta(A))]$. □

2. Extended p-Adic Spectral Theory

Theorem 2.1. (Perturbation Theory) *Let $\mathcal{A}$ be a p-adic Banach algebra. Let $A \in \mathcal{A}$ have spectrum σ_A. Then the following statements hold.*

(a)*If A is invertible and $B \in \mathcal{A}$ with $\|A - B\|_p < \|A^{-1}\|_p^{-1}$, then B is invertible and*

$$B^{-1} = A^{-1} \sum_{n=0}^{\infty} [(A - B)A^{-1}]^n; \tag{1}$$

$$\|B^{-1} - A^{-1}\|_p \leq \frac{\|A^{-1}\|_p^2 \|A - B\|_p}{1 - \|A - B\|_p \|A^{-1}\|_p}. \tag{2}$$

(b)*Let* $\sigma_A \subseteq G \subseteq \Omega_p$. *Let* $\epsilon > 0$ *be a positive real number.*

Assume that there exists a constant $N_\epsilon > 0$ *such that*

$$\|R(x;A)\|_p \leq N_\epsilon \text{ for all } x \in \overline{G}.$$

Then there exists a $\delta > 0$ *such that if* $B \in \mathcal{A}$ *and* $||A - B||_p < \delta$, *then*

$$\sigma_B \subseteq G \tag{3}$$

and

$$||R(x;A) - R(x;B)||_p < \epsilon \text{ for all } x \in \overline{G}. \tag{4}$$

(c)*Suppose that there exists an* $r > 0$ *such that* $\|R(x;A)\|_p$ *is bounded on* $\overline{D}_{\sigma_A}(r)$. *Moreover, suppose that there exists a sequence* (A_k) *in* $\mathcal{A}$ *such that each* $\sigma_{A_k} \neq \emptyset$ *and* $\lim_{k\to\infty} \|A - A_k\|_p = 0$. *Then* $\sigma_A \neq \emptyset$.

(d)*Let* $A \in \mathcal{A}$ *be* $\mathcal{A}$*-analytic. Then for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that if* $B \in \mathcal{A}$ *and* $||A - B||_p < \delta$, *then*

$$\sigma_B \subseteq D_{\sigma_A}(\epsilon^-) \tag{5}$$

and

$$||R(x;A) - R(x;B)||_p < \epsilon \text{ for all } x \in \overline{D}_{\sigma_A}(\epsilon). \tag{6}$$

Furthermore, $\sigma_A \neq \emptyset$.

Proof. By Theorem 4, page 108 of [8], if $X \in \mathcal{A}$ is such that the series $\sum_{n=0}^{\infty} X^n$ converges, then $1 - X$ is invertible, with

$$(1 - X)^{-1} = \sum_{n=0}^{\infty} X^n. \tag{$*$}$$

To prove (a), suppose that $A \in \mathcal{A}$ is invertible. Let $B \in \mathcal{A}$, with $\|A - B\|_p < \|A^{-1}\|_p^{-1}$. Then

$$\|1 - BA^{-1}\|_p = \|(A - B)A^{-1}\|_p \leq \|A - B\|_p \|A^{-1}\|_p < 1,$$

and hence the series $\sum_{n=0}^{\infty} (1 - BA^{-1})^n$ is convergent. Therefore $(*)$ implies that

$$1 - (1 - BA^{-1}) = BA^{-1}$$

is invertible, with

$$(BA^{-1})^{-1} = \sum_{n=0}^{\infty}(1 - BA^{-1})^n = \sum_{n=0}^{\infty}[(A - B)A^{-1}]^n.$$

But then because BA^{-1} is invertible, so is B, and we get

$$AB^{-1} = (BA^{-1})^{-1} = \sum_{n=0}^{\infty}[(A - B)A^{-1}]^n,$$

that is,

$$B^{-1} = A^{-1}\sum_{n=0}^{\infty}[(A - B)A^{-1}]^n.$$

This proves (1). Now, (1) implies that

$$\begin{aligned}
\|B^{-1} - A^{-1}\|_p &= \|A^{-1}\sum_{n=0}^{\infty}[(A - B)A^{-1}]^n - A^{-1}\|_p \\
&= \|A^{-1}\Big(\sum_{n=0}^{\infty}[(A - B)A^{-1}]^n - 1\Big)\|_p \\
&= \|A^{-1}\sum_{n=1}^{\infty}[(A - B)A^{-1}]^n\|_p \\
&\le \|A^{-1}\|_p\|\sum_{n=1}^{\infty}[(A - B)A^{-1}]^n\|_p \\
&\le \|A^{-1}\|_p\sum_{n=1}^{\infty}\|(A - B)\|_p\|A^{-1}\|_p^n \\
&= \|A^{-1}\|_p\left[\sum_{n=0}^{\infty}\|(A - B)\|_p\|A^{-1}\|_p^n - 1\right] \\
&= \|A^{-1}\|_p\left[\frac{1}{1 - \|A^{-1}\|_p\|A - B\|_p} - 1\right] \\
&= \frac{\|A^{-1}\|_p^2\|A - B\|_p}{1 - \|A^{-1}\|_p\|A - B\|_p}.
\end{aligned}$$

This proves (2). To prove (b), let $\epsilon > 0$ be arbitrary, and assume that there is a constant $N_\epsilon > 0$ such that $\|R(x; A)\|_p \le N_\epsilon$ for all $x \in \overline{G}$. Define

$$\delta = \min\left\{\frac{1}{N_\epsilon}, \frac{\epsilon}{N_\epsilon^2 + \epsilon N_\epsilon}\right\}.$$

Now let $B \in \mathcal{A}$ with $\|A - B\|_p < \delta$. To prove (3) it suffices to show that

$$\overline{G} \subseteq \rho(B). \tag{7}$$

To this end, let $x \in \overline{G}$. Then $\|R(x;A)\|_p \leq N_\epsilon$, and hence

$$N_\epsilon^{-1} \leq \|R(x;A)\|_p^{-1} = \|(x-A)^{-1}\|_p^{-1}.$$

Therefore,

$$\|(x-A) - (x-B)\|_p = \|A - B\|_p < \frac{1}{N_\epsilon} \leq \|(x-A)^{-1}\|_p^{-1}. \tag{8}$$

It follows from (a) that $x - B$ is invertible, i.e., $x \in \rho(B)$. This proves (7). To prove (4), define

$$\mu = \frac{\epsilon}{N_\epsilon^2 + \epsilon N_\epsilon}.$$

A short calculation shows that

$$\frac{N_\epsilon^2 \mu}{1 - N_\epsilon \mu} = \epsilon.$$

Now let $x \in \overline{G}$. By assumption, $\|A - B\|_p < \mu$. Because (8) holds, (a) implies that

$$\begin{aligned}
\|R(x;B) - R(x;A)\|_p &= \|(x-B)^{-1} - (x-A)^{-1}\|_p \\
&\leq \frac{\|(x-A)^{-1}\|_p^2 \|(x-A) - (x-B)\|_p}{1 - \|(x-A)^{-1}\|_p \|(x-A) - (x-B)\|_p} \\
&= \frac{\|R(x;A)\|_p^2 \|B - A\|_p}{1 - \|R(x;A)\|_p \|B - A\|_p} \\
&\leq \frac{N_\epsilon^2 \|A - B\|_p}{1 - N_\epsilon \|A - B\|_p} \\
&< \frac{N_\epsilon^2 \mu}{1 - N_\epsilon \mu} = \epsilon.
\end{aligned}$$

This proves (4), and hence the proof of (b) is complete. To prove (c), let $r > 0$ and suppose that $\|R(x;A)\|_p$ is bounded on $\overline{D}_{\sigma_A}(r)$ and suppose that there exists a sequence (A_k) in $\mathcal{A}$ such that each $\sigma_{A_k} \neq \emptyset$ and $\lim_{k\to\infty} \|A - A_k\|_p = 0$. Assume that $\sigma_A = \emptyset$. Then $D_{\sigma_A}(r^-) = D_\emptyset(r^-) = \emptyset$ (see Definition 1.1), consequently, $\overline{D}_{\sigma_A}(r) = \Omega_p$, from which it follows that $\|R(x;A)\|_p$ is bounded on Ω_p. Therefore there exists an $N_r > 0$ such that

$$\|R(x;A)\|_p \leq N_r \text{ for all } x \in \Omega_p. \tag{9}$$

Now let k be so large that $\|A - A_k\|_p < N_r^{-1}$. Let $x \in \Omega_p = \rho(A)$ be arbitrary. Then (9) implies that

$$\|(x - A_k) - (x - A)\|_p = \|A - A_k\|_p < N_r^{-1} \leq \|R(x; A)\|_p^{-1} = \|(x - A)^{-1}\|_p^{-1}.$$

It then follows from (a) that $x - A_k$ is invertible, i.e., $x \in \rho(A_k)$. Because $x \in \Omega_p$ is arbitrary, we see that $\rho(A_k) = \Omega_p$, i.e., $\sigma_{A_k} = \emptyset$. But this contradicts the assumption that $\sigma_{A_k} \neq \emptyset$. We conclude that $\sigma_A \neq \emptyset$. This proves (c). Finally, to prove (d), let $A \in \mathcal{A}$ be $\mathcal{A}$-analytic. Then by Definition 1.19 the following two conditions hold.

(e)For every $r > 0$, $\|R(x; A)\|_p$ is bounded on $\overline{D}_{\sigma_A}(r)$.

(f)There exists a sequence (A_k) in $\mathcal{A}$ and a sequence $(\mathcal{X}_k)$ of p-adic Banach spaces $\mathcal{X}_k$ of the form $\mathcal{X}_k \simeq \Omega_p(J_k)$, with $J_k \neq \emptyset$, such that $\sigma_{A_k} \neq \emptyset$ is compact, A_k is $\mathcal{X}_k$-analytic and $\lim_{k\to\infty} \|A - A_k\|_p = 0$.

Now let $\epsilon > 0$ be arbitrary. In (b), set $G = D_{\sigma_A}(r^-)$, then (e) implies that the A and ϵ satisfy the hypothesis of (b), form which we conclude that there exists a $\delta > 0$ such that if $B \in \mathcal{A}$ and $||A - B||_p < \delta$, then (5) and (6) above hold. Finally, (e) and (f) together imply that A satisfies the hypothesis of (c), and hence $\sigma_A \neq \emptyset$. This completes the proof of the theorem. □

Lemma 2.2. *Let $\mathcal{A}$ be a p-adic Banach algebra, and let $A \in \mathcal{A}$ have compact spectrum $\sigma_A \neq \emptyset$. Let $\mathcal{X}$ be a p-adic Banach space such that $\mathcal{X} \simeq \Omega_p(J)$, with $J \neq \emptyset$, and suppose that A be $\mathcal{X}$-analytic. Let $0 < r_1 \leq r_2$, with $r_1, r_2 \in |\Omega_p|_p$. Let Γ_1, Γ_2 be in Ω_p, with $|\Gamma_1|_p = r_1$, $|\Gamma_2|_p = r_2$. Assume that*

$$D_{\sigma_A}(r_1) = \bigcup_{i=1}^{M} D_{a_i}(r_1), \quad \sigma_A \subseteq \bigcup_{i=1}^{M} D_{a_i}({r_1}^-); (1)$$

$$D_{\sigma_A}(r_2) = \bigcup_{i=1}^{N} D_{b_i}(r_2), \quad \sigma_A \subseteq \bigcup_{i=1}^{N} D_{b_i}({r_2}^-), \quad (1)$$

where the $D_{a_i}(r_1), D_{b_i}(r_2)$ are disjoint. If $f \in B_{r_2}(\sigma_A)$, then $f \in B_{r_1}(\sigma_A)$, and the following sums exist and are equal.

$$\sum_{i=1}^{M} \int_{a_i,\Gamma_1} f(x)(x - a_i)R(x; A)\, dx = \sum_{i=1}^{N} \int_{b_i,\Gamma_2} f(x)(x - b_i)R(x; A)\, dx. \quad (2)$$

Proof. Let $\theta : \mathcal{A} \to B(\mathcal{X})$ be and embedding of $\mathcal{A}$ into $B(\mathcal{X})$ such that $\theta(A)$ is analytic. We have $\sigma_{\theta(A)} \subseteq \sigma_A$, hence (1) implies that

$$\sigma_{\theta(A)} \subseteq \bigcup_{i=1}^{M} D_{a_i}(r_1{}^-) \text{ and } \sigma_{\theta(A)} \subseteq \bigcup_{i=1}^{N} D_{b_i}(r_2{}^-). \tag{3}$$

By Lemma 1.3, (3) implies that there exist $c_1, \ldots, c_K \in \sigma_{\theta(A)}$ and $\emptyset \neq I \subseteq \{1, \ldots, M\}$ such that the $D_{c_i}(r)$ are disjoint and

$$D_{\sigma_{\theta(A)}}(r_1{}^-) = \bigcup_{i=1}^{K} D_{c_i}(r_1{}^-) = \bigcup_{i \in I} D_{a_i}(r_1{}^-)\,; \tag{4}$$
$$D_{\sigma_{\theta(A)}}(r_1) = \bigcup_{i=1}^{K} D_{c_i}(r_1) = \bigcup_{i \in I} D_{a_i}(r_1).$$

Likewise, by Lemma 1.3, (3) implies that there exist $d_1, \ldots, d_L \in \sigma_{\theta(A)}$ and $\emptyset \neq J \subseteq \{1, \ldots, N\}$ such that the $D_{d_i}(r)$ are disjoint and

$$D_{\sigma_{\theta(A)}}(r_2{}^-) = \bigcup_{i=1}^{L} D_{d_i}(r_2{}^-) = \bigcup_{i \in J} D_{b_i}(r_2{}^-)$$
$$D_{\sigma_{\theta(A)}}(r_2) = \bigcup_{i=1}^{L} D_{d_i}(r_2) = \bigcup_{i \in J} D_{b_i}(r_2). \tag{5}$$

Now let $f \in B_{r_2}(\sigma_A)$. Let $a \in \Omega_p$, with $D_a(r) \subseteq D_{\sigma_{\theta(A)}}(r_2)$. Then (1) and (5) imply that

$$D_a(r) \subseteq D_{\sigma_{\theta(A)}}(r_2) = \bigcup_{i \in J} D_{b_i}(r_2) \subseteq D_{\sigma_A}(r_2).$$

It follows that f is Krasner analytic on $D_a(r)$, and consequently, $f \in B_{r_2}(\sigma_{\theta(A)})$. Because $B_{r_2}(\sigma_{\theta(A)}) \subseteq B_{r_1}(\sigma_{\theta(A)})$, we see that $f \in B_{r_1}(\sigma_{\theta(A)})$. Let $y \in \mathcal{X}$ and $h \in \mathcal{X}^*$. Then the function $\varphi_{h,y}(x) = h(R(x;\theta(A))y)$ is in $H_0(\overline{\sigma}_{\theta(A)})$. By Lemma 1.11, (4) and (5) imply that

$$\sum_{i=1}^{K} \int_{c_i,\Gamma_1} f(x)(x - c_i)\varphi_{h,y}(x)\,dx = \sum_{i=1}^{L} \int_{d_i,\Gamma_2} f(x)(x - d_i)\varphi_{h,y}(x)\,dx. \tag{6}$$

Hence, by Lemma 1.16, (6) implies that

$$h\Big(\Big(\sum_{i=1}^{K}\int_{c_i,\Gamma_1} f(x)(x-c_i)R(x;\theta(A))\,dx - \sum_{i=1}^{L}\int_{d_i,\Gamma_2} f(x)(x-d_i)R(x;\theta(A))\,dx\Big)(y)\Big) = 0.$$

Because $h \in \mathcal{X}^*$ is arbitrary, we see that

$$\Big(\sum_{i=1}^{K}\int_{c_i,\Gamma} f(x)(x-c_i)R(x;\theta(A))\,dx - \sum_{i=1}^{L}\int_{d_i,\Gamma} f(x)(x-d_i)R(x;\theta(A))\,dx\Big)(y) = 0. \quad (7)$$

Here, again, we have used the fact that if $z \in \mathcal{X}$ is such that $k(z) = 0$ for all $k \in \mathcal{X}^*$, then $z = 0$ (see [7], Lemma 2, page 144). Finally, because $y \in \mathcal{X}$ is arbitrary, (7) implies that

$$\sum_{i=1}^{K}\int_{c_i,\Gamma} f(x)(x-c_i)R(x;\theta(A))\,dx = \sum_{i=1}^{L}\int_{d_i,\Gamma} f(x)(x-d_i)R(x;\theta(A))\,dx. \quad (8)$$

Similarly, we see that by Lemma 1.12 and Lemma 1.16, (3), (4), and (5) together imply that

$$\sum_{i=1}^{K}\int_{c_i,\Gamma_1} f(x)(x-c_i)R(x;\theta(A))\,dx = \sum_{i=1}^{M}\int_{a_i,\Gamma_1} f(x)(x-a_i)R(x;\theta(A))\,dx;$$

$$\sum_{i=1}^{L}\int_{d_i,\Gamma_2} f(x)(x-d_i)R(x;\theta(A))\,dx = \sum_{i=1}^{N}\int_{b_i,\Gamma_2} f(x)(x-b_i)R(x;\theta(A))\,dx. \quad (9)$$

Statements (8) and (9) then imply that

$$\sum_{i=1}^{M}\int_{a_i,\Gamma_1} f(x)(x-a_i)R(x;\theta(A))\,dx = \sum_{i=1}^{N}\int_{b_i,\Gamma_2} f(x)(x-b_i)R(x;\theta(A))\,dx \ (10).$$

Let $\xi \in \Omega_p$ be arbitrary with $|\xi|_p = 1$; for $1 \le i \le M$ define $x_\xi^i = a_i + \xi\Gamma_1$; for $1 \le i \le N$ define $y_\xi^i = b_i + \xi\Gamma_2$. Then $x_\xi^i, y_\xi^i \in \overline{\sigma}_A$, and therefore

$R(x^i_\xi; A), R(y^i_\xi; A) \in \mathcal{A}$, form which it follows that

$$R(x^i_\xi; \theta(A)) = \theta(R(x^i_\xi; A)) \in \theta(\mathcal{A}); \qquad (11)$$
$$R(y^i_\xi; \theta(A)) = \theta(R(y^i_\xi; A)) \in \theta(\mathcal{A}).$$

We have

$$\sum_{i=1}^{M} \int\limits_{a_i,\Gamma_1} f(x)(x-a_i)R(x;\theta(A))dx = \lim_{\substack{n\to\infty \\ p \nmid n}} \sum_{i=1}^{M} \frac{1}{n} \sum_{\xi^n=1} f(x^i_\xi)(x^i_\xi - a_i)R(x^i_\xi; \theta(A));$$
$$(12)$$
$$\sum_{i=1}^{N} \int\limits_{b_i,\Gamma_2} f(x)(x-b_i)R(x;\theta(A))dx = \lim_{\substack{n\to\infty \\ p \nmid n}} \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(y^i_\xi)(y^i_\xi - b_i)R(y^i_\xi; \theta(A)).$$

Because $\theta(\mathcal{A})$ is a p-adic Banach subalgebra of $B(\mathcal{X})$ and $\theta^{-1} : \theta(\mathcal{A}) \to \mathcal{A}$ is bounded, (11) and (12) imply that the following sums exist and are equal.

$$\sum_{i=1}^{M} \int\limits_{a_i,\Gamma_1} f(x)(x-a_i)R(x;A)\,dx = \sum_{i=1}^{N} \int\limits_{b_i,\Gamma_2} f(x)(x-b_i)R(x;A)\,dx.$$

This proves statement (2), and hence the proof of the lemma is complete. □

Lemma 2.3. *Let $\mathcal{A}$ be a p-adic Banach algebra, and let $A \in \mathcal{A}$ have compact spectrum $\sigma_A \neq \emptyset$. Let $\mathcal{X}$ be a p-adic Banach space such that $\mathcal{X} \simeq \Omega_p(J)$, with $J \neq \emptyset$, and suppose that A be $\mathcal{X}$-analytic. Let $0 < r \in |\Omega_p|_p$, and let $\Gamma \in \Omega_p$, with $|\Gamma|_p = r$. Assume that*

$$D_{\sigma_A}(r) = \bigcup_{i=1}^{M} D_{b_i}(r), \quad \sigma_A \subseteq \bigcup_{i=1}^{M} D_{b_i}(r^-), \qquad (1)$$

where $b_1, \ldots, b_M$ are in Ω_p and the $D_{b_i}(r)$ are disjoint. Finally, suppose that

$$\sigma_A \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-), \qquad (2)$$

where $a_1, \ldots, a_N$ are in Ω_p and the $D_{a_i}(r)$ are disjoint. If f is Krasner analytic on each $D_{a_i}(r)$, then we have $f \in B_r(\sigma_A)$ and

$$\sum_{i=1}^{M} \int\limits_{b_i,\Gamma} f(x)(x-b_i)R(x;A)\,dx = \sum_{i=1}^{N} \int\limits_{a_i,\Gamma} f(x)(x-a_i)R(x;A)\,dx.$$

Proof. Let $\theta : \mathcal{A} \to B(\mathcal{X})$ be and embedding of $\mathcal{A}$ into $B(\mathcal{X})$ such that $\theta(A)$ is analytic. We have $\sigma_{\theta(A)} \subseteq \sigma_A$, hence (1) and (2) imply that

$$\sigma_{\theta(A)} \subseteq \bigcup_{i=1}^{M} D_{b_i}(r^-) \text{ and } \sigma_{\theta(A)} \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-). \tag{3}$$

By Lemma 1.3, (3) implies that there exist $c_1, \dots, c_L$ in $\sigma_{\theta(A)}$ and $\emptyset \neq I \subseteq \{1, \dots, N\}$ such that the $D_{c_i}(r)$ are disjoint and

$$D_{\sigma_{\theta(A)}}(r^-) = \bigcup_{i=1}^{L} D_{c_i}(r^-) = \bigcup_{i\in I} D_{a_i}(r^-); \tag{4}$$
$$D_{\sigma_{\theta(A)}}(r) = \bigcup_{i=1}^{L} D_{c_i}(r) = \bigcup_{i\in I} D_{a_i}(r).$$

Now let f be Krasner analytic on each $D_{a_i}(r)$. Let $a \in \Omega_p$, with $D_a(r) \subseteq D_{\sigma_A}(r)$. Because $a \in D_{\sigma_A}(r)$ and σ_A is compact, there exists $b \in \sigma_A$ such that $|a-b|_p \leq r$. Then by (2) there exists $1 \leq i \leq N$ such that $|b - a_i|_p < r$. We then have

$$|a - a_i|_p \leq \max\{ |a-b|_p, |b-a_i|_p \} \leq r.$$

Therefore $a \in D_{a_i}(r)$, and hence $D_a(r) = D_{a_i}(r)$. Because f is Krasner analytic on $D_{a_i}(r)$, Lemma 1.4 implies that f is Krasner analytic on $D_a(r)$. This proves that f is in $B_r(\sigma_A)$. Let $y \in \mathcal{X}$ and $h \in \mathcal{X}^*$ be arbitrary. Then by assumption, the function $\varphi_{h,y}(x) = h(R(x;\theta(A))y)$ is in $H_0(\overline{\sigma}_{\theta(A)})$. Now, by (1), for all $1 \leq i \leq M$, we have $D_{b_i}(r) \subseteq D_{\sigma_A}(r)$, and hence f is Krasner analytic on $D_{b_i}(r)$. Because each $c_i \in \sigma_{\theta(A)}$, it follows from (3), (4) and Lemma 1.12 that

$$\sum_{i=1}^{L} \int_{c_i,\Gamma} f(x)(x-c_i)\varphi_{h,y}(x)\,dx = \sum_{i=1}^{M} \int_{b_i,\Gamma} f(x)(x-b_i)\varphi_{h,y}(x)\,dx; \tag{5}$$
$$\sum_{i=1}^{L} \int_{c_i,\Gamma} f(x)(x-c_i)\varphi_{h,y}(x)\,dx = \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)\varphi_{h,y}(x)\,dx.$$

We conclude from (5) that

$$\sum_{i=1}^{M} \int_{b_i,\Gamma} f(x)(x-b_i)\varphi_{h,y}(x)\,dx = \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)\varphi_{h,y}(x)\,dx. \tag{6}$$

Hence, by Lemma 1.16, (6) implies that

$$h\Bigg(\Bigg(\sum_{i=1}^{M}\int\limits_{b_i,\Gamma_1} f(x)(x-b_i)R(x;\theta(A))\,dx \\ -\sum_{i=1}^{N}\int\limits_{a_i,\Gamma_2} f(x)(x-a_i)R(x;\theta(A))\,dx\Bigg)(y)\Bigg)=0.$$

Because $h\in\mathcal{X}^*$ is arbitrary, we see that

$$\Bigg(\sum_{i=1}^{M}\int\limits_{b_i,\Gamma} f(x)(x-b_i)R(x;\theta(A))\,dx \\ -\sum_{i=1}^{N}\int\limits_{a_i,\Gamma} f(x)(x-a_i)R(x;\theta(A))\,dx\Bigg)(y)=0. \quad (7)$$

Here, once more, we have used the fact that if $z\in\mathcal{X}$ is such that $k(z)=0$ for all $k\in\mathcal{X}^*$, then $z=0$ (see [7], Lemma 2, page 144). Finally, because $y\in\mathcal{X}$ is arbitrary, (7) implies that

$$\sum_{i=1}^{M}\int\limits_{b_i,\Gamma} f(x)(x-b_i)R(x;\theta(A))\,dx=\sum_{i=1}^{N}\int\limits_{a_i,\Gamma} f(x)(x-a_i)R(x;\theta(A))\,dx. \quad (8)$$

Now, for $x\in\rho(A)$, we have $x\in\rho(\theta(A))$, with $\theta((x-A)^{-1})=(x-\theta(A))^{-1}$, and hence $(x-\theta(A))^{-1}\in\theta(\mathcal{A})$. Because $\theta^{-1}:\theta(\mathcal{A})\to\mathcal{A}$ is bounded, (8) implies that the following sums exist and are equal.

$$\sum_{i=1}^{M}\int\limits_{b_i,\Gamma} f(x)(x-b_i)R(x;A)\,dx=\sum_{i=1}^{N}\int\limits_{a_i,\Gamma} f(x)(x-a_i)R(x;A)\,dx.$$

This completes the proof of the lemma. □

Lemma 2.4. *Let $\mathcal{A}$ be a p-adic Banach algebra. Let $A\in\mathcal{A}$ be $\mathcal{A}$-analytic. Let (A_k) be a sequence in $\mathcal{A}$ and $(\mathcal{X}_k)$ a sequence of p-adic Banach spaces $\mathcal{X}_k$ of the form $\mathcal{X}_k\simeq\Omega_p(J_k)$, with $J_k\neq\emptyset$, such that $\sigma_{A_k}\neq\emptyset$ is compact, A_k is $\mathcal{X}_k$-analytic and $\lim_{k\to\infty}\|A-A_k\|_p=0$ (see Definition 1.19). Let A have spectrum σ_A. Let $0<r\in|\Omega_p|_p$, and let $\Gamma\in\Omega_p$, with $|\Gamma|_p=r$. Assume that*

$$\sigma_A\subseteq\bigcup_{i=1}^{N}D_{a_i}(r^-), \quad (1)$$

where $a_1, \ldots, a_N \in \Omega_p$ and the $D_{a_i}(r)$ are disjoint. Assume that for each k there exists $a_{k,1}, \ldots, a_{k,N_k}$ in σ_{A_k} such that

$$D_{\sigma_{A_k}}(r) = \bigcup_{i=1}^{N_k} D_{a_{k,i}}(r);$$
$$\sigma_{A_k} \subseteq \bigcup_{i=1}^{N_k} D_{a_{k,i}}(r^-),$$

where the $D_{a_{k,i}}(r)$ are disjoint. If f is Krasner analytic on the $D_{a_i}(r)$,then the following limits exist and are equal.

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x;A_k)\,dx = \lim_{k\to\infty} \sum_{i=1}^{N_k} \int_{a_{k,i},\Gamma} f(x)(x-a_{k,i})R(x;A_k)\,dx.$$

Proof. Because A is $\mathcal{A}$-analytic, (d) of Theorem 2.1 implies that there exists a $\delta_0 > 0$ such that if $B \in \mathcal{A}$ and $\|A - B\|_p < \delta_0$, then $\sigma_B \subseteq D_{\sigma_A}(r^-)$. By (1) we have

$$D_{\sigma_A}(r^-) \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-). \tag{3}$$

Let k_0 be so large that for $k \geq k_0$, we have $\|A - A_k\|_p < \delta_0$. Then for $k \geq k_0$, (3) implies that

$$\sigma_{A_k} \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-). \tag{4}$$

For the rest of the proof we may assume that for all k, $k \geq k_0$. Because f is Krasner analytic on each $D_{a_i}(r)$, it follows from (2), (4), and Lemma 2.3 that for all k, f is in $B_r(\sigma_{A_k})$ and

$$\sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x;A_k)\,dx = \sum_{i=1}^{N_k} \int_{a_{k,i},\Gamma} f(x)(x-a_{k,i})R(x;A_k)\,dx. \tag{5}$$

Also, because f is Krasner analytic on each $D_{a_i}(r)$, Lemma 1.5 implies that for each $1 \leq i \leq N$, $\max\limits_{x\in D_{a_i}(r)} |f(x)|_p$ is attained when $|x - a_i|_p = r$, and hence we may write

$$\rho_i = \max_{x\in D_{a_i}(r)} |f(x)|_p = \max_{|x-a_i|_p = r} |f(x)|_p.$$

Now let $0 < \epsilon < r$ be arbitrary. Let $\mu > 0$ be a positive number such that

$$\mu + \max_j \rho_j > \frac{1}{r}.$$

By (d) of Theorem 2.1 there exists a positive number $\delta_1 > 0$ such that if $B \in \mathcal{A}$ and $\|A - B\|_p < \delta_1$, then

$$\sigma_B \subseteq D_{\sigma_A}(\epsilon^-) \tag{6}$$

and

$$\|R(x; A) - R(x; B)\|_p < \frac{\epsilon}{r(\mu + \max_j \rho_j)} \text{ for all } x \in \overline{D}_{\sigma_A}(\epsilon).$$

Now let $k_1 \geq k_0$ be so large that $\|A - A_k\|_p < \delta_1$ for $k \geq k_1$. Then for $k \geq k_1$, we have $\sigma_{A_k} \subseteq D_{\sigma_A}(\epsilon^-) \subseteq D_{\sigma_A}(r^-)$. Therefore (5) implies that, for all $k, l \geq k_1$ we get, with $x^i_\xi = a_i + \xi\Gamma$,

$$\sum_{i=1}^{N_k} \int_{a_{k,i},\Gamma} f(x)(x - a_{k,i})R(x; A_k)\, dx - \sum_{i=1}^{N_l} \int_{a_{l,i},\Gamma} f(x)(x - a_{l,i})R(x; A_l)\, dx = \tag{7}$$

$$\sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x - a_i)R(x; A_k)\, dx - \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x - a_i)R(x; A_l)\, dx =$$

$$\lim_{\substack{n\to\infty \\ p \nmid n}} \left\{ \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x^i_\xi)(x^i_\xi - a_i)(R(x^i_\xi; A_k) - R(x^i_\xi; A)) - \right.$$
$$\left. \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x^i_\xi)(x^i_\xi - a_i)(R(x^i_\xi; A_l) - R(x^i_\xi; A)) \right\}.$$

Now, we claim that for $1 \leq i \leq N$ and $\xi \in \Omega_p$, with, $|\xi|_p = 1$,

$$x^i_\xi \in \overline{D}_{\sigma_A}(\epsilon). \tag{8}$$

To prove this claim, suppose that $x^i_\xi \in D_{\sigma_A}(\epsilon^-)$, then $x^i_\xi \in D_{\sigma_A}(r^-)$, hence (3) implies that for some $1 \leq j \leq N$, $x^i_\xi \in D_{a_j}(r^-)$. We must have $i = j$, for if $i \neq j$, then because $|x^i_\xi - a_i|_p = |\xi\Gamma|_p = r$, we would have $x^i_\xi \in D_{a_i}(r) \cap D_{a_j}(r) = \emptyset$. Thus, $i = j$, which implies that $x^i_\xi \in D_{a_i}(r^-)$. But then we have the contradiction that $r = |x^i_\xi - a_i|_p < r$. Thus (8) holds. If $p \nmid n$, $1 \leq i \leq N$ and $m \geq k_0$, then (6) and (8) imply that

$$\left\| \frac{1}{n} \sum_{\xi^n=1} f(x^i_\xi)(x^i_\xi - a_i)(R(x^i_\xi; A_m) - R(x^i_\xi; A)) \right\|_p \tag{9}$$

$$\begin{aligned}&\leq \max_j \rho_j \cdot |\Gamma|_p \cdot \max_{\xi^n=1} \|R(x_\xi^i; A_m) - R(x_\xi^i; A)\|_p \\ &\leq (r \max_j \rho_j)\left(\frac{\epsilon}{r(\mu + \max_j \rho_j)}\right) \\ &< \epsilon.\end{aligned}$$

Therefore, (7) and (9) imply that for $k, l \geq k_1$,

$$\left\| \sum_{i=1}^{N_k} \int_{a_{k,i},\Gamma} f(x)(x - a_{k,i}) R(x; A_k)\, dx - \sum_{i=1}^{N_l} \int_{a_{l,i},\Gamma} f(x)(x - a_{l,i}) R(x; A_l)\, dx \right\|_p \leq \epsilon.$$

Because $0 < \epsilon < r$ is arbitrary, we see that the terms

$$\sum_{i=1}^{N_k} \int_{a_{k,i},\Gamma} f(x)(x - a_{k,i}) R(x; A_k)\, dx = \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x - a_i) R(x; A_k)\, dx$$

form a Cauchy sequence. Hence the following limits exist and are equal.

$$\lim_{k\to\infty} \sum_{i=1}^{N_k} \int_{a_{k,i},\Gamma} f(x)(x - a_{k,i}) R(x; A_k)\, dx = \lim_{k\to\infty} \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x - a_i) R(x; A_k)\, dx.$$

This completes the proof of the lemma. □

Lemma 2.5. *Let $\mathcal{A}$ be a p-adic Banach algebra. Let $A \in \mathcal{A}$ be $\mathcal{A}$-analytic. Let (A_k) be a sequence in $\mathcal{A}$ and $(\mathcal{X}_k)$ a sequence of p-adic Banach spaces $\mathcal{X}_k$ of the form $\mathcal{X}_k \simeq \Omega_p(J_k)$, with $J_k \neq \emptyset$, such that $\sigma_{A_k} \neq \emptyset$ is compact, A_k is $\mathcal{X}_k$-analytic and $\lim_{k\to\infty} \|A - A_k\|_p = 0$. Let A have spectrum σ_A. Let $0 < r_2 \leq r_1$ be in $|\Omega_p|_p$, and let Γ_1, Γ_2 be in Ω_p, with $|\Gamma_1|_p = r_1$, $|\Gamma_2|_p = r_2$. Assume that $a_1, \ldots, a_M; b_1, \ldots, b_N \in \Omega_p$ are given, with*

$$\sigma_A \subseteq \bigcup_{i=1}^{M} D_{a_i}(r_1^-) \text{ and } \sigma_A \subseteq \bigcup_{i=1}^{N} D_{b_i}({r_2}^-), \tag{1}$$

where the $D_{a_i}(r_1)$ are disjoint and the $D_{b_i}(r_2)$ are disjoint. Let f be Krasner analytic on the $D_{a_i}(r_1)$ and on the $D_{b_i}(r_2)$. Then the following limits exist and are equal:

$$\lim_{k\to\infty} \sum_{i=1}^{M} \int_{a_i,\Gamma_1} f(x)(x - a_i) R(x; A_k)\, dx = \lim_{k\to\infty} \sum_{i=1}^{N} \int_{b_i,\Gamma_2} f(x)(x - b_i) R(x; A_k)\, dx.$$

Proof. Because A is $\mathcal{A}$-analytic, (d) of Theorem 2.1 implies that $\sigma_A \neq \emptyset$ and that there exists a $\delta > 0$ such that if $B \in \mathcal{A}$ with $\|A - B\|_p < \delta$, then

$$\sigma_B \subseteq D_{\sigma_A}(r_i^-), \text{ for } i = 1, 2.$$

Hence, because $\lim_{k\to\infty} \|A - A_k\|_p = 0$, by (1) we may assume that for all k,

$$\sigma_{A_k} \subseteq \bigcup_{i=1}^{M} D_{a_i}(r_1^-) \text{ and } \sigma_{A_k} \subseteq \bigcup_{i=1}^{N} D_{b_i}(r_2^-). \tag{2}$$

Then by Lemma 1.3, (2) implies that for all k there exist $I_k, J_k \neq \emptyset$ such that $I_k \subseteq \{1, \dots, M\}$, $J_k \subseteq \{1, \dots, N\}$ and

$$\sigma_{A_k} \subseteq D_{\sigma_{A_k}}(r_1^-) = \bigcup_{i\in I_k} D_{a_i}(r_1^-) \text{ and } D_{\sigma_{A_k}}(r_1) = \bigcup_{i\in I_k} D_{a_i}(r_1); \tag{3}$$

$$\sigma_{A_k} \subseteq D_{\sigma_{A_k}}(r_2^-) = \bigcup_{i\in J_k} D_{b_i}(r_2^-) \text{ and } D_{\sigma_{A_k}}(r_2) = \bigcup_{i\in J_k} D_{b_i}(r_2).$$

Now let k be arbitrary. By Lemma 2.3, (2) and (3) together imply that

$$\sum_{i=1}^{M} \int_{a_i,\Gamma_1} f(x)(x - a_i)R(x; A_k)\, dx = \sum_{i\in I_k} \int_{a_i,\Gamma_1} f(x)(x - a_i)R(x; A_k)\, dx; \tag{4}$$

$$\sum_{i=1}^{N} \int_{b_i,\Gamma_2} f(x)(x - b_i)R(x; A_k)\, dx = \sum_{i\in J_k} \int_{b_i,\Gamma_2} f(x)(x - b_i)R(x; A_k)\, dx.$$

By Lemma 2.2, (2) and (3) together also imply that

$$\sum_{i\in I_k} \int_{a_i,\Gamma_1} f(x)(x - a_i)R(x; A_k)\, dx = \sum_{i\in J_k} \int_{b_i,\Gamma_2} f(x)(x - b_i)R(x; A_k)\, dx. \tag{5}$$

Consequently, (4) and (5) together imply that for all k,

$$\sum_{i=1}^{M} \int_{a_i,\Gamma_1} f(x)(x - a_i)R(x; A_k)\, dx = \sum_{i=1}^{N} \int_{b_i,\Gamma_2} f(x)(x - b_i)R(x; A_k)\, dx. \tag{6}$$

It follows from (1), (3), (6), and Lemma 2.4 that the following limits exist and are equal.

$$\lim_{k\to\infty} \sum_{i=1}^{M} \int_{a_i,\Gamma_1} f(x)(x - a_i)R(x; A_k)\, dx = \lim_{k\to\infty} \sum_{i=1}^{N} \int_{b_i,\Gamma_2} f(x)(x - b_i)R(x; A_k)\, dx.$$

This completes the proof of the lemma. $\square$

Lemma 2.6. *Let $\mathcal{A}$ be a p-adic Banach algebra. Let $A \in \mathcal{A}$ be $\mathcal{A}$-analytic with spectrum σ_A. Let (A_k) be a sequence in $\mathcal{A}$ and $(\mathcal{X}_k)$ a sequence of p-adic Banach spaces $\mathcal{X}_k$ of the form $\mathcal{X}_k \simeq \Omega_p(J_k)$, with $J_k \neq \emptyset$, such that $\sigma_{A_k} \neq \emptyset$ is compact, A_k is $\mathcal{X}_k$-analytic and $\lim_{k\to\infty} \|A - A_k\|_p = 0$. Let $0 < r \in |\Omega_p|_p$, and let $\Gamma \in \Omega_p$, with $|\Gamma|_p = r$. Assume that $a_1, \ldots, a_N \in \Omega_p$ are given, with*

$$\sigma_A \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-), \tag{1}$$

where the $D_{a_i}(r)$ are disjoint. Assume that f is Krasner analytic on the $D_{a_i}(r)$. Then the following limits exist and are equal.

$$\lim_{\substack{n\to\infty \\ p \nmid n}} \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x^i_\xi)(x^i_\xi - a_i)R(x^i_\xi; A) = \lim_{k\to\infty} \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x; A_k)\, dx,$$

where for each $1 \leq i \leq N$ and $\xi \in \Omega_p$, we set $x^i_\xi = a_i + \xi\Gamma$. Therefore the sum

$$\sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x; A)\, dx$$

exits and we have

$$\sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x; A)\, dx = \lim_{k\to\infty} \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x; A_k)\, dx.$$

Proof. Because f is Krasner analytic on each of the $D_{a_i}(r)$, it follows from Lemma 1.5 that for $1 \leq i \leq N$, $\max_{x \in D_{a_i}(r)} |f(x)|_p$ is attained when $|x - a_i|_p = r$, and hence we may write

$$\rho_i = \max_{x \in D_{a_i}(r)} |f(x)|_p = \max_{|x-a_i|_p = r} |f(x)|_p.$$

Now let $0 < \epsilon < r$ be arbitrary. Let $\mu > 0$ be a positive number such that

$$\mu + \max_j \rho_j > \frac{1}{r}.$$

Because A is $\mathcal{A}$-analytic, (d) of Theorem 2.1 implies that $\sigma_A \neq \emptyset$ and that there exists a positive number $\delta > 0$ such that if $B \in \mathcal{A}$ and $\|A - B\|_p < \delta$, then

$$\sigma_B \subseteq D_{\sigma_A}(\epsilon^-)$$

and

$$\|R(x;A) - R(x;B)\|_p < \frac{\epsilon}{r(\mu + \max_j \rho_j)} \text{ for all } \in \overline{D}_{\sigma_A}(\epsilon). \tag{2}$$

Then by Lemma 1.3, (1) implies that for all k there exist $I_k \neq \emptyset$ such that $I_k \subseteq \{1, \ldots, N\}$ and

$$\sigma_{A_k} \subseteq D_{\sigma_{A_k}}(r^-) = \bigcup_{i \in I_k} D_{a_i}(r^-) \text{ and } D_{\sigma_{A_k}}(r) = \bigcup_{i \in I_k} D_{a_i}(r). \tag{3}$$

By Lemma 2.4, (1) and (3) together imply that the following limits exist and are equal.

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x - a_i)R(x;A_k)\,dx = \lim_{k\to\infty} \sum_{i\in I_k}^{N} \int_{a_i,\Gamma} f(x)(x - a_i)R(x;A_k)\,dx.$$

Now let k_0 be so large that $\|A - A_{k_0}\|_p < \delta$ and

$$\left\| \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x - a_i)R(x;A_{k_0})\,dx - \lim_{k\to\infty} \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x - a_i)R(x;A_k)\,dx \right\|_p < \epsilon. \tag{4}$$

Observe that if $1 \le i \le N$ and $\xi \in \Omega_p$, with $|\xi|_p = 1$, then as in statement (8) of Lemma 2.4, we have $x_\xi^i = a_i + \xi\Gamma \in \overline{D}_{\sigma_A}(\epsilon)$. Now let n_0 be so large that if $n \ge n_0$ and $p \not| \, n$, then

$$\left\| \sum_{i=1}^{N} \left\{ \frac{1}{n} \sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i - a_i)R(x_\xi^i;A_{k_0}) - \int_{a_i,\Gamma} f(x)(x - a_i)R(x;A_{k_0})\,dx \right\} \right\|_p < \epsilon. \tag{5}$$

For $n \ge n_0$ and $p \not| \, n$, we have

$$\sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i - a_i)R(x_\xi^i;A) - \lim_{k\to\infty} \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x - a_i)R(x;A_k)\,dx \tag{6}$$

$$= \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i - a_i)\big\{R(x_\xi^i;A) - R(x_\xi^i;A_{k_0})\big\} +$$

$$\sum_{i=1}^{N}\left\{\frac{1}{n}\sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i-a_i)R(x_\xi^i;A_{k_0})-\int\limits_{a_i,\Gamma} f(x)(x-a_i)R(x;A_{k_0})\,dx\right\}+$$
$$\sum_{i=1}^{N}\int\limits_{a_i,\Gamma} f(x)(x-a_i)R(x;A_{k_0})\,dx-\lim_{k\to\infty}\sum_{i=1}^{N}\int\limits_{a_i,\Gamma} f(x)(x-a_i)R(x;A_k)\,dx.$$

Thus, for $n \geq n_0$ and $p \not| \, n$, (1)-(6) together imply that

$$\left\|\sum_{i=1}^{N}\frac{1}{n}\sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i-a_i)R(x_\xi^i;A)-\lim_{k\to\infty}\sum_{i=1}^{N}\int\limits_{a_i,\Gamma} f(x)(x-a_i)R(x;A_k)\,dx\right\|_p<\epsilon.$$

Because $\epsilon > 0$ is arbitrary, this completes the proof of the lemma. □

Definition 2.7. Let $\mathcal{A}$ be a p-adic Banach algebra. Let $A \in \mathcal{A}$ be $\mathcal{A}$-analytic with spectrum σ_A. Let (A_k) be a sequence in $\mathcal{A}$ and $(\mathcal{X}_k)$ a sequence of p-adic Banach spaces $\mathcal{X}_k$ of the form $\mathcal{X}_k \simeq \Omega_p(J_k)$, with $J_k \neq \emptyset$, such that $\sigma_{A_k} \neq \emptyset$ is compact, A_k is $\mathcal{X}_k$-analytic and $\lim_{k\to\infty}\|A-A_k\|_p = 0$. Let $f \in \mathcal{F}(A)$. Let $0 < r \in |\Omega_p|_p$, and let $\Gamma \in \Omega_p$, with $|\Gamma|_p = r$. Assume that $a_1, \ldots, a_N \in \Omega_p$ are given, with

$$\sigma_A \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-), \tag{1}$$

where the $D_{a_i}(r)$ are disjoint. Assume that f is Krasner analytic on the $D_{a_i}(r)$. By Lemma 2.6, (1) implies that

$$\sum_{i=1}^{N}\int\limits_{a_i,\Gamma} f(x)(x-a_i)R(x;A)\,dx=\lim_{k\to\infty}\sum_{i=1}^{N}\int\limits_{a_i,\Gamma} f(x)(x-a_i)R(x;A_k)\,dx.$$

Now, Lemma 2.5 implies that

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int\limits_{a_i,\Gamma} f(x)(x-a_i)R(x;A_k)\,dx$$

does not depend on $a_1, \ldots, a_N, r, \Gamma$, hence we may define $f(A)$ by

$$f(A)=\sum_{i=1}^{N}\int\limits_{a_i,\Gamma} f(x)(x-a_i)R(x;A)\,dx.$$

Lemma 2.8. *Let $\mathcal{A}$ be a p-adic Banach algebra. Let $A \in \mathcal{A}$ have spectrum σ_A, and assume that A is $\mathcal{A}$-analytic. Let $f \in \mathcal{F}(A)$. Let $0 < r \in |\Omega_p|_p$, and let Γ be in Ω_p, with $|\Gamma|_p = r$. Assume that $a_1, \ldots, a_N$ in Ω_p are given, with*

$$\sigma_A \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-), \tag{1}$$

where the $D_{a_i}(r)$ are disjoint and f is Krasner analytic on the $D_{a_i}(r)$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $B \in \mathcal{A}$ is $\mathcal{A}$-analytic with $\|A - B\|_p < \delta$, then

$$\sigma_B \subseteq \bigcup_{i=1}^{N} D_{a_i}(r),$$

$f \in \mathcal{F}(B)$ and $\|f(A) - f(B)\|_p < \epsilon$.

Proof. Because f is Krasner analytic on each $D_{a_i}(r)$, Lemma 1.5 implies that for each $1 \leq i \leq N$, $\max\limits_{x \in D_{a_i}(r)} |f(x)|_p$ is attained when $|x - a_i|_p = r$, and hence we may write

$$\rho_i = \max_{x \in D_{a_i}(r)} |f(x)|_p = \max_{|x - a_i|_p = r} |f(x)|_p.$$

Now let $0 < \epsilon < r$ be arbitrary. Let $\mu > 0$ be a positive number such that

$$\mu + \max_j \rho_j > \frac{1}{r}.$$

Because A is $\mathcal{A}$-analytic, (d) of Theorem 2.1 implies that $\sigma_A \neq \emptyset$ and that there exists a positive number $\delta > 0$ such that if $B \in \mathcal{A}$ and $\|A - B\|_p < \delta$, then

$$\sigma_B \subseteq D_{\sigma_A}(\epsilon^-) \tag{2}$$

and

$$\|R(x; A) - R(x; B)\|_p < \frac{\epsilon}{r(\mu + \max\limits_j \rho_j)} \text{ for all } x \in \overline{D}_{\sigma_A}(\epsilon). \tag{3}$$

Now assume that $B \in \mathcal{A}$ is $\mathcal{A}$-analytic, and that $\|A - B\|_p < \delta$. It is easy to see that (1) implies

$$D_{\sigma_A}(r^-) \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-). \tag{4}$$

Then by (2), we have $\sigma_B \subseteq D_{\sigma_A}(\epsilon^-)$, and hence (4) implies that

$$\sigma_B \subseteq D_{\sigma_A}(\epsilon^-) \subseteq D_{\sigma_A}(r^-) \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-). \tag{5}$$

Because B is $\mathcal{A}$-analytic, $\sigma_B \neq \emptyset$, and hence (5) implies that $f \in \mathcal{F}(B)$. Then by (1), (5) and Definition 2.7, $f(A)$ and $f(B)$ are well defined, with

$$f(A) = \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x;A)\,dx = \lim_{\substack{n\to\infty \\ p\nmid n}} \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i - a_i)R(x_\xi^i;A);$$

$$f(B) = \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x-a_i)R(x;B)\,dx = \lim_{\substack{n\to\infty \\ p\nmid n}} \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i - a_i)R(x_\xi^i;B).$$

Then we get

$$f(A) - f(B) = \lim_{\substack{n\to\infty \\ p\nmid n}} \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i - a_i)\big[R(x_\xi^i;A) - R(x_\xi^i;B)\big]. \tag{6}$$

Now, we claim that for $1 \leq i \leq N$ and $\xi \in \Omega_p$, with, $|\xi|_p = 1$,

$$x_\xi^i \in \overline{D}_{\sigma_A}(\epsilon). \tag{7}$$

The proof of this claim is similar to the proof of statement (8) in Lemma 2.4. It then follows from (3), (6) and (7) that

$$\begin{aligned} \|f(A) - f(B)\|_p &= \lim_{\substack{n\to\infty \\ p\nmid n}} \left\| \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} f(x_\xi^i)(x_\xi^i - a_i)\big[R(x_\xi^i;A) - R(x_\xi^i;B)\big] \right\|_p \\ &\leq (r \max_j \rho_j)\left(\frac{\epsilon}{r(\mu + \max_j \rho_j)}\right) \\ &< \epsilon. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.9. *Let $\mathcal{A}$ be a p-adic Banach algebra. Let $A \in \mathcal{A}$ be $\mathcal{A}$-analytic with spectrum σ_A. Let (A_k) be a sequence in $\mathcal{A}$ and $(\mathcal{X}_k)$ a sequence of p-adic Banach spaces $\mathcal{X}_k$ of the form $\mathcal{X}_k \simeq \Omega_p(J_k)$, with $J_k \neq \emptyset$, such that $\sigma_{A_k} \neq \emptyset$ is* *compact, A_k is $\mathcal{X}_k$-analytic and $\lim_{k\to\infty} \|A - A_k\|_p = 0$. Let $0 < r \in |\Omega_p|_p$, and let $\Gamma \in \Omega_p$, with $|\Gamma|_p = r$. Assume that $a_1, \ldots, a_N \in \Omega_p$ are given, with*

$$\sigma_A \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-), \tag{1}$$

where the $D_{a_i}(r)$ are disjoint. Assume that f is Krasner analytic on the $D_{a_i}(r)$. Then there exists a k_0 such that for all $k \geq k_0$, $f \in B_r(\sigma_{A_k})$ and

$$f(A) = \lim_{k\to\infty} f(A_k),$$

where $f(A_k)$ is given by Theorem 1.21.

Proof. Because A is $\mathcal{A}$-analytic, (d) of Theorem 2.1 implies that there exists a $\delta > 0$ such that if $B \in \mathcal{A}$ and $\|A - B\|_p < \delta$, then $\sigma_B \subseteq D_{\sigma_A}(r^-)$. By (1) we have

$$D_{\sigma_A}(r^-) \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-). \tag{2}$$

Let k_0 be so large that for $k \geq k_0$, we have $\|A - A_k\|_p < \delta$. Then for $k \geq k_0$, (2) implies that

$$\sigma_{A_k} \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-). \tag{3}$$

For the rest of the proof we may assume that for all k, $k \geq k_0$. Then by Lemma 1.3, (3) implies that for all k there exists $a_{k,1}, \ldots, a_{k,N_k}$ in σ_{A_k} such that the $D_{a_{k,1}}(r), \ldots, D_{a_{k,N_k}}(r)$ are disjoint with

$$D_{\sigma_{A_k}}(r^-) = \bigcup_{i=1}^{N_k} D_{a_{k,i}}(r^-) \text{ and } D_{\sigma_{A_k}}(r) = \bigcup_{i=1}^{N_k} D_{a_{k,i}}(r). \tag{4}$$

It follows from (3), (4), and Lemma 2.4 that

$$\begin{aligned} f(A) &= \lim_{k\to\infty} \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x - a_i) R(x; A_k)\, dx \\ &= \lim_{k\to\infty} \sum_{i=1}^{N_k} \int_{a_{k,i},\Gamma} f(x)(x - a_{k,i}) R(x; A_k)\, dx. \end{aligned} \tag{5}$$

Now let $k \geq k_0$. We claim that $f \in B_r(\sigma_{A_k})$. To see this let $D_a(r) \subseteq D_{\sigma_{A_k}}(r)$. Then by (4) there exists $1 \leq i \leq N_k$ such that $|a - a_{k,i}|_p \leq r$. Then by (3) There exists $1 \leq j \leq N$ such that $|a_{k,i} - a_j|_p < r$. It follows that $|a - a_j|_p \leq \max |a_{k,i} - a_j|_p, |a - a_j|_p \leq r$, and hence we have $a \in D_{a_j}(r)$. Because f is Krasner analytic on $D_{a_j}(r)$, Lemma 1.4 implies that f is Krasner analytic on $D_a(r)$. This proves the claim. Therefore by (4) and Theorem 1.21 we see that

$$f(A_k) = \sum_{i=1}^{N_k} \int\limits_{a_{k,i},\Gamma} f(x)(x - a_i)R(x; A_k)\,dx.$$

Hence (5) implies that

$$f(A) = \lim_{k\to\infty} f(A_k).$$

This completes the proof of the lemma. □

Extended 2.10. p-adic Spectral Theorem. Let $\mathcal{A}$ be a p-adic Banach algebra. Let $A \in \mathcal{A}$ be $\mathcal{A}$-analytic, with spectrum σ_A. Let $f, g \in \mathcal{F}(A)$. Let $0 < r \in |\Omega_p|_p$, and let Γ be in Ω_p, with $|\Gamma|_p = r$. Assume that $a_1, \ldots, a_N$ in Ω_p are given, with

$$\sigma_A \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-), \tag{1}$$

where the $D_{a_i}(r)$ are disjoint. Finally, suppose that f and g are Krasner analytic on each $D_{a_i}(r)$. By (1) and Definition 2.7, $f(A)$ and $g(A)$ are well-defined, with

$$f(A) = \sum_{i=1}^{N} \int\limits_{a_i,\Gamma} f(x)(x - a_i)R(x; A)\,dx,$$
$$g(A) = \sum_{i=1}^{N} \int\limits_{a_i,\Gamma} g(x)(x - a_i)R(x; A)\,dx.$$

Let $\alpha, \beta \in \Omega_p$. Then $\alpha f + \beta g, f \cdot g \in \mathcal{F}(A)$, and

(2) $(\alpha f + \beta g)(A) = \alpha f(A) + \beta g(A)$;

(3) $(f \cdot g)(A) = f(A)g(A)$;

(4) The mapping $h : B_{r,a_1,\ldots,a_N} \to \mathcal{A}$ given by $h \mapsto h(A)$ is norm continuous.

Proof. Let $\alpha, \beta \in \Omega_p$. Then it is clear that $\alpha f + \beta g, f \cdot g \in \mathcal{F}(A)$. Because A is $\mathcal{A}$-analytic, there exists a sequence (A_k) in $\mathcal{A}$ and a sequence $(\mathcal{X}_k)$ of p-adic Banach spaces $\mathcal{X}_k$ of the form $\mathcal{X}_k \simeq \Omega_p(J_k)$, with $J_k \neq \emptyset$, such that $\sigma_{A_k} \neq \emptyset$ is compact, A_k is $\mathcal{X}_k$-analytic and $\lim_{k\to\infty} \|A - A_k\|_p = 0$. Then by Lemma 2.9 and Theorem 1.21 we have

$$\begin{aligned}(\alpha f + \beta g)(A) &= \lim_{k\to\infty} (\alpha f + \beta g)(A_k) \\ &= \lim_{k\to\infty} (\alpha f(A_k) + \beta g(A_k)) \\ &= \alpha \lim_{k\to\infty} f(A_k) + \beta \lim_{k\to\infty} g(A_k) \\ &= \alpha f(A) + \beta g(A);\end{aligned}$$

and we also have

$$\begin{aligned}(f \cdot g)(A) &= \lim_{k\to\infty} (f \cdot g)(A_k) \\ &= \lim_{k\to\infty} f(A_k) g(A_k) \\ &= f(A)g(A).\end{aligned}$$

This proves (2) and (3). To prove (4), let $h_0 \in B_{r,a_1,\ldots,a_N}$. Because A is $\mathcal{A}$-analytic, by Definition 1.19 there exists a constant $N_r > 0$ such that $\|R(x;A)\|_p \leq N_r$ for all $x \in \overline{D}_{\sigma_A}(r)$. Now let $\epsilon > 0$ be given, and define $\delta = \dfrac{\epsilon}{rN_r}$. Let $h \in B_{r,a_1,\ldots,a_N}$, with $\|h - h_0\|_u < \delta$. For $1 \leq i \leq N$ and $\xi \in \Omega_p$, with $|\xi|_p = 1$, set $x^i_\xi = a_i + \xi\Gamma$. Then for $1 \leq i \leq N$ and $|\xi|_p = 1$, $x^i_\xi \in \overline{D}_{\sigma_A}(r)$. Hence by (2) and Lemma 2.6, we have

$$\begin{aligned}\|h(A) - h_0(A)\|_p &= \|(h - h_0)(A)\|_p \\ &= \lim_{\substack{n\to\infty \\ p \nmid n}} \left\| \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} [h(x^i_\xi) - h_0(x^i_\xi)](x^i_\xi - a_i) R(x^i_\xi; A) \right\|_p \\ &= \lim_{\substack{n\to\infty \\ p \nmid n}} \left\| \sum_{i=1}^{N} \frac{1}{n} \sum_{\xi^n=1} [h(x^i_\xi) - h_0(x^i_\xi)](\xi\Gamma) R(x^i_\xi; A) \right\|_p \\ &\leq \|h - h_0\|_u (rN_r) \\ &< \frac{\epsilon}{rN_r}(rN_r) = \epsilon.\end{aligned}$$

This proves (4). ∎

Definition 2.11. Let $\mathcal{X}$ be a p-adic Banach space over Ω_p. Let $\emptyset \neq X \subseteq \mathcal{X}$. Then X is *weakly bounded* iff for each $h \in \mathcal{X}^*$ the set $\{\, |h(x)|_p \,|\, x \in X \,\}$ is bounded.

Lemma 2.12. *Let $\mathcal{X} \simeq \Omega_p(J)$, where $J \neq \emptyset$. Let $\emptyset \neq X \subseteq \mathcal{X}$ be weakly bounded. Then the set $\{\, \|x\|_p \,|\, x \in X \,\}$ is bounded.*

Proof. For each $x = (x_j)_{j\in J} \in \Omega_p(J)$ and each $\epsilon > 0$ let

$$I_x = \{\, j \in J \,|\, x_j \neq 0 \,\} \text{ and } I_x(\epsilon) = \{\, j \in J \,|\, \|x_j\|_p > \epsilon \,\}.$$

By definition of $\Omega_p(J)$, for each $x \in \Omega_p(J)$ and $\epsilon > 0$ the set $I_x(\epsilon)$ is finite; and because $I_x = \bigcup\limits_{n=1}^{\infty} I_x(\frac{1}{n})$, we see that I_x is countable. Now assume that the $\{\, \|x\|_p \,|\, x \in X \,\}$ is not bounded. Then we can find a sequence (x_n) in X such that

$$\lim_{n\to\infty} \|x_n\|_p = \infty.$$

Let I be the countable set $\bigcup_{n=1}^{\infty} I_{x_n}$, and define $\mathcal{Y} = \{\, y \in \Omega_p(J) \,|, I_y \subseteq I \,\}$. Then $\mathcal{Y}$ is a p-adic Banach subspace $\Omega_p(J)$ such that $X \subseteq \mathcal{Y}$. Moreover, the mapping

$$\theta : \mathcal{Y} \to \Omega_p(I)$$

defined by $y = (y_j)_{j\in J} \mapsto (y_j)_{j\in I} \in \Omega_p(I)$ is an isometric isomorphism of $\mathcal{Y}$ onto $\Omega_p(I)$. Because each $x_n \in \mathcal{Y}$ we may define a sequence (y_n) in $\Omega_p(I)$ by $y_n = \theta(x_n)$. Then

$$\lim_{n\to\infty} \|y_n\|_p = \lim_{n\to\infty} \|\theta(x_n)\|_p = \lim_{n\to\infty} \|x_n\|_p = \infty.$$

Let $g \in \Omega_p(I)^*$, and define $h \in \mathcal{X}^*$ by

$$h(x) = g((x_j)_{j\in I}), \quad x = (x_j)_{j\in J} \in \mathcal{X}.$$

Write $x_n = (x_{nj})_{j\in J}$, then we have

$$g(y_n) = g(\theta(x_n)) = g((x_{nj})_{j\in I}) = h((x_{nj})_{j\in J}) = h(x_n).$$

By assumption the sequence $(|h(x_n)|_p)$ is bounded, and hence the sequence $(|g(y_n)|_p)$ is bounded. Because I is countably infinite we may identify I with the set $\mathbf{N}$ of positive integers. We have that $\|y_n\|_p = \max\limits_j |y_{nj}|_p$. Let α_n denote the first j for which $|y_{nj}|_p = \|y_n\|_p$, and let β_n denote the last j for which $|y_{nj}|_p = \|y_n\|_p$. We have the following two cases to consider.

Case 1. In this case the sequence (α_n) is bounded. Then there exists some j_0 such that $\alpha_n = j_0$ for infinitely many n. Let $h \in \Omega_p(\mathbf{N})^*$ be defined by

$$h((z_j)) = z_{j_0}, \quad (z_j) \in \Omega_p(\mathbf{N}).$$

Then for infinitely many n we have

$$|h(y_n)|_p = |y_{nj_0}|_p = |y_{n\alpha_n}|_p = \|y_n\|_p.$$

Because the sequence $(|h(y_n)|_p)$ is bounded and $\lim_{n\to\infty} \|y_n\|_p = \infty$, we have a contradiction.

Case 2. In this case the sequence (α_n) is unbounded. Then we can find a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that

$$\alpha_1 \leq \beta_1 < \alpha_{n_1} \leq \beta_{n_1} < \alpha_{n_2} \leq \beta_{n_2} < \cdots \leq \beta_{n_{k-1}} < \alpha_{n_k} \leq \beta_{n_k}.$$

Define $j_0 = \alpha_1$ and for $k \geq 1$, set $j_k = \alpha_{n_k}$. Let $h \in \Omega_p(\mathbf{N})^*$ be defined by

$$h((z_j)) = \sum_{k=1}^{\infty} z_{n_k}, \quad (z_j) \in \Omega_p(\mathbf{N}).$$

Then for all $m \in \mathbf{N}$ we have

$$|h(y_{n_m})|_p = \left| \sum_{k=1}^{\infty} y_{n_m n_k} \right|_p = |y_{n_m \alpha_{n_m}}|_p = \|y_{n_m}\|_p.$$

Because the sequence $(|h(y_{n_m})|_p)$ is bounded and $\lim_{m\to\infty} \|y_{n_m}\|_p = \infty$, we again have a contradiction. This completes the proof of the lemma. □

Theorem 2.13. *Let $\mathcal{X} \simeq \Omega_p(J)$, where $J \neq \emptyset$. Let $\mathcal{A}$ be a p-adic Banach algebra. If $A \in \mathcal{A}$ is $\mathcal{X}$-analytic with compact spectrum $\sigma_A \neq \emptyset$, then A is $\mathcal{A}$-analytic.*

Proof. We first prove that if $A \in B(\mathcal{X})$ is analytic with compact spectrum $\sigma_A \neq \emptyset$, then for all $r > 0$ there exists a constant $M_r > 0$ such that

$$\|R(x; A)\|_p \leq M_r \text{ for all } x \in \overline{D}_{\sigma_A}(r).$$

Fix $y \in \mathcal{X}$ and $h \in \mathcal{X}^*$. Because A is analytic, Definition 1.14 implies that the function

$$\phi(x) = h(R(x; A)y), \quad x \in \overline{\sigma}_A,$$

is in $H_0(\overline{\sigma}_A)$. Hence by Definition 1.10, $\lim_{|x|_p \to \infty} \phi(x) = 0$ and $\phi(x)$ is the uniform limit on $\overline{D}_{\sigma_A}(r)$ of rational functions with poles contained in σ_A. Let $L > 0$ be so large that $|\phi(x)|_p \leq 1$ for $|x|_p \geq L$. Now let $R(x) = \dfrac{P(x)}{Q(x)}$ be a rational function with poles in σ_A such that

$$|\phi(x) - R(x)|_p \leq 1 \text{ for all } x \in \overline{D}_{\sigma_A}(r).$$

Now write

$$R(x) = \frac{a_0 + a_1 x + \cdots + a_n x^n}{(x - b_1)^{n_1} \cdots (x - b_k)^{n_k}},$$

where $\{ b_1, \ldots, b_{n_k} \} \subseteq \sigma_A$. Let $x \in \overline{D}_{\sigma_A}(r)$, with $|x|_p \leq L$. Then for $1 \leq j \leq k$, $|x - b_j|_p \geq r$, because $b_j \in \sigma_A$ and $\operatorname{dist}(x, \sigma_A) \geq r$; hence we have

$$\frac{1}{|x - b_j|_p^{n_j}} \leq \frac{1}{r^{n_j}}.$$

Therefore we get

$$\begin{aligned} |R(x)|_p &= \frac{|a_0 + a_1 x + \cdots + a_n x^n|_p}{|x - b_1|_p^{n_1} \cdots |x - b_k|_p^{n_k}} \\ &\leq \frac{\max\{ |a_0|_p, |a_1|_p |x|_p, \ldots, |a_n|_p |x|_p^n \}}{r^{n_1} \cdots r^{n_k}} \\ &\leq \frac{\max\{ |a_0|_p, |a_1|_p L, \ldots, |a_n|_p L^n \}}{r^{n_1} \cdots r^{n_k}} \\ &= M. \end{aligned}$$

It follows that

$$|\phi(x)|_p \leq \begin{cases} |\phi(x) - R(x)|_p + |R(x)|_p \leq 1 + M, & \text{if } |x|_p \leq L \text{ and } x \in \overline{D}_{\sigma_A}(r); \\ 1, & \text{if } |x|_p \geq L. \end{cases}$$

Therefore if we set $L_r = M + 1$ then $|\phi(x)|_p \leq L_r$ for all $x \in \overline{D}_{\sigma_A}(r)$. It follows that the set

$$\{ |h(R(x; A)y)|_p \mid x \in \overline{D}_{\sigma_A}(r) \}$$

is bounded. Because $h \in \mathcal{X}^*$ is arbitrary, Lemma 2.12 implies that the set

$$\{ \|R(x; A)y\|_p \mid x \in \overline{D}_{\sigma_A}(r) \}$$

is bounded. Hence, because $y \in \mathcal{X}$ is arbitrary, it follows from the Uniform Boundedness Theorem ([13],Theorem 3.12, p. 65) that the set

$$\{ \|R(x; A)\|_p \mid x \in \overline{D}_{\sigma_A}(r) \}$$

is bounded above by some $M_r > 0$.

Now let $\mathcal{A}$ be a p-adic Banach algebra and let $A \in \mathcal{A}$ be $\mathcal{X}$-analytic with compact $\sigma_A \neq \emptyset$. Then by definition there exists an embedding $\theta : \mathcal{A} \to B(\mathcal{X})$ such that $\theta(A)$ is analytic in $B(\mathcal{X})$-analytic. Let $r > 0$. Then from what we proved above, there exists a constant $M_r > 0$ such that $\|R(x; \theta(A))\|_p \leq M_r$ for all $x \in \overline{D}_{\sigma_{\theta(A)}}(r)$. Let $N_r = (M_r + 1)\|\theta^{-1}\|_p$. Then because $\sigma_{\theta(A)} \subseteq \sigma_A$, it follows that $\overline{D}_{\sigma_A}(r) \subseteq \overline{D}_{\sigma_{\theta(A)}}(r)$, and hence for all $x \in \overline{D}_{\sigma_A}(r)$,

$$\begin{aligned} \|R(x; A)\|_p &= \|\theta^{-1}(R(x; \theta(A)))\|_p \\ &\leq \|\theta^{-1}\|_p \|R(x; \theta(A))\|_p \leq \|\theta^{-1}\|_p (M_r + 1) = N_r. \end{aligned}$$

This completes the proof of the theorem. □

Corollary 2.14. *Let $\mathcal{X} \simeq \Omega_p(J)$, where $J \neq \emptyset$. Let $\mathcal{A}$ be a p-adic Banach algebra. Then for the $\mathcal{X}$-analytic elements in $\mathcal{A}$, Theorem 2.10 (the Extended p-adic Spectral Theorem) reduces to Theorem 1.21 (Vishik's Spectral Theorem II).*

Proof. Let $A \in \mathcal{A}$ have compact spectrum $\sigma_A \neq \emptyset$, and assume that A is $\mathcal{X}$-analytic. Let $f \in L(\sigma_A)$. Let $0 < r \in |\Omega_p|_p$ be such that $f \in B_r(\sigma_A)$, where

$$D_{\sigma_A}(r) = \bigcup_{i=1}^{N} D_{a_i}(r), \quad \sigma_A \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-). \tag{1}$$

Here the union is assumed to be disjoint and each $a_i \in \sigma_A$. Let $\Gamma \in \Omega_p$, with $|\Gamma|_p = r$. By Theorem 2.13, A is $\mathcal{A}$-analytic. By (f) of Lemma 1.8, $L(\sigma_A) = \mathcal{L}(\sigma_A) = \mathcal{F}(\sigma_A)$, and hence we have $f \in \mathcal{F}(\sigma_A)$. It follows from (1) and Theorem 2.10 that

$$f(A) = \sum_{i=1}^{N} \int_{a_i, \Gamma} f(x)(x - a_i) R(x; A)\, dx.$$

Hence Theorem 1.21 and Theorem 2.10 assign the same element of $\mathcal{A}$ to $f(A)$. By (g) of Lemma 1.8, we have $B_{r,a_1,\ldots,a_N} = B_r(\sigma)$, and hence statement (4) of Theorem 2.10 reduces to statement (3) of Theorem 1.21. □

3. p-Adic UHF and TUHF Banach Algebras

Definition 3.1. A *p-adic UHF algebra* over Ω_p is a unital p-adic Banach algebra of the form

$$\mathcal{A} = \overline{\bigcup_{n=1}^{\infty} \mathcal{M}_n},$$

where $(\mathcal{M}_n)$ is an increasing sequence of p-adic Banach subalgebras of $\mathcal{A}$, such that each $\mathcal{M}_n$ contains the identity of $\mathcal{A}$ and is algebraically isomorphic as an Ω_p-algebra to the Ω_p-algebra $M_{p_n}(\Omega_p)$ of $p_n \times p_n$ matrices over Ω_p. A *p-adic TUHF algebra* over Ω_p is a unital p-adic Banach algebra of the form

$$\mathcal{T} = \overline{\bigcup_{n=1}^{\infty} \mathcal{T}_n},$$

where $(\mathcal{T}_n)$ is an increasing sequence of p-adic Banach subalgebras of $\mathcal{T}$, such that each $\mathcal{T}_n$ contains the identity of $\mathcal{T}$ and is algebraically isomorphic as an Ω_p-algebra to the Ω_p-algebra $T_{p_n}(\Omega_p)$ of $p_n \times p_n$ upper triangular matrices over Ω_p. Here, for any positive integer m, the Ω_p-algebra $M_m(\Omega_p)$ of $m \times m$ matrices over Ω_p is identified with $B(\Omega_p^m)$ and $T_m(\Omega_p)$ is the Ω_p-algebra of $m \times m$ upper triangular matrices in $M_m(\Omega_p)$. Finally, Ω_p^m is given the norm $\|(x_1, \ldots, x_m)\| = \max_{1 \le i \le m} |x_i|_p$.

Lemma 3.2. *Let $\mathcal{X}$ be a p-adic Banach space. with $\mathcal{X} \simeq \Omega_p(J)$, where $J \neq \emptyset$ is a finite set. Let $\mathcal{A}$ be a p-adic Banach algebra and assume that there exists a unital Ω_p-monomorphism $\theta : \mathcal{A} \to B(\mathcal{X})$. Then any $A \in \mathcal{A}$ is $\mathcal{X}$-analytic.*

Proof. Let $A \in \mathcal{A}$. Because $B(\mathcal{X})$ is finite-dimensional, the results of finite-dimensional spectral theory apply to operators in $B(\mathcal{X})$. Therefore there exist $x_1, \ldots, x_N$ in Ω_p such that $\sigma_{\theta(A)} = \{ x_1, \ldots, x_N \}$. Moreover, there exist positive integers $\nu(x_1), \ldots, \nu(x_N)$ and idempotents $E(x_1), \ldots, E(x_N)$ in $B(\mathcal{X})$ such that for $x \in \overline{\sigma}_{\theta(A)}$,

$$R(x; \theta(A)) = \sum_{j=1}^{N} \sum_{\nu=0}^{\nu(x_j)-1} \frac{(\theta(A) - x_j)^{\nu}}{(x - x_j)^{\nu+1}} E(x_j). \tag{1}$$

For a proof of (1), see [5], Theorem 10, page 560. Now fix $y \in \mathcal{X}$ and $h \in \mathcal{X}^*$. Then for $x \in \overline{\sigma}_{\theta(A)}$, (1) implies that

$$h(R(x; \theta(A))y) = \sum_{j=1}^{N} \sum_{\nu=0}^{\nu(x_j)-1} \frac{h_{j\nu}}{(x - x_j)^{\nu+1}}, \tag{2}$$

where for $1 \leq j \leq N$ and $0 \leq \nu \leq \nu(x_j) - 1$, $h_{j\nu} = h((\theta(A) - x_j)^\nu E(x_j)y)$. Now, each of the functions $\dfrac{h_{j\nu}}{(x - x_j)^{\nu+1}}$ in (2) is clearly in $H_0(\overline{\sigma}_{\theta(A)})$, hence (2) implies that the function $R_{h,y}(x) = h(R(x;\theta(A))y)$ belongs to $H_0(\overline{\sigma}_{\theta(A)})$. Because $y \in \mathcal{X}$ and $h \in \mathcal{X}^*$ are arbitrary, we see that $\theta(A)$ is analytic. Because $\theta(\mathcal{A})$ is finite-dimensional, we see that $\theta^{-1} : \theta(\mathcal{A}) \to \mathcal{A}$ is bounded and $\theta(\mathcal{A})$ is a p-adic Banach subalgebra of $B(\mathcal{X})$ (see [13], Theorem 3.15, page 69). It follows that A is $\mathcal{X}$-analytic. $\mathcal{X}$-analytic. □

Spectral 3.3. Theorem for p-adic UHF and TUHF Algebras. Let $\mathcal{A}$ be a p-adic UHF or a p-adic TUHF Banach algebra. Let $A \in \mathcal{A}$ have spectrum σ_A. Assume that for all $r > 0$, $\|R(x;A)\|_p$ is bounded on the complement $\overline{D}_{\sigma_A}(r)$ of $D_{\sigma_A}(r^-)$. Then A is $\mathcal{A}$-analytic. Let $f, g \in \mathcal{F}(\sigma_A)$. Let $0 < r \in |\Omega_p|_p$, and let Γ be in Ω_p, with $|\Gamma|_p = r$. Assume that $a_1, \ldots, a_N$ in Ω_p are given, with

$$\sigma_A \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-), \tag{1}$$

where the $D_{a_i}(r)$ are disjoint. Finally, suppose that f and g are Krasner analytic on each $D_{a_i}(r)$. By (1) and Definition 2.7, $f(A)$ and $g(A)$ are well-defined, with

$$f(A) = \sum_{i=1}^{N} \int_{a_i,\Gamma} f(x)(x - a_i)R(x;A)\,dx,$$

$$g(A) = \sum_{i=1}^{N} \int_{a_i,\Gamma} g(x)(x - a_i)R(x;A)\,dx.$$

Let $g \in \mathcal{F}(A)$, and let $\alpha, \beta \in \Omega_p$. Then $\alpha f + \beta g, f \cdot g \in \mathcal{F}(A)$, and

(2) $(\alpha f + \beta g)(A) = \alpha f(A) + \beta g(A)$;

(3) $(f \cdot g)(A) = f(A)g(A)$;

(4) The mapping $h \mapsto h(A)$ is continuous on $B_{r,a_1,\ldots,a_N}$.

Proof. $\mathcal{A}$ has the form

$$\mathcal{A} = \overline{\bigcup_{n=1}^{\infty} \mathcal{A}_n},$$

where for each n, there exists a finite nonempty set J_n and unital Ω_p-monomorphism θ_n from $\mathcal{A}_n$ into $B(\mathcal{X}_n)$, with $\mathcal{X}_n \simeq \Omega_p(J_n)$; because $B(\mathcal{X}_n)$ is finite-dimensional, it follows form Lemma 3.2 that the members of $\mathcal{A}_n$ are $\mathcal{X}_n$-analytic. There exists a sequence (A_n) in

$$\bigcup_{n=1}^{\infty} \mathcal{A}_n$$

such that $\lim_{n\to\infty} \|A - A_n\|_p = 0$. Therefore A satisfies condition (b) of Definition 1.19. By assumption, for all $r > 0$, $\|R(x; A)\|_p$ is bounded on the complement $\overline{D}_{\sigma_A}(r)$ of $D_{\sigma_A}(r^-)$, and hence A satisfies condition (a) of Definition 1.19. It follows that A is $\mathcal{A}$-analytic. Properties (2)-(4) are then consequences of applying Theorem 2.10 to A. □

Theorem 3.4. *Let $\mathcal{A}$ be a p-adic* UHF *or a p-adic* TUHF *Banach algebra. Let $A \in \mathcal{A}$ have spectrum σ_A. Assume that for all $r > 0$, $\|R(x; A)\|_p$ is bounded on the complement $\overline{D}_{\sigma_A}(r)$ of $D_{\sigma_A}(r^-)$. Then A is $\mathcal{A}$-analytic. Let $f \in \mathcal{F}(A)$. Let $0 < r$ be in $|\Omega_p|_p$, and let Γ be in Ω_p, with $|\Gamma|_p = r$. Assume that $a_1, \ldots, a_N$ in Ω_p are given, with*

$$\sigma_A \subseteq \bigcup_{i=1}^{N} D_{a_i}(r^-), \tag{1}$$

where the $D_{a_i}(r)$ are disjoint and f is Krasner analytic on the $D_{a_i}(r)$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $B \in \mathcal{A}$ is $\mathcal{A}$-analytic, with $\|A - B\|_p < \delta$, then

$$\sigma_B \subseteq \bigcup_{i=1}^{N} D_{a_i}(r),$$

$f \in \mathcal{F}(B)$, and $\|f(A) - f(B)\|_p < \epsilon$.

Proof. Because $\mathcal{A}$ and A satisfy the hypothesis of Theorem 3.3 it follows that A is $\mathcal{A}$-analytic. Therefore the present lemma follows from an application of Lemma 2.8. □

Lemma 3.5. *Let $\mathcal{A}$ be a p-adic Banach algebra and let $E \in \mathcal{A}$ be an idempotent. Then for all $r > 0$, $\|R(x; E)\|_p$ is bounded on $\overline{D}_{\sigma_E}(r)$. If $\mathcal{A}$ is a p-adic* UHF *Banach algebra or a p-adic* TUHF *Banach algebra, then E satisfies the hypothesis of Theorem 3.3 and the hypothesis of Theorem 3.4.*

Proof. Let $r > 0$ be arbitrary. Suppose that $E = 0$, then $\sigma_E = \{0\}$. If $x \in \overline{D}_{\sigma_E}(r)$, then $|x|_p \geq r$, and hence $\|R(x; E)\|_p = |x|_p^{-1} \leq \dfrac{1}{r}$. This shows that

$\|R(x;E)\|_p$ is bounded on $\overline{D}_{\sigma_E}(r)$. Now suppose that $E = 1$, then $\sigma_E = \{1\}$. If $x \in \overline{D}_{\sigma_E}(r)$, then $|x-1|_p \geq r$, and hence $\|R(x;E)\|_p = |x-1|_p^{-1} \leq \dfrac{1}{r}$. We conclude that $\|R(x;E)\|_p$ is bounded on $\overline{D}_{\sigma_E}(r)$. Now assume that $E \neq 0, 1$, then $\sigma_E = \{0,1\}$. Let $x \in \Omega_p$, with $|x|_p > \|E\|_p$. Then by (a) of Theorem 2.1, $\left(1 - \dfrac{E}{x}\right)^{-1}$ exists. Hence $(x-E)^{-1} = x^{-1}\left(1 - \dfrac{E}{x}\right)^{-1}$ exists and

$$\begin{aligned} \|(x-E)^{-1}\|_p = |x|_p^{-1}\left\|\left(1-\frac{E}{x}\right)^{-1}\right\|_p &= |x|_p^{-1}\left\|\sum_{n=0}^{\infty}\left(\frac{E}{x}\right)^n\right\|_p \\ &\leq |x|_p^{-1}\sum_{n=0}^{\infty}\left(\frac{\|E\|_p}{|x|_p}\right)^n \\ &= |x|_p^{-1}\left[1 - \frac{\|E\|_p}{|x|_p}\right]^{-1}. \end{aligned}$$

It follows that $\lim_{|x|_p \to \infty} \|R(x;E)\|_p = 0$. Let $M \geq 1, r^2, \|E\|_p$ be so large that

$$\|R(x;E)\|_p \leq 1 \text{ for } |x|_p \geq M.$$

Let $x \in \overline{D}_{\sigma_E}(r)$, with $|x|_p \leq M$. Then $|x|_p = |x-0|_p \geq r$ and $|x-1|_p \geq r$, hence we have

$$\frac{1}{|x|_p} \cdot \frac{1}{|x-1|_p} \leq \frac{1}{r^2}.$$

Because $E \neq 0, 1$ it is easy to see that for $x \neq 0, 1$,

$$R(x;E) = (x-E)^{-1} = \frac{(x-1)+E}{x(x-1)}.$$

Therefore we get

$$\begin{aligned} \|R(x;E)\|_p &= \frac{\|(x-1)+E\|_p}{|x|_p|x-1|_p} \\ &\leq \frac{\max\{|x-1|_p, \|E\|_p\}}{r^2} \\ &\leq \frac{\max\{1, |x|_p, \|E\|_p\}}{r^2} \\ &\leq \frac{\max\{1, M, \|E\|_p\}}{r^2} \\ &= \frac{M}{r^2}. \end{aligned}$$

Thus, if we define $N_r = \dfrac{M}{r^2}$, we see that

$$\|R(x;E)\|_p \leq N_r \text{ for all } x \in \overline{D}_{\sigma_E}(r).$$

Hence for all $r > 0$, $\|R(x;E)\|_p$ is bounded on $\overline{D}_{\sigma_E}(r)$.

Now assume that $\mathcal{A}$ is a p-adic UHF or TUHF algebra, and $E \in \mathcal{A}$ is an idempotent. $\mathcal{A}$ has the from

$$\mathcal{A} = \overline{\bigcup_{n=1}^{\infty} \mathcal{A}_n},$$

where for each n, there exists a finite nonempty set J_n such that $\mathcal{X}_n \simeq \Omega_p(J_n)$ and an embedding $\theta_n : \mathcal{A}_n \to B(\mathcal{X}_n)$; because $B(\mathcal{X}_n)$ is finite-dimensional, it follows form Lemma 3.2 that the members of $\mathcal{A}_n$ are $\mathcal{X}_n$-analytic with nonempty spectrum. There exists a sequence (A_n) in

$$\bigcup_{n=1}^{\infty} \mathcal{A}_n$$

such that $\lim_{n\to\infty} \|E - A_n\|_p = 0$. Now, we proved above that for all $r > 0$, $\|R(x;E)\|_p$ is bounded on $\overline{D}_{\sigma_E}(r)$, hence by Definition 1.19, E is $\mathcal{A}$-analytic. This completes the proof of the theorem. □

As mentioned in the introduction of the present article, Theorem 3.3, Theorem 3.4, and Lemma 3.5 can be used to transfer, *mutatis mutandis*, the results of [2] to p-adic TUHF algebras, which is the content of [4]. The following key lemma is used in [4]; this lemma can be proved using Theorem 3.3, Theorem 3.4, and Lemma 3.5.

Lemma 3.6. *Let $\mathcal{A}$ be a p-adic* UHF *or a p-adic* TUHF *algebra over Ω_p. Let $\epsilon > 0$ be positive number, and let $I = \{\, e_i \mid 1 \leq i \leq n \,\}$ be an orthogonal family of idempotents in $\mathcal{A}$. Then there exists a positive number $\delta(\epsilon, I) > 0$ with the following property. Let $\mathcal{B}$ be a unital p-adic Banach subalgebra of $\mathcal{A}$ such that each member of $\mathcal{B}$ is $\mathcal{A}$-analytic, and suppose that $\{\, a_i \mid 1 \leq i \leq n \,\}$ is a family of elements in $\mathcal{B}$ such that $\|e_i - a_i\| \leq \delta(\epsilon, I)$ for $1 \leq i \leq n$. Then there exists an orthogonal family $\{\, f_i \mid 1 \leq i \leq n \,\}$ of idempotents in $\mathcal{B}$ such that $\|e_i - f_i\| < \epsilon$ for $1 \leq i \leq n$.*

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