International Journal of Pure and Applied Mathematics
Volume 75 No. 2 2012, 237-246
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

NEW STABILITY CRITERIA FOR TAKAGI-SUGENO FUZZY HOPFIELD NEURAL NETWORKS

Choon Ki Ahn
Seoul National University of Science and Technology
172, Gongneung 2-Dong, Nowon-Gu, Seoul, 139-743, KOREA

Abstract: In this paper, new stability criteria are derived for Takagi-Sugeno (T-S) fuzzy Hopfield neural networks via the input/output-to-state stability (IOSS) approach. Based on matrix norm and linear matrix inequality (LMI), these stability criteria guarantee input/output-to-state stability for external input vector. Moreover, the criteria for asymptotic stability of T-S fuzzy Hopfield neural networks without external input vector are presented.

AMS Subject Classification: 92B20, 34A07, 34D23
Key Words: input/output-to-state stability (IOSS), Takagi-Sugeno (T-S) fuzzy Hopfield neural networks, linear matrix inequality (LMI)

1. Introduction

The Takagi-Sugeno (T-S) fuzzy models have been very important in academic research and industrial applications. The T-S fuzzy models use a set of fuzzy rules to describe complex nonlinear systems in terms of a set of local linear systems that are smoothly connected by fuzzy membership functions [12]. These T-S fuzzy models can be also used to describe various complex nonlinear systems by having a set of Hopfield neural networks as its consequent parts. Some stability problems for T-S fuzzy Hopfield neural networks have been investigated in [6, 8, 2, 1].

Received: November 24, 2011

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The input/output-to-state stability (IOSS) approach [10, 7] is accepted as an important concept to analyze the stability of nonlinear systems. It means that no matter what the initial state is, if the inputs and the observed outputs are small, then eventually also the state of the system will become small. The IOSS concept is a nice method for the stability analysis of neural networks because some general conclusions on stability using only input-output characteristics can be obtained. A natural question arises: Can we obtain an IOSS condition for T-S fuzzy Hopfield neural networks? This paper give an answer to this interesting question. To the best of our knowledge, the IOSS analysis of T-S fuzzy Hopfield neural networks has not been published in the literature so far.

In this paper, we propose new stability criteria for T-S fuzzy Hopfield neural networks based on the IOSS approach. The proposed criteria in this paper are a new contribution to the stability analysis of fuzzy neural networks. These criteria are presented based on matrix norm and linear matrix inequality (LMI). The proposed criteria ensure that T-S fuzzy Hopfield neural networks are asymptotically stable and input/output-to-state stable for an external input vector. This paper is organized as follows. In Section 2, new IOSS conditions for T-S fuzzy Hopfield neural networks are derived, and finally, conclusions are presented in Section 3.

2. New Stability Criteria

We consider the following T-S fuzzy Hopfield neural network [6, 2]:

Fuzzy Rule i:

IF ω_1 is μ_{i1} and $\ldots$ ω_s is μ_{is} **THEN**

$$\dot{x}(t) = A_i x(t) + W_i \phi(x(t)) + J(t), \tag{1}$$

$$y(t) = C_i x(t), \tag{2}$$

where $x(t) = [x_1(t) \ldots x_n(t)]^T \in R^n$ is the state vector, $y(t) = [y_1(t) \ldots y_m(t)]^T \in R^m$ is the output vector, $A_i = diag\{-a_{(i,1)}, \ldots, -a_{(i,n)}\} \in R^{n\times n}$ ($a_{(i,k)} > 0, k = 1, \ldots, n$) is the self-feedback matrix, $W_i \in R^{n\times n}$ is the connection weight matrix, $\phi(x(t)) = [\phi_1(x(t)) \ldots \phi_n(x(t))]^T : R^n \to R^n$ is the nonlinear function vector satisfying the global Lipschitz condition with Lipschitz constant $L_\phi > 0$, $J(t) \in R^n$ is an external input vector, $C_i \in R^{m\times n}$ is a known constant matrix, ω_j ($j = 1, \ldots, s$) is the premise variable, μ_{ij} ($i = 1, \ldots, r, j = 1, \ldots, s$) is the fuzzy set that is characterized by membership function, r is the number of the

IF-THEN rules, and s is the number of the premise variables. Using a standard fuzzy inference method, the system (1) is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(\omega)[A_i x(t) + W_i \phi(x(t)) + J(t)], \tag{3}$$

$$y(t) = \sum_{i=1}^{r} h_i(\omega) C_i x(t), \tag{4}$$

where $\omega = [\omega_1, \ldots, \omega_s]$, $h_i(\omega) = w_i(\omega)/\sum_{j=1}^{r} w_j(\omega)$, $w_i : R^s \to [0,1]$ $(i = 1, \ldots, r)$ is the membership function of the system with respect to the fuzzy rule i. h_i satisfies $h_i(\omega) \geq 0$ and $\sum_{i=1}^{r} h_i(\omega) = 1$. Now, we introduce the following definitions:

Definition 1. A function $\gamma : R_{\geq 0} \to R_{\geq 0}$ is a $\mathcal{K}$ function if it is continuous, strictly increasing and $\gamma(0) = 0$. It is a $\mathcal{K}_\infty$ function if it is a $\mathcal{K}$ function and $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0}$ is a $\mathcal{KL}$ function if, for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$ function, and for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s,t) \to 0$ as $t \to \infty$.

Definition 2. The nonlinear system $\dot{X}(t) = f(X(t), U(t))$ and $Y(t) = h(X(t))$, where $X(t) \in R^n, Y(t) \in R^m, U(t) \in R^p$, is said to be input/output-to-state stable if there exist $\mathcal{K}$ functions $\gamma_1(s)$, $\gamma_2(s)$ and a $\mathcal{K}L$ function $\beta(s,t)$, such that, for each input $U(t)$, each output $Y(t)$, and each initial state $X(0)$, it holds that

$$\|X(t)\| \leq \max\left\{\beta(\|X(0)\|, t),\ \gamma_1\left(\sup_{0\leq\tau\leq t}\|U(\tau)\|\right),\ \gamma_2\left(\sup_{0\leq\tau\leq t}\|Y(\tau)\|\right)\right\} \tag{5}$$

for each $t \geq 0$.

First, a new IOSS criterion for the T-S fuzzy Hopfield neural network (3)-(4) is proposed in the following theorem:

Theorem 1. *The T-S fuzzy Hopfield neural network (3)-(4) is input/output-to-state stable if*

$$\|W_i\| < \frac{1}{L_\phi}\sqrt{\frac{\gamma_i - \|P\|\|C_i\|\|C_j\| - \|P\| - \|P\|^2}{\|P\|}}, \tag{6}$$

$$\|P\| < \frac{-\|C_i\|\|C_j\| - 1 + \sqrt{(\|C_i\|\|C_j\| + 1)^2 + 4\gamma_i}}{2}, \tag{7}$$

$$\gamma_i > 0, \qquad P = P^T > 0, \tag{8}$$

where P satisfies the Lyapunov inequality $A_i^T P + PA_i < -\gamma_i I$ for $i = 1, ..., r$ and $j = 1, ..., r$.

Proof. First, consider the Lyapunov function $V(t) = x^T(t)Px(t)$. This Lyapunov function $V(t)$ satisfies the following Rayleigh inequality [11]:

$$\lambda_{\min}(P)\|x(t)\|^2 \le V(t) \le \lambda_{\max}(P)\|x(t)\|^2, \tag{9}$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the maximum and minimum eigenvalues of the matrix. The time derivative of $V(t)$ along the trajectory of the T-S fuzzy Hopfield neural network (3) satisfies

$$\dot{V}(t) < \sum_{i=1}^{r} h_i(\omega)\bigg\{ -\gamma_i x^T(t)x(t) + 2x^T(t)PW_i\phi(x(t)) + 2x^T(t)PJ(t)\bigg\}. \tag{10}$$

If we add and subtract $\sum_{i=1}^{r} h_i(\omega)\sum_{j=1}^{r} h_j(\omega)x^T(t)C_i^T PC_j x(t)$, we obtain the following inequality:

$$\begin{aligned}\dot{V}(t) < \sum_{i=1}^{r} h_i(\omega)\sum_{j=1}^{r} h_j(\omega)\bigg\{ &-x^T(t)(\gamma_i I + C_i^T PC_j)x(t) + 2x^T(t)PW_i\phi(x(t)) \\ &+ 2x^T(t)PJ(t)\bigg\} + y^T(t)Py(t).\end{aligned} \tag{11}$$

By Young's inequality [3], we have

$$\begin{aligned}2x^T(t)PW\phi(x(t)) &\le x^T(t)Px(t) + (PW\phi(x(t)))^T P^{-1}(PW\phi(x(t))) \\ &\le \|P\|\|x(t)\|^2 + L_\phi^2\|P\|\|W\|^2\|x(t)\|^2\end{aligned} \tag{12}$$

and

$$\begin{aligned}2x^T(t)PJ(t) &\le x^T(t)PP^T x(t) + J^T(t)J(t) \\ &\le \|P\|^2\|x(t)\|^2 + \|J(t)\|^2.\end{aligned} \tag{13}$$

If we substitute (12) and (13) into (10), we obtain

$$\begin{aligned}\dot{V}(t) &< \sum_{i=1}^{r} h_i(\omega)\sum_{j=1}^{r} h_j(\omega)\bigg\{ -\big(\gamma_i - \|P\|\|C_i\|\|C_j\| - \|P\| - \|P\|^2 - L_\phi^2\|P\|\|W_i\|^2\big) \\ &\quad \times \|x(t)\|^2 + \|J(t)\|^2 + \|P\|\|y(t)\|^2\bigg\} \\ &= -\sum_{i=1}^{r} h_i(\omega)\sum_{j=1}^{r} h_j(\omega)\left[\gamma_i - \|P\|\|C_i\|\|C_j\| - \|P\| - \|P\|^2 - L_\phi^2\|P\|\|W_i\|^2\right]\|x(t)\|^2\end{aligned}$$

$$+\|J(t)\|^2+\|P\|\|y(t)\|^2 \tag{14}$$
$$=-\alpha_1(\|x(t)\|)+\theta_1(\|J(t)\|)+\mu_1(\|y(t)\|), \tag{15}$$

where

$$\alpha_1(r)=\sum_{i=1}^{r}h_i(\omega)\sum_{j=1}^{r}h_j(\omega)\left[\gamma_i-\|P\|\|C_i\|\|C_j\|-\|P\|-\|P\|^2-L_\phi^2\|P\|\|W_i\|^2\right]r^2,$$
$$\theta_1(r)=r^2,$$
$$\mu_1(r)=\|P\|r^2.$$

If $\alpha_1(\cdot)$, $\theta_1(\cdot)$, and $\mu_1(\cdot)$ are class $\mathcal{K}_\infty$ functions, $V(t)$ is an IOSS-Lyapunov function [9] from (9) and (15). Note that $\theta_1(\cdot)$ and $\mu_1(\cdot)$ are class $\mathcal{K}_\infty$ functions. Thus, if $\gamma_i-\|P\|\|C_i\|\|C_j\|-\|P\|-\|P\|^2-L_\phi^2\|P\|\|W_i\|^2>0$ for $i=1,...,r$ and $j=1,...,r$, the T-S fuzzy Hopfield neural network (3)-(4) is IOSS. This inequality implies

$$\|W_i\|^2<\frac{\gamma_i-\|P\|\|C_i\|\|C_j\|-\|P\|-\|P\|^2}{L_\phi^2\|P\|},$$
$$\|P\|<\frac{-\|C_i\|\|C_j\|-1+\sqrt{(\|C_i\|\|C_j\|+1)^2+4\gamma_i}}{2},$$

for $i=1,...,r$ and $j=1,...,r$. This completes the proof. □

Corollary 1. *When $J(t)=0$, the T-S fuzzy Hopfield neural network (3)-(4) is asymptotically stable if*

$$\|W_i\|<\frac{1}{L_\phi}\sqrt{\frac{\gamma_i-2\|P\|\|C_i\|\|C_j\|-\|P\|-\|P\|^2}{\|P\|}}, \tag{16}$$
$$\|P\|<\frac{-2\|C_i\|\|C_j\|-1+\sqrt{(2\|C_i\|\|C_j\|+1)^2+4\gamma_i}}{2}, \tag{17}$$
$$\gamma_i>0, \qquad P=P^T>0, \tag{18}$$

where P satisfies the Lyapunov inequality $A_i^TP+PA_i<-\gamma_iI$ for $i=1,...,r$ and $j=1,...,r$.

Proof. When $J(t)=0$, from (14), we have

$$\dot V(t)<-\sum_{i=1}^{r}h_i(\omega)\sum_{j=1}^{r}h_j(\omega)\left[\gamma_i-\|P\|\|C_i\|\|C_j\|-\|P\|-\|P\|^2-L_\phi^2\|P\|\|W_i\|^2\right]\|x(t)\|^2$$
$$+\|P\|\|y(t)\|^2$$

$$\leq -\sum_{i=1}^{r} h_i(\omega) \sum_{j=1}^{r} h_j(\omega) \left[\gamma_i - 2\|P\|\|C_i\|\|C_j\| - \|P\| - \|P\|^2 - L_\phi^2 \|P\|\|W_i\|^2\right] \times \|x(t)\|^2. \tag{19}$$

If $\gamma_i - 2\|P\|\|C_i\|\|C_j\| - \|P\| - \|P\|^2 - L_\phi^2\|P\|\|W_i\|^2 > 0$ for $i = 1, ..., r$ and $j = 1, ..., r$, the T-S fuzzy Hopfield neural network (3)-(4) is asymptotically stable from Lyapunov stability theory. This inequality implies

$$\|W_i\|^2 < \frac{\gamma_i - 2\|P\|\|C_i\|\|C_j\| - \|P\| - \|P\|^2}{L_\phi^2\|P\|},$$

$$\|P\| < \frac{-2\|C_i\|\|C_j\| - 1 + \sqrt{(2\|C_i\|\|C_j\| + 1)^2 + 4\gamma_i}}{2},$$

for $i = 1, ..., r$ and $j = 1, ..., r$. This completes the proof. □

Next, a new LMI criterion for the IOSS of the T-S fuzzy Hopfield neural network (3)-(4) is proposed. This LMI criterion can be checked easily by standard numerical algorithms [4, 5].

Theorem 2. *The T-S fuzzy Hopfield neural network (3)-(4) is input/output-to-state stable if there exist positive symmetric matrices P, S_1, S_2, S_3 and a positive scalar ϵ such that*

$$\begin{bmatrix} A_i^T P + PA_i + S_1 - C_i^T S_3 C_j + \epsilon L_\phi^2 & PW_i & P \\ W_i^T P & -\epsilon I & 0 \\ P & 0 & -S_2 \end{bmatrix} < 0, \tag{20}$$

for $i = 1, ..., r$ and $j = 1, ..., r$.

Proof. Consider the Lyapunov function $V(t) = x^T(t)Px(t)$. By Young's inequality [3], we have the following inequality:

$$\epsilon[L_\phi^2 x^T(t)x(t) - \phi^T(x(t))\phi(x(t))] \geq 0. \tag{21}$$

By using (21), the time derivative of $V(t)$ along the trajectory of the T-S fuzzy Hopfield neural network (3)-(4) satisfies

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{r} h_i(\omega)\Big\{x^T(t)[A_i^T P + PA_i]x(t) + 2x^T(t)PW_i\phi(x(t)) + 2x^T(t)PJ(t)\Big\} \\ &\leq \sum_{i=1}^{r} h_i(\omega)\Big\{x^T(t)[A_i^T P + PA_i + \epsilon L_\phi^2]x(t) + 2x^T(t)PW_i\phi(x(t)) \qquad (22) \\ &\quad + 2x^T(t)PJ(t) \end{aligned}$$

$$- \epsilon \phi^T(x(t))\phi(x(t))\Big\}.$$

If we add and subtract $\sum_{i=1}^{r} h_i(\omega) \sum_{j=1}^{r} h_j(\omega) x^T(t) C_i^T S_3 C_j x(t)$, we obtain

$$\begin{aligned}
\dot{V}(t) \leq & \sum_{i=1}^{r} h_i(\omega) \sum_{j=1}^{r} h_j(\omega) \Big\{ x^T(t)[A_i^T P + PA_i - C_i^T S_3 C_j + \epsilon L_\phi^2] x(t) \\
& + 2x^T(t) PW_i \phi(x(t)) \\
& + 2x^T(t) PJ(t) - \epsilon \phi^T(x(t))\phi(x(t)) \Big\} \\
& + \sum_{i=1}^{r} h_i(\omega) \sum_{j=1}^{r} h_j(\omega) x^T(t) C_i^T S_3 C_j x(t) \\
& = \sum_{i=1}^{r} h_i(\omega) \sum_{j=1}^{r} h_j(\omega) \begin{bmatrix} x(t) \\ \phi(x(t)) \\ J(t) \end{bmatrix}^T \\
& \times \begin{bmatrix} A_i^T P + PA_i + S_1 - C_i^T S_3 C_j + \epsilon L_\phi^2 & PW_i & P \\ W_i^T P & -\epsilon I & 0 \\ P & 0 & -S_2 \end{bmatrix} \\
& \times \begin{bmatrix} x(t) \\ \phi(x(t)) \\ J(t) \end{bmatrix} - x^T(t) S_1 x(t) + J^T(t) S_2 J(t) \\
& + \sum_{i=1}^{r} h_i(\omega) \sum_{j=1}^{r} h_j(\omega) x^T(t) C_i^T S_3 C_j x(t). \qquad (23)
\end{aligned}$$

If the LMI (20) is satisfied for $i = 1, ..., r$ and $j = 1, ..., r$, we have

$$\begin{aligned}
\dot{V}(t) < & - x^T(t) S_1 x(t) + J^T(t) S_2 J(t) \\
& + \sum_{i=1}^{r} h_i(\omega) \sum_{j=1}^{r} h_j(\omega) x^T(t) C_i^T S_3 C_j x(t) \qquad (24) \\
& = -x^T(t) S_1 x(t) + J^T(t) S_2 J(t) + y^T(t) S_3 y(t) \\
& \leq -\lambda_{\min}(S_1)\|x(t)\|^2 + \lambda_{\max}(S_2)\|J(t)\|^2 + \lambda_{\max}(S_3)\|y(t)\|^2 \\
& = -\alpha_2(\|x(t)\|) + \theta_2(\|J(t)\|) + \mu_2(\|y(t)\|), \qquad (25)
\end{aligned}$$

where

$$\alpha_2(r) = \lambda_{\min}(S_1) r^2,$$

$$\theta_2(r) = \lambda_{\max}(S_2)r^2,$$
$$\mu_2(r) = \lambda_{\max}(S_3)r^2.$$

Because $\alpha_2(\cdot)$, $\theta_2(\cdot)$, and $\mu_2(\cdot)$ are class $\mathcal{K}_\infty$ functions, $V(t)$ is an IOSS-Lyapunov function [9] from (9) and (25). Thus, the T-S fuzzy Hopfield neural network (3)-(4) is IOSS. This completes the proof. □

Corollary 2. *When $J(t) = 0$, the T-S fuzzy Hopfield neural network (3)-(4) is asymptotically stable if there exist positive symmetric matrices P, S_1, S_2, S_3 and a positive scalar ϵ satisfying*

$$\begin{bmatrix} A_i^T P + PA_i + S_1 - C_i^T S_3 C_j + \epsilon L_\phi^2 & PW_i & P \\ W_i^T P & -\epsilon I & 0 \\ P & 0 & -S_2 \end{bmatrix} < 0, \tag{26}$$

$$S_1 - C_i^T S_3 C_j > 0, \tag{27}$$

for $i = 1, ..., r$ and $j = 1, ..., r$.

Proof. When $J(t) = 0$, from (24), we have

$$\begin{aligned} \dot{V}(t) &< -x^T(t)S_1x(t) + \sum_{i=1}^{r} h_i(\omega) \sum_{j=1}^{r} h_j(\omega)x^T(t)C_i^T S_3 C_j x(t) \\ &= -\sum_{i=1}^{r} h_i(\omega) \sum_{j=1}^{r} h_j(\omega)x^T(t)(S_1 - C_i^T S_3 C_j)x(t). \end{aligned} \tag{28}$$

If the LMI (27) is satisfied for $i = 1, ..., r$ and $j = 1, ..., r$, $\dot{V}(t) < 0, \forall x(t) \neq 0$. This ensures that the T-S fuzzy Hopfield neural network (3)-(4) is asymptotically stable from Lyapunov stability theory. This completes the proof. □

3. Conclusion

In this paper, new stability criteria for T-S fuzzy Hopfield neural networks have been proposed based on the IOSS approach. The presented criteria were represented by matrix norm and LMI. These criteria ensure that T-S fuzzy Hopfield neural networks are asymptotically stable and input/output-to-state stable for the external input vector.

Acknowledgments

This work was supported by the Grant of the Korean Ministry of Education, Science and Technology (The Regional Core Research Program/Center for Healthcare Technology Development).

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