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WEAK IDEAL CONVERGENCE IN l_p SPACES

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Abstract: In this paper, we see that how weak ideal convergence 'looks like' in l_p spaces. We extend the recently introduced concept of weak* statistical convergence to have a new concept of weak* ideal convergence. We give the necessary and sufficient conditions for a sequence of bounded linear functionals on a Banach space to be weak* ideal convergent.

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1. Introduction

The idea of statistical convergence was given in the first edition (published in Warsaw in 1935) of the monograph of Zygmund [22]. Formally the concept of statistical convergence was introduced by Steinhaus [21] and Fast [7] and has been studied by many authors(see, for instance [4], [8], [9], [10], [13], [14], [18] and [22]). In [5], Connor et al. introduced the concept of weak statistical convergence and characterized Banach spaces with seperable duals via weak statistical convergence. Bhardwaj and Bala [3] have introduced a new concept of weak statistically Cauchy sequence in a normed space and it has been shown that in a reflexive space, weak statistically Cauchy sequences are the same as weakly

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statistically convergent sequences. Pehlivan and Karaev [16] have also used the idea of weak statistical convergence in strengthening a result of Gokhberg and Krein on compact operators. Quite recently, Bala [1] has introduced a new concept of weak* statistical convergence of sequence of functionals.

Using the notion of the ideal I of subsets of the set N of positive integers, Kostyrko et al. [11] generalized the concept of statistical convergence and introduced a new concept called ideal convergence(briefly I-convergence). This convergence is very important in the sense that it includes many types of convergence studied in contemporary analysis. For a detailed discussion of I-convergence, one may refer to [11], [12], [19] and [20].

Following [3], [5], Pehlivan et al. [17] have introduced the concepts of weak I-convergence and weak I-Cauchy sequence in a normed space, and investigated their basic properties. In this paper, we extend some notions and results known for the weak statistical convergence to the weak ideal convergence. Our main interest in this paper is to see that how weak ideal convergence 'looks like' in l_p spaces. Moreover, following Bala [1], we introduce the concept of weak* ideal convergence for a sequence of functionals and give the necessary and sufficient conditions for a sequence of bounded linear functionals on a Banach space to be weak* ideal convergent.

Before proceeding with the main results, we recall some terminology and notations.

Let X be a non-empty set. Then $\phi \neq I \subset 2^X$ (power set of X) is said to be an ideal in X if I is finitely additive i.e. $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e. $A \in I$, $B \subset A \Rightarrow B \in I$.

An ideal $I \subset 2^X$ is said to be non-trivial if $X \notin I$ i.e. $I \neq 2^X$.

A non-trivial ideal $I \subset 2^X$ is said to be admissible if I contains all the finite subsets of X.

A non-empty family $F \subset 2^X$ is said to be a filter on X if $\phi \notin F$; for $A, B \in F$, we have $A \cap B \in F$ and for each $A \in F$ and $A \subset B$ implies $B \in F$.

For each non-trivial ideal I in X, there corresponds a filter $F(I)$, having elements as the complements of elements of I i.e. $F(I) = \{K \subset X : K^c \in I\}$, where $K^c = X - K$.

Throughout this paper, I will denote an admissible ideal in $N; X, X'$ will denote a normed linear space $(X, \|\cdot\|)$ and its continuous dual, respectively and $w(X)$ will denote the set of all sequences with elements from X.

A sequence $(x_n) \in w(X)$ is said to be I-convergent [20] to $x \in X$ if for each $\varepsilon > 0$, the set $\{n \in N : \|x_n - x\| \geq \varepsilon\} \in I$. The element x is called the I-limit of the sequence (x_n) and we write $I - \lim_{n\to\infty} x_n = x$.

If we consider the class I_f of all finite subsets of N, then I_f is an admissible ideal and I_f-convergence coincides with the usual convergence in a normed space X.

If we take the family $I_d = \{A \subset N : \delta(A) = 0\}$ where $\delta(A) = \lim_{n\to\infty} \frac{card(\{1,2,3,\dots n\} \cap A)}{n}$ is the natural density of A (see [15]) provided that the limit exists, then I_d forms an admissible ideal and for $I = I_d$, I-convergence coincides with the (norm) statistical convergence (see [3], [5] and [10]).

For further examples of I-convergence, one may refer to [11].

A sequence $(x_n) \in w(X)$ is said to be I-bounded [20] if there exists $M > 0$ such that the set

$$\{n \in N : \|x_n\| > M\} \in I.$$

Following [6] we call a sequence $(x_n) \in w(X)$ to be I-Cauchy sequence if for a given $\varepsilon > 0$, there exists $p(\varepsilon) \in N$ such that the set

$$\{n \in N : \|x_n - x_{p(\varepsilon)}\| \geq \varepsilon\} \in I.$$

Finally we recall [17] the concepts of weak ideal convergence and weak ideal Cauchy sequence in a normed space.

A sequence $(x_n) \in w(X)$ is weakly I-convergent to some $x \in X$ if for each $\varepsilon > 0$ and for each $f \in X'$, the set $\{n \in N : |f(x_n) - f(x)| \geq \varepsilon\} \in I$. In this case, x is called the weak I-limit of (x_n) and we write $w - I - \lim x_n = x$.

A sequence $(x_n) \in w(X)$ is said to be weak I-Cauchy if for each $\varepsilon > 0$ and $f \in X'$, the sequence $(f(x_n))$ is I-Cauchy sequence of scalars.

2. Supporting Lemmas

The following elementary lemmas are well-known to the people working with ideal convergence. We just formulate them in the way we need.

Lemma 2.1. (see [12]) *For $(x_n), (y_n) \in w(X)$, we have*

(i) *if $\lim_{n\to\infty} x_n = x$, then $I - \lim_{n\to\infty} x_n = x$.*

(ii) *if $I - \lim_{n\to\infty} x_n = x, I - \lim_{n\to\infty} y_n = y$, then $I - \lim_{n\to\infty} (x_n + y_n) = x + y$.*

(iii) *for any scalar $\alpha, I - \lim_{n\to\infty} x_n = x$ implies that $I - \lim_{n\to\infty} \alpha x_n = \alpha x$.*

Lemma 2.2. (see [6]) *If a sequence $(x_n) \in w(X)$ is I-convergent, then it is I-Cauchy.*

In strict analogy with this lemma we have a simple

Lemma 2.3. *A weakly I-convergent sequence $(x_n) \in w(X)$ is weak I-Cauchy.*

Lemma 2.4. (see [11]) *A function $g : X \to X$ preserves I-convergence in X if and only if g is continuous on X.*

We now establish some more lemmas which will form a basis of our study in the forthcoming sections.

Lemma 2.5. *An I-Cauchy sequence $(x_n) \in w(X)$ is I-bounded.*

Proof being straightforward is omitted.

Motivated from Theorem 3 of Tripathy [22, p.261], we have an interesting

Lemma 2.6. *If there are two real sequences (x_n) and (y_n) for which there exists $K \in F(I)$ such that $x_n \leq y_n$ for every $n \in K$ and $I - \lim x_n, I - \lim y_n$ exist, then $I - \lim x_n \leq I - \lim y_n$.*

Proof. Suppose $I - \lim x_n = \xi$ and $I - \lim y_n = \eta$.

By given condition, $\{n \in N : x_n \leq y_n\} = K \in F(I)$.

Let, if possible, $\xi > \eta$ and take $\varepsilon = \dfrac{\xi - \eta}{2}$.

Then the sets $\{n \in N : |x_n - \xi| < \varepsilon\}$ and $\{n \in N : |y_n - \eta| < \varepsilon\}$ belong to $F(I)$. Also $\{n \in N : x_n > \xi - \varepsilon\} \supset \{n \in N : |x_n - \xi| < \varepsilon\}$ and $\{n \in N : y_n < \eta + \varepsilon\} \supset \{n \in N : |y_n - \eta| < \varepsilon\}$ and therefore $\{n \in N : y_n < x_n\} \cap \{n \in N : x_n \leq y_n\} \in F(I)$. It is easy to see that $\{n \in N : x_n > \xi - \varepsilon\} \cap \{n \in N : y_n < \eta + \varepsilon\} \subset \{n \in N : y_n < x_n\}$ which implies that $\{n \in N : y_n < x_n\} \in F(I)$ and hence $\phi = \{n \in N : y_n < x_n\} \cap \{n \in N : x_n \leq y_n\} \in F(I)$ which is a contradiction. □

We now present a characterization of I-bounded sequences in the following

Lemma 2.7. *A sequence $(x_n) \in w(X)$ I-bounded if and only if there exists a set $K = \{n_1 < n_2 < \ldots < n_k < \ldots\} \in F(I)$ such that the sequence (x_{n_k}) is bounded in the ordinary sense.*

Proof. Suppose that the sequence (x_n) is I-bounded. Then there exists a real number $M > 0$ such that the set $A = \{n \in N : \|x_n\| > M\} \in I$ and therefore, the set $B = N - A = \{n \in N : \|x_n\| \leq M\} \in F(I)$. Now B is infinite, because otherwise $B \in I$, I being admissible and therefore $A \cup B = N \in I$ and consequently I is a trivial ideal which is not so. Let $\{n_1 < n_2 < \ldots < n_k < \ldots\}$ be the enumeration of B. Then $\|x_{n_k}\| \leq M$ for all $k \in N$ and hence (x_{n_k}) is bounded in the ordinary sense.

Conversely, suppose that there exists a set $K = \{n_1 < n_2 < \ldots < n_k < \ldots\} \in F(I)$ such that (x_{n_k}) is bounded in the ordinary sense. By hypothesis, there exists a real number $M > 0$ such that $\|x_{n_k}\| \leq M$ for all $k \in N$. The set $\{n \in N : \|x_n\| > M\} \subset \{n \in N : n \leq n_1\} \cup (N - K) \in I$ and therefore the sequence (x_n) is I-bounded. □

As an application of lemma 2.4 and 2.6, we have

Lemma 2.8. *Every weak I-Cauchy sequence is I-bounded.*

Proof. Let $(x_n) \in w(X)$ be a weak I-Cauchy sequence. Then for each $f \in X'$, the sequence $(f(x_n))$ is I-Cauchy. Then, by lemma 2.4, the sequence $(f(x_n))$ is I-bounded and then by using lemma 2.6, there exists a set $K = \{n_1 < n_2 < \ldots < n_k < \ldots\} \in F(I)$ such that the sequence $(f(x_{n_k}))$ is bounded. Consider the canonical mapping $C : X \to X''$ defined by $C(x) = g_x$ for each $x \in X$ where $g_x \in X''$ is given by $g_x(f) = f(x)$ for every $f \in X'$. Therefore

$$\|g_x\| = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|g_x(f)|}{\|f\|} = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{|f(x)|}{\|f\|} = \|x\|$$

and

$$\sup_k |g_{x_{n_k}}(f)| = \sup_k |f(x_{n_k})| < \infty \quad \text{for each} \quad f \in X'.$$

Since X' is a Banach space, by using uniform boundedness theorem, we have $\sup_k \|g_{x_{n_k}}\| < \infty$ which implies that $\sup_k \|x_{n_k}\| < \infty$. Then, by lemma 2.7, the sequence (x_n) is I-bounded. □

3. Weak Ideal Convergence in l_p $(1 < p < \infty)$

Loosely speaking, in this section, we see that how weak ideal convergence 'looks like'in l_p spaces.

Theorem 3.1. *In the space $l_p(1 < p < \infty)$, we have $w - I - \lim x_k = x$ if and only if*

(i) *the sequence $(\|x_k\|)$ is I-bounded;*

(ii) *for every fixed j, we have $I - \lim x_j^{(k)} = x_j$; here $x_k = (x_j^{(k)})$ and $x = (x_j)$.*

The proof is completely analogous to the classical theorem (see [2, p. 236]), once we establish the following lemma.

Lemma 3.2. *For any sequence* $(x_k) \in w(X)$*, we have* $w - I - \lim x_k = x$ *if and only if*

(i) *the sequence* $(\|x_k\|)$ *is* I*-bounded;*

(ii) *for every element* f *of a total subset* $M \subset X'$*, we have* $I - \lim f(x_k) = f(x)$*.*

Proof. First suppose that $w - I - \lim x_k = x$. Then, by using lemma 2.3 and 2.5, we have the sequence $(\|x_k\|)$ is I -bounded. Thus (i) holds.

Since $w - I - \lim x_k = x$, therefore $I - \lim f(x_k) = f(x)$ for all $f \in X'$. In particular, $I - \lim f(x_k) = f(x)$ for all $f \in M \subset X'$ where M is a total subset of X'. Thus (ii) holds.

Conversely, suppose that the conditions (i) and (ii) hold. We prove that $w - I - \lim x_k = x$. Let $h \in X'$. We will show that $I - \lim h(x_k) = h(x)$. This will be done in two steps. First, it will be shown that the result is true for all $h \in spanM$ and then for $h \in \overline{spanM}$. To prove the first conclusion, let $g \in spanM$. Then $g = \sum_{i=1}^{n} \alpha_i f_i$ for $f_1, f_2, \ldots, f_n \in M$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$. By hypothesis (ii), $I - \lim f_i(x_k) = f_i(x)$ for all $i, 1 \leq i \leq n$ and hence $I - \lim g(x_k) = g(x)$ by lemma 2.1. Thus the first conclusion is established.

For the second conclusion, suppose $h \in \overline{spanM}$. By hypothesis (i), there exists a constant $c > 0$ such that the set $\{n \in N : \|x_n\| > c\} \in I$ i.e. the set $A = \{n \in N : \|x_n\| \leq c\} \in F(I)$. Therefore for any $f \in M \subset X'$, there exists $A \in F(I)$ such that $|f(x_k)| \leq \|f\|\|x_k\| < c\|f\|$ for every $k \in A$, which by lemma 2.6 gives that $I - \lim |f(x_k)| \leq c\|f\|$. Again by using lemma 2.4, we have $|f(x)| \leq c\|f\|$ which implies that $\|x\| \leq c$. Since $h \in \overline{spanM}$, for a given $\varepsilon > 0$, there exists some $g_j \in spanM$ such that $\|h - g_j\| < \frac{\epsilon}{3c}$.

Consider

$$\begin{aligned} |h(x_k) - h(x)| &\leq \|h - g_j\|\|x_k\| + |g_j(x_k) - g_j(x)| + \|g_j - h\|\|x\| \\ &< \frac{\varepsilon}{3c}\|x_k\| + |g_j(x_k) - g_j(x)| + \frac{\varepsilon}{3c}\|x\| \\ &< \frac{\varepsilon}{3c}c + |g_j(x_k) - g_j(x)| + \frac{\epsilon}{3c}c \quad \text{for all} \quad k \in A. \end{aligned}$$

Since $g_j \in spanM$, so by first part of the proof, $I - \lim g_j(x_k) = g_j(x)$ and therefore the set

$$B = \left\{ k \in N : |g_j(x_k) - g_j(x)| < \frac{\varepsilon}{3} \right\} \in F(I).$$

Thus $|h(x_k) - h(x)| < \varepsilon$ for $k \in A \cap B \in F(I)$.
Hence $w - I - \lim x_k = x$ and the proof is complete. □

A simple application of Riesz representation theorem for bounded linear functionals on a Hilbert space yields the following

Proposition 3.3. *In a Hilbert space H, $w - I - \lim x_k = x$ if and only if*

$$I - \lim\langle x_k, y\rangle = \langle x, y\rangle \quad \text{for all } \ y \in H.$$

4. Weak* Ideal Convergence

Following Bala [1], we now introduce the concept of weak* ideal convergence of sequence of functionals and present some results.

Definition 4.1. A sequence (f_n) in X' is said to be weak* I-convergent to $f \in X'$ if for each $x \in X$, the sequence $(f_n(x))$ is I-convergent to $f(x)$. It means that for each $x \in X$ and $\varepsilon > 0$, the set $\{n \in N : |f(x_n) - f(x)| \geq \varepsilon\} \in I$. In this case, f is called the weak* I-limit of (f_n) and we write $w^* - I - \lim f_n = f$.

The next theorem shows that weak* I-convergence is a generalization of the usual notion of weak* convergence.

Theorem 4.2. *Let (f_k) be a weak* convergent sequence in X' and $w^* - \lim f_k = f$. Then (f_k) is weak* I-convergent to f. The converse is not generally true.*

Proof. Since $w^* - \lim f_k = f$, therefore for each $x \in X$, the sequence $(f_k(x))$ is convergent to $f(x)$. But usual convergence implies I-convergence, by lemma 2.1(i). Therefore $I - \lim f_k(x) = f(x)$ and hence $w^* - I - \lim f_k = f$.

To show that the converse need not be true, we proceed as in [1, p. 650]. For $x = (x_k) \in l_1$, define

$$f_k(x) = \begin{cases} x_1 & \text{if } k = m^2 \\ x_k & \text{if } k \neq m^2 \end{cases}$$

Then $f_k \in l_1'$ for all $k \in N$. It is easy to see that $w^* - I - \lim f_k = 0$ for $I = I_d$ but $w^* - \lim f_k \neq 0$. □

A slight modification of theorem 2.3 and 2.4 of [1, p.650] gives

Theorem 4.3. (i) *If a sequence (f_k) in X' is weakly I-convergent to $f \in X'$, then it is weak* I-convergent.*

(ii) *Let X be a reflexive normed space. If a sequence (f_k) in X' is weak* I-convergent to $f \in X'$, then it is weakly I-convergent.*

Our next result shows that in a Banach space, every weak* I-convergent is I-bounded.

Lemma 4.4. *Let X be a Banach space. If a sequence (f_k) in X' is weak* I-convergent, then $(\|f_k\|)$ is I-bounded.*

Proof. Let $\varepsilon > 0$ be given. Since (f_k) is weak* I-convergent, there exists $f \in X'$ such that for each $x \in X$, the sequence $(f_n(x))$ is I-convergent to $f(x)$. By lemma 2.2 and 2.5, the sequence $(f_n(x))$ is I-bounded which in turn by lemma 2.7 implies that there exists a set $K = \{n_1 < n_2 < \ldots < n_k < \ldots\} \in F(I)$ such that the sequence $(f_{n_k}(x))$ is bounded. By uniform boundedness theorem, the sequence $\{\|f_{n_k}\| : k \in N\}$ is bounded and again by lemma 2.7 the sequence $(\|f_k\|)$ is I-bounded. □

Finally we give a characterization of sequences of bounded linear functionals defined on a Banach space which are weak* ideal convergent.

Theorem 4.5. *For a Banach space X and a sequence (f_k) in X', we have*

$$w^* - I - \lim f_k = f \quad \textit{if and only if}$$

(i) *the sequence $(\|f_k\|)$ is I-bounded*

(ii) *for every element x in a total subset $M \subset X$, we have $I - \lim f_k(x) = f(x)$.*

The proof is similar to as that of lemma 3.2 and hence is omitted.

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