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SOME COMMON FIXED POINT THEOREMS OF MULTIVALUED MAPPINGS AND FUZZY MAPPINGS IN ORDERED METRIC SPACES

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Abstract: Heilpern [9] introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mappings. Generalizing Heilpern's result, Bose and Sahani [5] proved a common fixed point theorem for a pair of generalized fuzzy contraction mappings and also a fixed point theorem for nonexpansive fuzzy mappings. Since then, many authors have generalized Bose and Sahani's results in different directions. Also Bose and mukherjee (see [2], [3]) considered common fixed points of a pair of multivalued mappings and a sequence of single valued mappings. We present several theorems which are generalized to ordered metric space setting.In Section 3, we present our remarks concerning some generalizations of the main theorm of Bose and Sahani.Three such results, of Vijayaraju and Marudai [18], Azam and Arshad [1], and B.S. Lee et al [13] are discussed and a correct proof of the main theorem of Vijayaraju and Marudai has ben presented using a tecnique of Bose and Mukherjee [2]. In Section 4, we present several new theorems in ordered metric space setting. One is a version of the fixed point theorem for a pair of multivalued mappings of Bose and Mukherjee in ordered metric space setting and the other is a new version of the main theorem of Bose and Sahani in orderd metric space setting. Also we present a few results concerning common fixed point of a sequence of such mappings in ordered metric space setting.

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1. Introduction

Heilpern (1981, see [9]) introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mappings. Bose and Sahani (1987, see [5]) proved a common fixed point theorems for a pair of generalized fuzzy contraction mappings and also for nonexpansive mappings. Since then many authors have genralized Bose and Sahani's result in different directions. We discuss three such results in this paper as a prelude to proving their generalizations to ordered metric space setting. First one is by Vijayaraju and Marudai (2003, see [18]).Second one is by Azam and Arshad (2010, see [1]) and third one is by B.S. Lee, G.M. Lee, S.J. Cho, and D.S. Kim (1998, see [13]). The proof of the main theorem of Vajayaraju and Marudai is wrong. I obtained a correct proof by using the technique of Bose and Mukherjee (1977, see [2]) but did not publish it. Azam and Arshad concluded that the theorem of Vijayrajaju and Marudai cannot be proved and proved a diluted version of the same. This is not correct. I shall present the correct proof of the theorem stated by Vijayaraju and Marudai using the technique used in Bose and Mukherjee [2]. Lee et al (1998, see [13]) presented a different generalization whose proof was unfortunately wrong. But from the results obtained by Shi Chuan (1997, see [6]), we can show that, though their (Lee et al) result is not true for a pair of fuzzy mappings, the theorem can be proved for an infinite sequenc of such mappings with specified conditions.

Ran and Reurings [17] first investigated the existence of fixed points in a partially ordered metric spaces and they were followed by Nieto and Lopez [16], Hajrani and Sadarangani [8], Kadelberg et al [11] (in a ordered cone metric space),etc. Also recently L. Ciric et al [7] have considered fuzzy common fixed theorems in ordered metric spaces and we can modify the last part of their proof following the proof of Theorem 55 of Bose and Roychowdhury [4] where we proved two fixed point theorems concerning a pair of fuzzy weakly (generalized) contractive mappings. We shall consider the ordered metric space version of these in another paper.

We shall present some new common fixed point theorems for a pair of multivalued mappings and for a pair of fuzzy mappings in an ordered metric space setting. These can be considered as generalizations of theorems A, B, and E. Next we prove some theorems concerning the existence of a common fixed point of a sequence of such mappings (multivalued/fuzzy mappings) in ordered metric spaces. Also we briefly consider the conditions under which the fixed point can be unique. Many of these results are true for ordered cone metric spaces, but those results will be presented elsewhere.

2. Preliminaries

Let (X, d) be a metric linear space. A fuzzy set A in is a function from X into $I = [0, 1]$. If $x \in X$, the function value $A(x)$ is called the grade of membership of X in A. The $\alpha-$level set ($\alpha - \texttt{cut}$) of A, denoted by A_α, is defined by $A_\alpha = \{x : A(x) \geq \alpha\}$ if $\alpha \in (0, 1]$ and $A_0 = \overline{\{x : A(x) > 0\}}$, where $\overline{B}$ denotes the closure of the set B.

Definition 2.1. A fuzzy set A is said to be an approximate quantity iff A_α is compact and convex for each $\alpha \in [0, 1]$, and $\texttt{sup}_{x\in X} A(x) = 1$. When A is an approximate quantity and $A(x_0) = 1$ for some $x_0 \in X$, A is identified with an approximation of x_0. Let $\mathcal{F}(X)$ be the collection of all fuzzy sets in X and $W(X)$ be a subcollection of all approximate quantities. Then we difine the distance between two approximate quantities.

Definition 2.2. Let $A, B \in W(X)$, $\alpha \in [0, 1]$. Then we define $p_\alpha(A, B) = \texttt{inf}_{x\in A_\alpha, y\in B_\alpha} d(x, y)$, $D_\alpha(A, B) = H(A_\alpha, B_\alpha)$ where H is the Hausdorff distance. $D(A, B) = \texttt{sup}_\alpha D_\alpha(A, B)$, and $p(A, B) = \texttt{sup}_\alpha p_\alpha(A, B)$. The function D_α is called an $\alpha-$distance, and D is a distance between A and B, i.e., D is a metric on $W(X)$. We note that $p_\alpha$is a nondecreasing function of α. Next we define an order on the family $W(X)$ which charecterizes the accuracy of a given quantity.

Definition 2.3. Let $A, B \in W(X)$. Then A is said to be more accurate than B, denoted by$A \subset B$, iff $A(x) \leq B(x)$ for each $x \in X$. The relation $\subset$ induces a partial order on the family $W(X)$.

Definition 2.4. Let X be an arbitrary set and Y be a metric linear space. F is called a fuzzy mapping if and only if F is a mapping from the set X into $W(Y)$.

Definition 2.5. For $\alpha \in (0, 1]$, the fuzzy point x_α of X is the fuzzy set defined by $x_\alpha(y) = \alpha$ if $y = x$ and $x_\alpha(y) = 0$ if $y \neq x$. A fuzzy point x_α in X is called a fixed fuzzy point of the fuzzy mapping F if $x_\alpha \subset Fx$. If $\{x\} \subset Fx$, then x is a fixed point of the fuzzy mapping F.

Definition 2.6. Let $\alpha \in [0,1]$. Then the family $W_\alpha(X)$ is given by the set $\{A \in \mathcal{F}(X) : A_\alpha \texttt{is nonempty, compact and convex}\}$. We shall consider also fuzzy mappings $F : X \to W_\alpha(X)$, $\alpha \in [0,1]$.

Lemma 2.7. (see Heilpern [9]) *Let $x, \in X, A \in W(X)$, and let $\{x\}$ be a fuzzy set with membership function equal to a charecteristic function of the set x. If $\{x\} \subset A$, then $p_\alpha(x, A) = 0$ for each $\alpha \in [0,1]$ and vice versa.*

Lemma 2.8. (see Heilpern [9]) *$p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$ for any $x, y \in X$.*

Lemma 2.9. (see Heilpern [9]) *If $\{x_0\} \subset A$ then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $B \in W(X)$.*

Let (X, d) be a metric space and let $CB(X)$ be the set of all closed bounded subsets of X. We denote the closure of a set$A \subseteq X$ as $\overline{A}$. The Hausdorff metric on $CB(X)$ is defined as

$$H(A,B) = \texttt{max}\left\{\sup_{x\in B}\inf_{y\in A} d(x,y), \sup_{x\in A}\inf_{y\in B} d(x,y)\right\}$$
$$= \texttt{max}\left\{\texttt{sup}_{\texttt{x}\in\texttt{A}}\texttt{d}(\texttt{x},\texttt{B}), \texttt{sup}_{\texttt{y}\in\texttt{B}}\texttt{d}(\texttt{y},\texttt{A})\right\}$$

where $A, B \in CB(X)$ and

$$d(x, A) = \texttt{inf}_{y\in A} d(x, y),$$

$H(A, B) = 0$ if and only if $A = B$.

Lemma 2.10. (see Nadler, Jr. [14]) *Let A and B be nonempty compact subsets of a metric space (X, d). If $a \in A$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Lemma 2.11. (see Nadler, Jr. [14]) *If $A, B \in CB(X)$ and $x \in A$, then for each positive number α there exists $y \in B$ such that $d(x, y) \leq H(A, B) + \alpha$, i.e.,$d(x, y) \leq qH(A, B)$ where $q > 1$.*

Lemma 2.12. *Let $A, B \in W(X)$. Then for each $\{x\} \subset A$, there exists $\{y\} \subset B$ such that $D(\{x\}, \{y\}) \leq D(A, B)$.*

Since $D(\{x\}, \{y\}) = d(x, y)$,we have $d(x, y) \leq D(A, B)$.

Lemma 2.13. (see Nadler Jr. [14]) *The metric space $(CB(X),H)$ is complete provided X is complete.*

Lemma 2.14. (see Lee and Cho [12]). *Let (X, d) be a complete metric linear space. If F is a mapping from X into $W(X)$ and $x_0 \in X$, then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

Lemma 2.15. (see [15]) *Let A be a set in X and let $\{A_\alpha : \alpha \in I\}$ be a family of subsets of A such that:*

(i) $A_0 = A$,

(ii) $\alpha \leq \beta$ *implies* $A_\beta \subseteq A_\alpha$,

(iii) $\alpha_1 \leq \alpha_2 \leq, ..., lim_{n\to\infty}\alpha_n = \alpha implies\ A_\alpha = \cap_{k=1}^{\infty} A_{\alpha_k}$.

Then the function $\phi : X \to I$ defined by $\phi(x) = sup\{\alpha \in I : x \in A_\alpha\}$ has the property that $[\phi]_\alpha = A_\alpha$. Conversely, for any fuzzy set μ in X, the family $\{[\mu]_\alpha\}$of α-level sets of μ satisfies the above conditions (i)-(iii).

The function ϕ is actually defined on the set A, but we can extend it to X by defining $\phi(x) = 0$ for all $x \in X - A$. Lemma 3 is known as Negoita-Ralescu representation theorem.

Definition 2.16. Let (X, d) be a metric space. A mapping $F : X \to X$ is said to be weakly contractive or a $\phi-$weak contraction if $d(Fx, Fy) \leq d(x, y) - \phi(d(x, y))$ for each $x, y \in X$ where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ϕ is positive on $(0, \infty)$, $\phi(0) = 0$.

Definition 2.17. Let X be a nonempty set. Then $(X, d, \leq)$ is called an ordered metric space if and only if

(i) (X, d) is a metric space,

(ii) $(X, \leq)$ is a partialy ordered set.

Definition 2.18. Let $(X, \leq)$ be a partially ordered set. A pair $(F_1, F_2)$of self maps of X is said to be weakly incresing (decreasing) if $F_1x \leq (\geq)F_2F_1x$ and $F_2x \leq (\geq)F_1F_2x$ for all $x \in X$.

The following is a sufficient condition for the uniqueness of fixed point in $(X, d, \leq)$:

For every $x, y \in X$ there exists a lower bound or an upper bound. In [16] it is shown to be equivalent to the following:

For $x, y \in X$ there exsts $z \in X$ which is comparable to x and y.

3. Common Fixed Point Theorems in Metric Spaces

Bose and Sahani(1987, [5]) proved a common fixed point theorem for a pair of generalized fuzzy contraction mappings. Since then many authors have genralized Bose and Sahani's result in different directions. We discuss three such results in this paper. First one is by Vijayaraju and Marudai (2003, [18]). Second one is by Azam and Arshad (2010,[1]) and third one is by B.S. Lee, G.M.

Lee, S.J. Cho, and D. S. Kim(1998, [13]). The proof of the main theorem of Vajayaraju and Marudai was wrong. I obtained a correct proof by using the technique of Bose and Mukherjee(1977, [2],[10]) but did not publish it. Azam and Arshad concluded that the theorem of Vijayrajaju and Marudai cannot be proved and proved a diluted version of the same. This is not correct. I shall present the correct proof of the theorem stated by Vijayaraju and Marudai using the technique of Bose and Mukherjee (see [2],[10]). Lee et al (1998, [13])presented a different generalization whose proof was unfortunately wrong. But from the results obtained by Shi Chuan (1997, [6]), we can show that, though their (Lee et al) result is not true for a pair of fuzzy mappings, the theorem can be proved for an infinite sequenc of such mappings with specified conditions. Since the proof of this is very similar to the proofs given for a sequence of such mappings in ordered metric spaces in Section 4, it will not be presented here separately. Here, first we list the relevant theorems mentioned above and give the correct proof of the theorem of Vijayaraju and Marudai [18].

Theorem A. (see Bose and Sahani [5]) *Let (X,d) be a complete metric linear space and let F_1 and F_2 be fuzzy mappings from X into $W(X)$ satisfying the following condition:*

$$D\left(F_1x, F_2y\right) \leq a_1p(x, F_1x) + a_2p(y, F_2y) + a_3p(y, F_1x) + a_4p(x, F_2y) \\ + a_5d(x, y) \quad \text{for all} \quad x, y \in X,$$

where $a_i, i = 1,2,3,4,5$ are non-negative real numbers and $\sum_0^5 a_i < 1$ and $a_1 = a_2$ or $a_3 = a_4$.Then F_1 and F_2 have a common fixed point.

The following theorem, obtainbed as a generalization of Theorem A, is the main result of Vijayaraju and Marudai [18].

Theorem B. *Let $(X,d))$ be a complete metric space and let F_1,F_2 be fuzzy mappings from X into $\mathcal{F}(X)$ satisfying the following conditions:*

(a) for each $x \in X$, there exists $\alpha(x) \in (0,1]$ such that $[F_1x]_{\alpha(x)}, [F_2x]_{\alpha(x)}$are nonempty closed bounded subsets of X, and

(b) $$H\left([F_1x]_{\alpha(x)}, [F_2x]_{\alpha(x)}\right) \leq a_1d(x, [F_1x]_{\alpha(x)}) + a_2d(y, [F_2y]_{\alpha(y)}) \\ +a_3d(y, [F_1x]_{\alpha(x)}) + a_4d(x, [F_2y]_{\alpha(y)}) + a_5d(x, y),$$
for all $x, y \in X$, where $a_i, i = 1,2,3,4,5$are non-negative real numbers and $\sum_0^5 a_i < 1$ and $a_1 = a_2$ or $a_3 = a_4$. Then there exists $z \in X$ such that $z \in [F_1z]_{\alpha(z)} \cap [F_2z]_{\alpha(z)}$.

Azam and Arshad [1] pointed out various errors in the proof of Theorem B and claimed that the following theorem is the right version of the generalization given by Vijayaraju and Marudai [18]. The proof (of Theorem B) did not use Lemma 2.5 correctly in the beginning of the proof and it was not a valid proof due to various other errors.

Theorem C. *Let $(X, d))$ be a complete metric space and let F_1, F_2 be fuzzy mappings from X into $\mathcal{F}(X)$ satisfying the following conditions:*

(a) for each $x \in X$, there exists $\alpha(x) \in (0,1]$ such that $[F_1x]_{\alpha(x)}, [F_2x]_{\alpha(x)}$ are nonempty closed bounded subsets of X and

(b) $$H\left([F_1x]_{\alpha(x)}, [F_2x]_{\alpha(x)}\right) \leq a_1 d(x, [F_1x]_{\alpha(x)}) + a_2 d(y, [F_2y]_{\alpha(y)}) + a_3[d(y, [F_1x]_{\alpha(x)}) + d(x, [F_2y]_{\alpha(y)})] + a_4 d(x, y),$$

for all $x, y \in X$, where $a_i, i = 1, 2, 3, 4$ are non-negative real numbers with $a_1 + a_2 + 2a_3 + a_4 < 1$.Then there exists $z \in X$ such that $z \in [F_1z]_{\alpha(z)} \cap [F_2z]_{\alpha(z)}$.

It is noted that one condition has been dropped (instead of $a_1 = a_2$or $a_3 = a_4$, only $a_3 = a_4$ is assumed). It will shown in the proof of theorem 3.1 that this implies $0 < r, s < 1$.

Generalizing the results of Bose and Sahani [5] and Lee and Cho [12], Lee et al [13] presented the following common fixed point theorem for a pair of fuzzy mappings:

Theorem D. *Let (X, d) be a complete metric linear space and let F_1 and F_2 be fuzzy mappings from X into $W(X)$ satisfying the following condition:*

For any $x \in X$, $\{u_x\} \subset F_1x$ and $y \in X$,there exists $\{v_y\} \subset F_2y$ such that

$$D(\{u_x\}, \{v_y\}) \leq a_1 d(x, u_x) + a_2 d(y, v_y) + a_3 d(y, u_x) + a_4 d(x, v_y) + a_5 d(x, y),$$

for all$x, y \in X$, where $a_i, i = 1, 2, 3, 4, 5$ are non-negative real numbers, $\sum_0^5 a_i < 1$ and $a_3 \geq a_4$.Then F_1 and F_2 have a common fixed point, that is, there exists $z \in X$ such that $z \subset F_1z$ and $z \subset F_2z$.

Theorem E. (see Bose and Mukherjee [2], [10]) *Let (X, d) be a complete metric space and let F_1 and F_2 be multivalued mappings from X into $CB(X)$ satisfying the following condition:*

$$H(F_1x, F_2y) \leq a_1 d(x, F_1x) + a_2 d(y, F_2y) + a_3 d(y, F_1x) + a_4 d(x, F_2y) + a_5 d(x, y),$$

for all $x, y \in X$, where $a_i, i = 1, 2, 3, 4, 5$ are non-negative numbers and $\sum_0^5 a_i < 1$ and $a_1 = a_2$ or $a_3 = a_4$.Then F_1 and F_2 have a common fixed point.

Corollary F. (see Bose and Mukherjee [2], [10]) *Let (X,d) be a complete metric space and let $F_i : X \to X, i = 1,2$ be mappings satisfying the following condition:*

$$d(F_1(x), F_2(y)) \\ \leq a_1 d(x, F_1x) + a_2 d(y, F_2y) + a_3 d(y, F_1x) + a_4 d(x, F_2y) + a_5 d(x,y),$$

for all$x, y \in X$, where $a_i, i = 1,2,3,4,5$ are non-negative numbers and $\sum_0^5 a_i < 1$ and $a_1 = a_2$ or $a_3 = a_4$.Then F_1 and F_2 have a common fixed point.

I noticed the errors in the proof of Theorem B given by Vijayaraju and Marudai [18] and found a correct proof based on the technique used in Theorem E of Bose and Mukherjee[2]. In fact, we used the same technique, adapted to the new setting, in the proof of Theorem A also. Hence the assertion of Azam and Arshad [1] was not correct.

Also Lee and Cho [12] made a generalization of one theorem of Bose and Sahani and proved the existence of a common fuzzy fixed point of an infinite family of fuzzy contractions. Following Lee and Cho [12], Lee et al[13] presented a generalization of Theorem A (for a pair of fuzzy mappings)with a different set of conditions on a_i.that is, $\sum_0^5 a_i < 1$ and $a_3 \geq a_4$. However the proof is not correct. Shi Chuan [6] proved a similar theorem (genralized in a different direction) with the same condition on a_i but for an infinite sequence of fuzzy mappings. A perusal of the proof given by Shi Chuan shows that the theorem of Lee et al [13] for an infinite sequence of fuzzy mappings satisfying the same inequality for F_i, F_j and with same conditions on a_i, goes through. Similar technique has been employed in Bose and Mukherjee [3] for a sequence of self-mappings of X and the same technique will be used here when we consider the common fixed point of a sequence of mappings later satisfying certain condition in ordered metric space setting (see Section 4).

We now present the correct proof of Theorem B following the techniques used in Bose and Mukherjee [2]. A variation of the same technique was also used in Bose and Sahani [5].

Theorem 3.1. *Let $(X,d))$ be a complete metric space and let $F_{1,}F_2$ be fuzzy mappings from X into $\mathcal{F}(X)$ satisfying the following conditions:*

(a) for each $x \in X$, there exists $\alpha(x) \in (0,1]$ such that $[F_1x]_{\alpha(x)}, [F_2x]_{\alpha(x)}$are nonempty closed bounded subsets of X and

(b) $$H\left([F_1x]_{\alpha(x)}, [F_2x]_{\alpha(x)}\right) \leq a_1 d(x, [F_1x]_{\alpha(x)}) + a_2 d(y, [F_2y]_{\alpha(y)}) \\ + a_3 d(y, [F_1x]_{\alpha(x)}) + a_4 d(x, [F_2y]_{\alpha(y)}) + a_5 d(x,y),$$

for all $x, y \in X$, *where* $a_i, i = 1, 2, 3, 4, 5$*are non-negative real numbers and* $\sum_0^5 a_i < 1$ `and` $a_1 = a_2$ `or` $a_3 = a_4$.*Then there exists* $z \in X$ *such that* $z \in [F_1 z]_{\alpha(z)} \cap [F_2 z]_{\alpha(z)}$.

Proof. Let $x_0 \in X$. For this $x_{0,}$ by condition (a), there exists $\alpha_1 \in (0, 1]$ such that $[F_1(x_0)]_{\alpha_1} \in CB(X)$. Choose $x_1 \in [F_1(x_0)]_{\alpha_1}$. For this x_1, there exists $\alpha_2 \in (0, 1]$ such that $[F_2(x_1)]_{\alpha_2} \in CB(X)$. By Lemma 2.11, there exists $x_2 \in [F_2(x_1)]_{\alpha_2}$ such that $d(x_1, x_2) \leq p^{-1} H\left([F_1(x_0)]_{\alpha_1}, [F_2(x_1)]_{\alpha_2}\right)$ where $p = (a_1 + a_2 + a_3 + a_4 + a_5)^{\frac{1}{2}}$.

Using condition (b) and simplifying:

$$d(x_1, x_2) \leq p^{-1} H\left([F_1(x_0)]_{\alpha_1}, [F_2(x_1)]_{\alpha_2}\right) \leq \frac{a_1 + a_4 + a_5}{p - a_2 - a_4} d(x_0, x_1).$$

Proceeding in a similar manner, there exists $x_3 \in [F_1(x_2)]_{\alpha_3}$ such that $d(x_2, x_3) \leq \frac{a_2+a_3+a_5}{p-a_1-a_3} d(x_1, x_2)$. Consider the sequence$\{x_n\}$ where $x_{2n+1} \in [F_1(x_{2n})]_{\alpha_{2n+1}}$ and $x_{2n+2} \in [F_2(x_{2n+1})]_{\alpha_{2n+2}}, n = 0, 1, 2, 3, ...$ We have $0 < r, s < 1$ if $a_3 = a_4$ and $0 < rs < 1$ when $a_1 = a_2$ or $a_3 = a_4$ where $r = \frac{a_1+a_4+a_5}{p-a_2-a_4}$ and $s = \frac{a_2+a_3+a_5}{p-a_1-a_3}$. Further

$$d(x_{2n+1}, x_{2n+2}) \leq r(rs)^n d(x_0, x_1) \texttt{ and } d(x_{2n}, x_{2n+1}) \leq (rs)^n d(x_0, x_1).$$

Also

$$\sum_{n=0}^{\infty} [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \leq (1 + r) d(x_0, x_1) \sum_{n+0}^{\infty} (rs)^n.$$

From this, it is easily seen that the sequence $\{x_n\}$is Cauchy and hence the sequence converges to some point $u \in X$. Let us consider $d\Big(u, [F_2(u)]_{\alpha(u)}\Big)$. We have

$$\begin{aligned} d\left(u, [F_2(u)]_{\alpha(u)}\right) &\leq d(u, x_{n+1}) + d\left(x_{n+1}, [F_2(u)]_{\alpha(u)}\right) \\ &\leq d(u, x_{n+1}) + H\Big([F_1(x_n)]_{\alpha_{n+1}}, [F_2(u)]_{\alpha(u)}\Big) \end{aligned}$$

(here n is taken to be even). Using condition (b) and simplifying, we have

$$\begin{aligned} &d\left(u, [F_2(u)]_{\alpha(u)}\right) \\ &\quad \leq d(u, x_{n+1}) + a_1 d(x_n, x_{n+1}) + a_2 d\left(u, [F_2(u)]_{\alpha(u)}\right) + a_3 d(u, x_{n+1}) \\ &\qquad + a_4 d(x_n, u) + a_4 d\left(u, [F_2(u)]_{\alpha(u)}\right) + a_5 d(x_n, u). \end{aligned}$$

Taking limit $n \to \infty$, we have $d\left(u, [F_2(u)]_{\alpha(u)}\right) \leq (a_2 + a_4)\left[d\left(u, F_2(u)\right)\right]_{\alpha(u)}$, i.e.,$d\left(u, [F_2(u)]_{\alpha(u)}\right) = 0$. Since $[F_2(u)]_{\alpha(u)}$ is closed, we have $u \in [F_2(u)]_{\alpha(u)}$. Similarly it can be shown that $u \in [F_1(u)]_{\alpha(u)}$.

Remark 3.2. Theorem C of Azam and Arshad [1] follows as a corollary of Theorem 3.1.

4. Fixed Point Theorems in Ordered Metric Spaces

We now present four new theorems concerning fixed points in ordered metric space setting. First one extends the theorem of Bose and Mukherjee (see [2], [10]) and second one extend the theorem of Bose and Sahani [5] to ordered metric space setting. Third and fourth are the sequential versions of the theorems of Bose and Mukherjee and Bose and Sahani.

Theorem 4.1. *Let $(X, d, \leq)$be a complete ordered metric space. Let $CB(X)$ denote the space of nonempty closed bounded subsets of X equipped with the Hausdorff metric H. Let F_1, F_2 be mappings from X into $CB(X)$ satisfying the following conditions:*

(a) $H(F_1x, F_2y) \leq a_1 d(x, F_1x) + a_2 d(y, F_2y) + a_3 d(y, F_1x) + a_4 d(x, F_2y) + a_5 d(x, y)$ for all comparable elements $x, y \in X$, where $a_i, i = 1, 2, 3, 4, 5$ are non-negative numbers and $\sum_0^5 a_i < 1$ and $a_1 = a_2$ or $a_3 = a_4$.

(b) Let $x_0 \in X$. Suppose if $x_1 \in F_1(x_0)$ then x_0 and x_1 are comparable. Suppose, if $x, y \in X$ are comparable, then every $u \in F_1(x)$ and every $v \in F_2(y)$ are comparable. Also suppose that if a sequence $\{x_n\} \to x$ and its consequtive terms are comparable, then $x_n, x \in X$ are comparable for all n.

Then there exists a point x such that $x \in F_1(x)$ and $x \in F_2(x)$, i.e., x is a common fixed point of F_1 and F_2.

Proof. Let $x_0 \in X$. Choose $x_1 \in F_1(x_0)$. Then x_0, x_1 are comparable. By Lemma 2.11, there exists $x_2 \in F_2(x_1)$ such that

$$d(x_1, x_2) \leq p^{-1} H\left(F_1(x_0), F(x_1)\right),$$

where $p = (a_1 + a_2 + a_3 + a_4 + a_5)^{\frac{1}{2}}$.

Using condition (a) and simplifying, we have

$$d(x_1, x_2) \leq p^{-1} H\left(F_1(x_0), F_2(x_1)\right) \frac{a_1 + a_4 + a_5}{p - a_2 - a_4} d(x_0, x_1).$$

Proceeding in a similar manner, there exists $x_3 \in F_1(x_2)$ such that

$$d(x_2, x_3) \leq \frac{a_2 + a_3 + a_5}{p - a_1 - a_3} d(x_1, x_2).$$

Consider the sequence$\{x_n\}$ where $x_{2n+1} \in F_1(x_{2n})$ and $x_{2n+2} \in F_2(x_{2n+1})$, $n = 0, 1, 2, 3, ...$

We have $0 < r, s < 1$ if $a_3 = a_4$ and $0 < rs < 1$ when $a_1 = a_2$ or $a_3 = a_4$ where $r = \frac{a_1+a_4+a_5}{p-a_2-a_4}$ and $s = \frac{a_2+a_3+a_5}{p-a_1-a_3}$. Further $d(x_{2n+1}, x_{2n+2}) \leq r(rs)^n d(x_0, x_1)$ and $d(x_{2n}, x_{2n+1}) \leq (rs)^n d(x_0, x_1)$.

Also

$$\sum_{n=0}^{\infty} [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \leq (1 + r) d(x_0, x_1) \sum_{n+0}^{\infty} (rs)^n.$$

From this, it is easily seen that the sequence $\{x_n\}$ is Cauchy and hence the sequence converges to some point $u \in X$. Let us consider $d(u, F_2(u))$. We have $d(u, F_2(u)) \leq d(u, x_{n+1}) + d(x_{n+1}, F_2(u)) \leq d(u, x_{n+1}) + H(F_1(x_n), F_2(u))$ (here n is taken to be even and note that x_n and u are comparable by condition (b)). Using conditions (a) and (b) and simplifying, we have $d(u, F_2(u)) \leq$.

$$\begin{aligned} &d(u, x_{n+1}) + a_1 d(x_n, x_{n+1}) + a_2 d(u, F_2(u)) + a_3 d(u, x_{n+1}) + a_4 d(x_n, u) \\ &\qquad + a_4 d(u, F_2(u)) + a_5 d(x_n, u). \end{aligned}$$

Taking limit $n \to \infty$, we have $d(u, F_2(u)) \leq (a_2 + a_4) d(u, F_2(u))$, i.e., $d(u, F_2(u)) = 0$. Since $F_2(u)$ is closed, we have $u \in F_2(u)$. Similarly it can be shown that $u \in F_1(u)$.

Corollary 4.2. *Let $(X, d, \leq)$ be a complete ordered metric space. Let f_1, f_2 be mappings from X into X satisfying the following conditions:*

(a) $d(f_1x, f_2y) \leq a_1 d(x, f_1x) + a_2 d(y, f_2y) + a_3 d(y, f_1x) + a_4 d(x, f_2y) + a_5 d(x, y)$ for all comparable elements $x, y \in X$, where $a_i, i = 1, 2, 3, 4, 5$ are non-negative numbers and $\sum_0^5 a_i < 1$ and $a_1 = a_2$ or $a_3 = a_4$.

(b) Let $x_0 \in X$. Suppose if $x_1 = f_1x_0$ then x_0 and x_1 are comparable. Suppose, if $x, y \in X$ are comparable, then $u = f_1x$ and $v = f_2y$ are comparable. Also suppose that if a sequence $\{x_n\} \to x$ and its consequtive terms are comparable, then $x_n, x \in X$ are comparable for all n or f_1 or f_2 is continuous.

Then f_1 and f_2 have a common fixed point $x \in X$, that is, there exists a point $x \in X$ such that $x = f_1x$ and $x = f_2x$.

Proof. Define a sequence $\{x_n\}$ by $x_{2n+1} = f_1 x_{2n}$ and $x_{2n+2} = f_2 x_{2n+1}$. It can be shown similarly that the defined sequence $\{x_n\}$ converges to say $x \in X$ and x is a common fixed point of $f_1$and f_2 when consequtive terms of the sequence are comparable and x_n, x are comparable for all n.

Suppose f_1 is continuous (in lieu of :suppose that if a sequence $\{x_n\} \to x$ and its consequtive terms are comparable, then $x_n, x \in X$ are comparable for all n). Then $x_{2n+1} = f_1 x_{2n}$ implies $x = f_1 x$ since the sequence $\{x_n\} \to x$ and by continuity of f_1. Also it can be shown that $d(x, f_2 x) = d(f_1 x, f_2 x) = 0$. Hence x is a common fixed point of f_1 and f_2.

Remark 4.3. If the pair of mappings (f_1, f_2) is weakly increasing, it can be easily shown that the sequence $\{x_n\}$defined in the above corollary is nondecreasing with respect to $\leq$,i.e., $x_0 \leq x_1 \leq ... \leq x_n \leq ...$.

Kadelberg et al [11] poroved a similar result in ordered cone metric space setting where the condition (b) is replaced by the following:(i) f_1 or f_2 is continuous or(ii) if a nondecreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \leq x$ for all $n \in N$.

We now discuss sufficient condition for the uniqueness of fixed point or common fixed point of results discussed above. The condition is as follows: For every $x, y \in (X, \leq)$ there exists a lower bound or an upper bound or equivalently, for every $x, y \in (X, \leq)$ there exists $z \in (X, \leq)$ which is comparable to x and y. This equivalence was proved in [16]. Also following example of an ordered metric space was presented in [16], where the above condition (ii) is satisfied(if a nondecreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \leq x$ for all $n \in N$.):

Let $C([0, T])$ denotes the space of continuous functions defined on $[0, T]$. This space with the metric defined by $d(x, y) = \sup\{| x(t) - y(t) |: t \in [0, T]\}$, for $x, y \in C([0, T])$, is a complete metric space. A partial order in this space is given by $x, y \in C([0, T])$, $x \leq y \Leftrightarrow x(t) \leq y(t)$ for $t \in [0, T]$. Also it was shown that if $\{x_n\}$ is a nondecreasing sequence in this space such that $x_n \to x$, then $x_n \leq x$ for all $n \in N$.

Corollary 4.4. *Let $(X, d, \leq)$be a complete ordered metric space. Let f be a mapping from X into X satisfying the following conditions:*

(a) $d(fx, fy) \leq a_1 d(x, fx) + a_2 d(y, fy) + a_3 d(y, fx) + a_4 d(x, fy) + a_5 d(x, y)$ for all comparable elements $x, y \in X$, where a_i, $i = 1, 2, 3, 4, 5$ are non-negative numbers and $\sum_0^5 a_i < 1$ and $a_1 = a_2$ and $a_3 = a_4$.

(b) Let $x_n \in X$. Suppose if $x_{n+1} = fx_n$ then x_n and x_{n+1} are comparable, i.e., x and fx are comparable for all $x \in X$. Also suppose that if a

sequence $\{x_n\} \to x$ and its consequtive terms are comparable, then $x_n, x \in X$ are comparable for all n or f is continuous.

Then f has a fixed point $x \in X$, that is, there exists a point $x \in X$ such that $x = fx$.

Further, if the space is totally ordered, the fixed point of f is unique.

Remark 4.5. Kadelberg et al [11] proved such a result in an odered cone metric space setting where f is a continuous self mapping of X such that $x \leq fx$ for all $x \in X$.

The next theorem generalizes Theorem A (Bose and Sahani[5]) dealing with the existence of a common fixed points of a pair of generalized fuzzy contractions in ordered metric space setting.

Theorem 4.6. *Let $(X, d, \leq)$ be complete ordered metric linear space and let F_1 and F_2 be fuzzy mappings from X into $W_\alpha(X)$ satisfying the following conditions:*

(a) $$D_\alpha(F_1x, F_2y) \leq a_1p_\alpha(x, F_1x) + a_2p_\alpha(y, F_2y) + a_3p_\alpha(y, F_1x) + a_4p_\alpha(x, F_2y) + a_5d(x, y)$$

for all comparable $x, y \in X$, where $a_i, i = 1, 2, 3, 4, 5$ are non-negative real numbers and $\sum_0^5 a_i < 1$ and $a_1 = a_2$ or $a_3 = a_4$.

(b) Let $x_0 \in X$. Suppose if $x_1 \subset F_1(x_0)$ then x_0 and x_1 are comparable. Suppose, if $x, y \in X$ are comparable, then every $u \in [F_1(x)]_\alpha$ and every $v \in [F_2(y)]_\alpha$ are comparable. Also suppose that if a sequence $\{x_n\} \to x$ and its consequtive terms are comparable, then $x_n, x \in X$ are comparable for all n.

Then F_1 and F_2 have a common fuzzy fixed point., i.e., there exists x_α such that $x_\alpha \subset F_1(x)$ and $x_\alpha \subset F_2(x)$.

Proof. Let $x_0 \in X$. By Lemma 2.14 $x_1 \subset F_1(x_0)$. Then x_0, x_1 are comparable. By Lemma 2.7, $p_\alpha(x_1, F_1(x_0)) = 0$ for each $\alpha \in [0, 1] \Leftrightarrow x \in [F_1(x_0)]_\alpha$.Then there exists $x_2 \in [F_2(x_1)]_\alpha$ such that

$$\begin{aligned} d(x_1, x_2) &\leq p^{-1}H([F_1(x_0)]_\alpha, [F_2(x_1)]_\alpha) = p^{-1}D_\alpha(F_1(x_0), F_2(x_1)) \\ &\leq p^{-1}[a_1p_\alpha(x_0, F_1(x_0)) + a_2p_\alpha(x_1, F_2(x_1)) + a_3p_\alpha(x_1, F_1(x_0)) \\ &\quad + a_4p_\alpha(x_0.F_2(x_1)) + a_5d(x_0, x_1)], \end{aligned}$$

where $p = (a_1 + a_2 + a_3 + a_4 + a_5)^{\frac{1}{2}}$ (see Lemma 2.11).

$$\leq p^{-1}[a_1d(x_0, x_1) + a_2d(x_1, x_2) + a_3d(x_1, x_1) + a_4d(x_0.x_2) + a_5d(x_0, x_1)],$$

by condition (a). Simplifying, we have $d(x_1, x_2) \leq p^{-1}H(F_1(x_0), F_2(x_1)) \leq \frac{a_1+a_4+a_5}{p-a_2-a_4}d(x_0, x_1)$.

Proceeding in a similar manner, there exists $x_3 \in [F_1(x_2)]_\alpha$ such that $d(x_2, x_3) \le \frac{a_2+a_3+a_5}{p-a_1-a_3} d(x_1, x_2)$.

Consider the sequence $\{x_n\}$, where

$$x_{2n+1} \in [F_1(x_{2n})]_\alpha \text{ and } x_{2n+2} \in [F_2(x_{2n+1})]_\alpha,\ n = 0, 1, 2, 3, ...$$

We have $0 < r, s < 1$ if $a_3 = a_4$ and $0 < rs < 1$ when $a_1 = a_2$ or $a_3 = a_4$ where $r = \frac{a_1+a_4+a_5}{p-a_2-a_4}$ and $s = \frac{a_2+a_3+a_5}{p-a_1-a_3}$. Further $d(x_{2n+1}, x_{2n+2}) \le r(rs)^n d(x_0, x_1)$ and $d(x_{2n}, x_{2n+1}) \le (rs)^n d(x_0, x_1)$.

Also

$$\sum_{n=0}^{\infty} [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \le (1+r)\, d(x_0, x_1) \sum_{n+0}^{\infty} (rs)^n .$$

From this, it is easily seen that the sequence $\{x_n\}$ is Cauchy and hence the sequence converges to some point $u \in X$. Let us consider $p_\alpha(u, F_2(u))$. By Lemma, we have $p_\alpha(u, F_2(u)) \le d(u, x_{n+1}) + p_\alpha(x_{n+1}, F_2(u)) \le d(u, x_{n+1}) + D_\alpha(F_1(x_n), F_2(u))$ (here n is taken to be even). Using conditions (a) and (b) and simplifying, we have $p_\alpha(u, F_2(u)) \le d(u, x_{n+1}) + a_1 d(x_n, x_{n+1}) + a_2 p_\alpha(u, F_2(u)) + a_3 d(u, x_{n+1}) + a_4 d(x_n, u) + a_4 p_\alpha(u, F_2(u)) + a_5 d(x_n, u)$.

Taking limit $n \to \infty$, we have $p_\alpha(u, F_2(u)) \le (a_2 + a_4)\, p_\alpha(u, F_2(u))$, i.e., $p_\alpha(u, F_2(u)) = 0$.

By Lemma 2.7, we have $u_\alpha \subset F_2(u)$. Similarly it can be shown that $u_\alpha \subset F_1(u)$.

Next we consider a sequence of multivalued mappings and a sequence of fuzzy mappings in ordered metric spaces satisfying similar conditions and prove some results concerning the existence of common fixed point of such mappings.

Theorem 4.7. *Let $(X, d, \le)$ be a complete ordered metric space. Let $CB(X)$ denote the space of nonempty closed bounded subsets of X equipped with the Hausdorff metric H. Let $\{F_n\}$ be a sequence of mappings from X into $CB(X)$ satisfying the following conditions:*

(a) $H(F_i x, F_j y) \le a_1 d(x, F_i x) + a_2 d(y, F_j y) + a_3 d(y, F_i x) + a_4 d(x, F_j y) + a_5 d(x, y)$ for all comparable elements $x, y \in X$, where $a_i, i = 1, 2, 3, 4, 5$ are non-negative numbers and $\sum_0^5 a_i < 1$ and $a_{3 \ge} a_4$.

(b) Let $x_0 \in X$. Suppose if $x_1 \in F_1(x_0)$ then x_0 and x_1 are comparable. Suppose, if $x, y \in X$ are comparable, then every $u \in F_i(x)$ and every $v \in F_j(y)$ are comparable. Also suppose that if a sequence $\{x_n\} \to x$ and its consequtive terms are comparable, then $x_n, x \in X$ are comparable for all n.

Then there exists a point x such that $x \in F_n(x)$for all $n \in N$.

Proof (brief). Choose $x_{k+1} \in F_{k+1}(x_k), k = 0, 1, 2, 3...$ such that

$$d(x_k, x_{k+1}) \leq p^{-1} H(F_k(x_{k-1}), F_{k+1}(x_k)).$$

Then by condition (a) and (b), we have $d(x_k, x_{k+1}) \leq \frac{a_1+a_4+a_5}{p-a_2-a_4} d(x_{k-1}, x_k) = rd(x_{k-1}, x_k)$, where $0 < r < 1$. It can be shown that the sequence $\{x_n\}$ is Cauchy and it converges to say $x \in X$. We have $d(x, F_n(x)) \leq d(x, x_k) + d(x_k, F_n(x)) \leq d(x, x_k) + H(F_k(x_{k-1}), F_n(x))$. Using condition (a) and (b) again and taking limita as $k \to \infty$, we get $d(x, F_n(x)) = 0$. This implies that $x \in F_n(x)$ for all $n \in N$.

When $\{f_n\}$ is a sequence of self mappings of X, we have the following:

Corollary 4.8. *Let $(X, d, \leq)$ be a complete ordered metric space. Let $\{f_n\}$ be a sequence of self-mappings of X satisfying the following conditions:*

(a) $d(f_i x, f_j y) \leq a_1 d(x, f_i x) + a_2 d(y, f_j y) + a_3 d(y, f_i x) + a_4 d(x, f_j y) + a_5 d(x, y)$ for all comparable elements $x, y \in X$, where $a_i, i = 1, 2, 3, 4, 5$ are non-negative numbers and $\sum_0^5 a_i < 1$ and $a_{3 \geq} a_4$.

(b) Let $x_0 \in X$. Suppose if $x_1 = f_1 x_0$ then x_0 and x_1 are comparable. Suppose, if $x, y \in X$ are comparable, then $u = f_i x$ and $v = f_j y$ are comparable. Also suppose that if a sequence $\{x_n\} \to x$ and its conseqtive terms are comparable, then $x_n, x \in X$ are comparable for all n.

Then there exists a point x such that $x = f_n x$ for all $n \in N$.

Remark 4.9. This corollary extends a result of Bose and Mukherjee (see [3], [10]) to ordered metric space setting.

Combining the techniques used in the proofs of Theorem 4.4 and Theorem 4.5 we can prove the following theorem.

Theorem 4.10. *Let $(X, d, \leq)$ be a complete ordered metric linear space and let $\{F_n\}$ be a sequnce of fuzzy mappings from X into $W_\alpha(X)$ satisfying the following conditions:*

(a) $D_\alpha(F_i x, F_j y) \leq a_1 p_\alpha(x, F_i x) + a_2 p_\alpha(y, F_j y) + a_3 p_\alpha(y, F_i x) + a_4 p_\alpha(x, F_j y) + a_5 d(x, y)$ for all comparable $x, y \in X$, where $a_i, i = 1, 2, 3, 4, 5$ are non-negative real numbers and $\sum_0^5 a_i < 1$ and $a_3 \geq a_4$.

(b) Let $x_0 \in X$. Suppose if $x_1 \subset F_1(x_0)$ then x_0 and x_1 are comparable. Suppose, if $x, y \in X$ are comparable, then every $u \in [F_i(x)]_\alpha$ and every $v \in [F_j(y)]_\alpha$ are comparable. Also suppose that if a sequence $\{x_n\} \to x$ and its consequtive terms are comparable, then $x_n, x \in X$ are comparable for all n.

Then $\{F_n\}$ *have a common fuzzy fixed point., i.e., there exists* x_α *such that* $x_\alpha \subset F_n(x)$ *for all* $n \in N$.

Remark 4.11. When $\alpha = 1$, this corresponds to the sequential version of Theorem A.

References

[1] A. Azam, M. Arshad, A note on "Fixed point theorems for fuzzy mappings" by P. Vijayaraju and M. Marudai, *Fuzzy Sets and Systems*, **161** (2010), 1145-1149.

[2] R.K. Bose, R. N. Mukherjee, Common fixed points of some multi-valued mappings, *Tamkang J. Math.*, **8**, No. 2 (1977), 245-249.

[3] R.K. Bose, R.N. Mukherjee, Stability of fixed point set and common fixed points of families of mappings, *Indian J. Pure and Applied Math.*, **11**, No. 9 (1980), 1130-1138.

[4] R.K. Bose, M.K. Roychowdhury, Fixed point theorems for generalized weakly contractive mappings, *Surveys in Mathematics and its Applications*, **4** (2009), 215-238.

[5] R.K. Bose, D. Sahani, Fuzzy mappings and fixed point theorems, *Fuzzy Sets and systems*, **21** (1987), 53-58.

[6] Shi Chuan, Common fuzzy hybrid fixed point theorems for a sequence of fuzzy mappings, *International J. Math. & Math Sciences*, **20**, No. 3 (1997), 451-456.

[7] L. Ciric, M. Abbas, B. Damjanovic, R. Saadati, Common fuzzy fixed point theoremss in ordered metric spaces, *Mathematical and Computer Modelling*, **53** (2011), 1737-1741.

[8] J.Hajrani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, *Nonlinear Analysis*, **72** (2010), 1188-1197.

[9] S. Heilpern, Fuzzy mappings and fixed point theorems, *J. Math. Anal. Appl.*, **83** (1981), 566-569.

[10] M.C. Joshi, R.K. Bose, *Some Topics in Nonlinear Functional Analysis*, John Wiley/ Wiley Eastern Ltd. (1985).

[11] Z. Kadelberg, M. Pavlovic, S. Radenovic, Common fixed point theorems for ordered contractions and quasicontraction in ordered cone metric spaces, *Computers and Mathematics with Applications*, **59** (2010), 3148-3159.

[12] B.S. Lee, S.J. Cho, Common fixed point theorems for a sequence of fuzzy mappings, *Internat. J. Math & Math Sciences,* **17**, No. 3 (1994), 423-428.

[13] B.S. Lee, G.M. Lee, S.J. Cho, D.S. Kim, a common fixed point thoerem for a pair of fuzzy mappings, *Fuzzy Sets and Systems,* **98** (1998), 133-136.

[14] S.B. Nadler Jr., Multivalued contraction mappings, *Pacific Journal of Math.*, **30** (1969), 475-487.

[15] C. V. Negoita, D. Ralescu, Representation theorem for fuzzy concepts, *Kybernets*, **4** (1975), 169-174.

[16] J.J. Nieto, R.R. Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22** (2005), 223-239.

[17] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some application to matrix equations, *Proc. Amer. Math. Soc.*, **132** (2004), 1435-1443.

[18] P. Vijayaraju, M. Marudai, Fixed point theorems for fuzzy mappings, *Fuzzy Sets and Systems*, **135** (2003), 401-408.