International Journal of Pure and Applied Mathematics
Volume 75 No. 2 2012, 221-227
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

SOME REMARKS ON THE QUASI F-MOMENT PROBLEM

A.S. Okb El Bab[1], Hossam A. Ghany[2 §], A. El Hadi[3]
[1,2,3]Mathematics Department
Faculty of Since
Azhar University
Naser City, Cairo, EGYPT
[2]Department of Mathematics
Faculty of Science
Taif University
Hawia, P.O. Box 888, Taif, SAUDI ARABIA

Abstract: This paper is devoted to give some properties of the so called quasi F-moment problem generated by orthogonal polynomials. Different relations between different types of quasi F-moment problem are explained. Some transformations between different types of quasi F-moment problems are given.

AMS Subject Classification: 43A62, 43A22, 43A10
Key Words: moment problem, orthogonal polynomial

1. Introduction

In his fundamental papers In [2] authors introduced the definition of F-moment problem, where F be a closed subset of $\mathbb{R}$, i.e. characterizing the real sequences $s = (s_n)_{n\geq 0}$ of the form

$$s_n = \int_F x^n d\mu(x), \quad n \geq 0,$$

where $\mu \in E_+(\mathbb{R})$ is supported by the closed set F. In this paper, we will

Received: November 12, 2011

§Correspondence author

consider the following two quasi F-moment problems: first the quasi Hausdorff moment sequence(QH):

$$h_n = \int_0^1 p_n(x)d\mu(x), \quad n \geq 0,$$

where $p_n(x)$ represents orthogonal polynomial on [0,1] and the quasi Stieltjes moment sequence(QS):

$$h_n = \int_0^\infty q_n(x)d\nu(x), \quad n \geq 0,$$

where $q_n(x)$ represents orthogonal polynomial on $[0,\infty[$. Suppose that the polynomials $p_n(x) = \sum_{n=0}^\infty a_n x^n$, $q_n(x) = \sum_{n=0}^\infty b_n x^n$ with positive coefficients i.e. $a_n, b_n \geq 0$. So the set QS can be considered as a subset of $[0,\infty[^{\mathbb{N}_0}$ with the product topology.

2. Some Properties of the Sets QH and QS

The following lemma are in fact, an adaption of whatever done for semigroups in [2, Berg et al]. We will not repeat the proof, whenever the proof for semigroups can be applied to the polynomials with necessary modification.

Lemma 1. *A sequence $g = (g_n)_{n\geq 0}$ of orthogonal polynomial is the sequence of moments of a measure $\nu \in E_+(\mathbb{R})$ where*

$$E_+(\mathbb{R}) = \{\mu \in M_+(\mathbb{R}); \sum_{n=0}^\infty b_n \int |x^n| d\nu(x) = \int |q_n(x)| d\nu(x) < \infty, \quad \text{for all} \quad n \geq 0\}$$

i.e., is of the form

$$g_n = \int q_n(x)d\nu(x), \quad n = 0, 1, 2, ...$$

if and only if $g \in \mathbb{P}(\mathbb{N}_0)$.

Proposition 2. *For a sequence of orthogonal polynomial $g = (g_n)_{n\geq 0}$ the following conditions are equivalent:*

(i) $g, E_1 g \in \mathbb{P}(\mathbb{N}_0)$;

(ii) $t = (g_0, 0, g_1, 0, g_2, 0, ...) \in \mathbb{P}(\mathbb{N}_0)$;

(iii) There exists $\nu \in M_+([0,\infty[)$ such that

$$g_n = \int_0^\infty q_n(x)d\nu(x), \quad n \geq 0.$$

Proof. The proof of the two directions "(i)⇒(ii)" and "(iii)⇒ (i)" are similar to Theorem 6.2.5 [2]. Now we sufficiently concern our efforts to prove the direction "(ii)⇒(iii)". By the above lemma there is $\sigma \in E_+(\mathbb{R})$ such that

$$t_{2n} = g_n = \int q_{2n}(x)d\sigma(x),$$

$$t_{2n+1} = \int q_{2n+1}(x)d\sigma(x) \quad n \geq 0.$$

hence

$$g_n = \int_0^\infty q_n(x)d\nu(x), \quad n \geq 0.$$

where ν is the image measure of σ under the continuous mapping $x \mapsto x^2$ of $\mathbb{R}$ into $[0,\infty[$.

Theorem 3. *The set QS is a closed set stable under point-wise sums, products and multiplication by non-negative scalars.*

Proof. The above proposition tells us that $(g_n) \in QS$ if and only if $(g_n) and (g_{n+1})$ are positive definite. This shows that QS is a closed set. Its stable under point-wise sum and multiplication by nonnegative scalars, but it also stable under point-wise products this is a real consequence of the following remark; let $(g_n), (f_n) \in QS$ with measures ν, η respectively i.e.,

$$g_n = \int_0^\infty q_n(x)d\nu(x),$$

$$f_n = \int_0^\infty q_n(x)d\eta(x).$$

Then $(g_n f_n)$ is the quasi moment sequence of the product convolution measure $\nu \diamond \eta$.

3. Main Result

Lemma 4. *Let* $p \geq 1, d_j > 0, 0 < q_j < 1, j = 1, ..., p$ *be given. Then* $g_0 = 1$,

$$g_n = \prod_{k=0}^{n-1} (1 + d_1 q_k(q_1) + ... + d_p q_k(q_p))^{-1}, n \geq 1$$

is a QS moment sequence.

Proof. Consider the entire function of p complex variables:

$$f(z_1, ..., z_p) = \prod_{k=0}^{\infty} (1 + z_1 q_k(q_1) + ... + z_p q_k(q_p))$$

The power series expansion of f can be written

$$f(z) = f(z_1, ..., z_p) = \sum_{\alpha} b_\alpha z^\alpha$$

where we use the multi-index notation

$$z = (z_1, ..., z_p), \alpha = (\alpha_1, \alpha_2, ..., \alpha_p), z^\alpha = z_1^\alpha ... z_p^\alpha$$

where the sum is over all integers $\alpha_1 \geq 0, ..., \alpha_p \geq 0$. The coefficients $b_\alpha = b_\alpha(q)$ of the power series are positive as sums of products of powers of $q_k(q_1), ..., q_k(q_p)$. Let

$$\nu = \frac{1}{f(d_1, ..., d_p)} \sum_{\alpha} b_\alpha d^\alpha \delta_{q^\alpha}$$

Then ν is a probability measure with compact support. The n^{th} quasi moment of ν is

$$g_n = \frac{1}{f(d)} \sum_{\alpha} b_\alpha d^\alpha q_k(q^\alpha)$$

$$= \frac{f(d_1 q_{k+n}(q_1), ..., d_p q_{k+n}(q_p))}{f(d_1, ..., d_p)}$$

$$= \prod_{k=0}^{n-1} (1 + d_1 q_k(q_1) + ... + d_p q_k(q_p))^{-1}.$$

Theorem 5. (Main Result) *Let* (h_n) *be a non vanishing quasi Hausdorff moment sequence. Then* (g_n) *defined by* $g_0 = 1$ *and* $g_n = \frac{1}{h_0 ... h_n}$ *for* $n \geq 1$ *is a normalized quasi Stieltjes moment sequence.*

Proof. Any non-negative measure μ on [0,1] is a weak limit of a sequence of discrete measures of the form $a_1\delta_{x_1} + ... + a_p\delta_{x_p}$ where $a_j > 0, j = 1, ..., p$ and $0 < x_1 < x_2 < ... < x_p < 1$[1]. By the closedness of QS (see, Prop 3), it is enough to prove Theorem 5 for discrete measures of this type i.e., to prove that

$$g_n = \prod_{k=1}^{n}(a_1 q_k(x_1) + ... + a_p q_k(x_p))^{-1}$$

with (g_0=1) belongs to QS. We have

$$g_n = (\frac{1}{a_p})^n [q_k(x_p]^{-\binom{n+1}{2}} \prod_{k=1}^{n}(1 + \frac{a_1 q_k(x_1)}{a_p q_k(x_p)} + ... + \frac{a_{p-1} q_k(x_{p-1})}{a_p q_k(x_p)})^{-1}$$

which is the point-wise product of three QS moment sequences, namely $(\frac{1}{a_p})^n$ and the moment sequence $[q_k(x_p]^{-\binom{n+1}{2}}$ and Lemma 4. A representing measure is the product convolution of 3 corresponding representing measures.

Theorem 6. *Let (g_n) be a normalized quasi Stieltjes moment sequence generated by a quasi Hausdorff moment sequence (h_n) i.e., $g_0 = 1$ and $g_n = \frac{1}{h_0 ... h_n}$ for $n \geq 1$. If $h_\infty = c > 0$ then (g_n) is determinate and the support S of the uniquely determined representing measure satisfies $\frac{1}{c} \in S \subseteq [0, \frac{1}{c}]$. The sequence (g_n) is a quasi Hausdorff moment sequence if and only if $h_\infty \geq 1$.*

Proof. Suppose that $h_\infty = c > 0$. Then clearly $g_n \leq \frac{1}{c^n}$, which shows that the support S of μ is contained in $[0, \frac{1}{c}]$, and then μ is determinate. On the other hand, since $h_n \to c$ there exists to any $\epsilon > 0$ an $N \in \mathbb{N}$ such that

$$g_n \geq \frac{(c+\epsilon)^N}{h_1 ... h_n}(\frac{1}{c+\epsilon})^n$$

So, $\frac{1}{c} \in S$. Finally if $h_\infty \geq 1$, then S is a subset of the unit interval, so (g_n) is a HS. Conversely, if (g_n) is a HS and in particular decreasing, we get from $s_n \leq s_{n-1}$ that $h_n \geq 1$ and hence $h_\infty \geq 1$.

As an application of the above two Theorems we get:

Corollary 7. sl For an arbitrary HS (h_n) the sequence (g_n) defined by
$g_0 = 1$ and
$g_n = 1/(1 + h_1)...(1 + h_n)$ for $n \geq 1$ is a HS.

4. General Transformations

In his fundamental paper [3] Berg introduced a non-linear injective transformation T from the set of non-vanishing normalized Hausdorff moment sequences (a_n)

$$a_n = \int_0^1 x^n d\mu(x), \quad n \geq 0, \quad a_0 = 1$$

to the set of normalized Stieltjes moment sequences (s_n) by the formula

$$T[(a_n)_n] = \frac{1}{a_1 ... a_n}$$

This section is devoted to present some general results about the multi-dimensional cases. Let

$$\tilde{a}_n = \int_{[0,1]\times[0,1]} x^n d\mu(x)$$

where $n = (n_1, n_2), x = (x_1, x_2)$ and μ is a non-negative Borel measure on $\mathbb{R}^2$. Using the notation $x^n = x_1^{n_1} x_2^{n_2}$ for $n \in \mathbb{N}_0^2$ the n-th moment of $\mu \in M$ is defined by

$$\tilde{s}_n = \int_{\mathbb{R}^{\not\vDash}} x^n d\mu(x), \quad n \geq 0, \quad a_0 = 1 \quad (1)$$

where M is the set of Borel measures on $\mathbb{R}^2$. We will denote by $\tilde{S}$ the set of all sequences of the form (1).

Theorem 8. *The set $\tilde{S}$ is a closed set stable under point-wise sums, products and multiplication by non-negative scalars.*

Theorem 9. *Let $p \geq 1, c_j > 0, 0 < q_j < 1, j = 1, ..., p$ be given. Then*

$$s(n) = \prod_{k=\{0\}}^{n\backslash\{1\}} (1 + c_1 q_1^k + ... + c_p q_p^k)^{-1}$$

$$= \prod_{k_1=0}^{n_1-1} \prod_{k_2=0}^{n_2-1} (1 + c_1 q_{1;1}^{k_1} q_{1;2}^{k_2} + ... + c_p q_{p;1}^{k_1} q_{p;2}^{k_2})^{-1}$$

belongs to is $\tilde{S}$, where $n = (n_1, n_2); k = (k_1, k_2); q_i = (q_{i,1}, q_{i,2})$.

Proof. Consider the entire function of p complex variables:

$$f(z_1, ..., z_p) = \prod_{k=\{0\}}^{\{\infty\}} (1 + z_1 q_1^k + ... + z_p q_p^k)$$

The power series expansion of f can be written

$$f(z) = f(z_1, ..., z_p) = \sum_{\alpha} b_\alpha z^\alpha$$

where we use the multi-index notation

$$z = (z_1, ..., z_p), \alpha = (\alpha_1, \alpha_2, ..., \alpha_p), z^\alpha = z_1^{\alpha_1} ... z_p^{\alpha_p}$$

where the sum is over all integers $\alpha_1 \geq 0, ..., \alpha_p \geq 0$. The coefficients $b_\alpha = b_\alpha(q)$ of the power series are positive as sums of products of powers of $q_1, ..., q_p$. Let

$$\mu = \frac{1}{f(c_1, ..., c_p)} \sum_{\alpha} b_\alpha c^\alpha \delta_{q^\alpha}$$

Then μ is a probability measure with compact support. The n^{th} quasi moment of μ is

$$s(n) = \frac{1}{f(c)} \sum_{\alpha} b_\alpha c^\alpha (q^\alpha)^n = \frac{f(c_1 q_1^n, ..., c_p q_p^n)}{f(c_1, ..., c_p)} = \prod_{k=\{0\}}^{\{n-1\}} (1 + c_1 q_1^k + ... + c_p q_p^k)^{-1}.$$

Theorem 10. *Let $\tilde{s}(n)$ belongs to the set $\tilde{S}$. Then the sequence $(T(n))$ defined by*

$$T(n) = \frac{1}{\tilde{s}(1) ... \tilde{s}(n)}$$

belongs to the set $\tilde{S}$.

References

[1] A.I. Akhiezer, *The Classical Moment Problem and Some Related Questions In Analysis*, Oliver and Boyd Ltd. (1965).

[2] C. Berg, J.P.R. Christensen, P. Ressel, *Harmonic Analysis on Semigroups. Theory of Positive Definite and Related Functions*, Graduated texts in Math., **100**, Springer-Verlag, Berlin-Heidelberg-New-York (1984).

[3] C. Berg, A. J. Dur'an, A transformation from Hausdorff to Stieltjes moment sequences, *Ark. Mat.*, **42** (2004), 239-257.

[4] A.S. Okb El Bab, H.A. Ghany, Harmonic analysis on hypergroups, *Int. J. Pure Appl. Math.*, **64**, No. 1 (2010), 9-19.