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PETERSEN PETAL GRAPHS

V. Kolappan[1 §], R. Selva Kumar[2]
[1,2]VIT University
Vellore, INDIA

Abstract: In this paper we define $(p_1, p_2, \ldots, p_r)$-petal graph and Petersen petal graph. We identify the petal graphs which are Petersen petal graphs. We also derive some results on the unit distance and domination number an of the Petersen petal graphs.

AMS Subject Classification: 05C10, 05C12, 05C69
Key Words: p-petal graph, $(p_1, p_2, \ldots, p_r)$-petal graph, generalized Petersen graph, Petersen petal graph

1. Introduction

The concept of petal graphs was introduced in [1]. A petal graph is a simple connected (possibly infinite) graph G such that

1. $\triangle(G) = 3$;
2. $\delta(G) = 2$;
3. $G_\triangle$ is 2-regular (possibly disconnected);
4. Each edge of G is incident with at least one vertex in $G_\triangle$.

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§Correspondence author

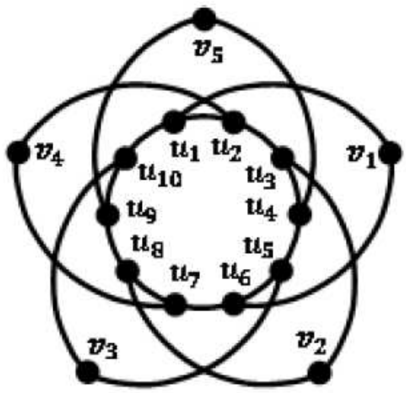

Figure 1: The 5-petal graph $G = P_{5,5}$

If $G_{\triangle}$ is disconnected, then each of its components is a cycle. In this paper, we consider petal graphs with a petals, $P_0,\ P_1,\ \ldots, P_{a-1}$.

The vertex set of G is given by $V = V_1 \cup V_2$, where $V_1 = \{u_i\},\ i = 0, 1, \ldots, 2a-1$ is the set of vertices of degree three, and $V_2 = \{v_j\},\ j = 0, 1, \ldots, a-1$ is the set of vertices of degree two. The subgraph G_δ is the totally disconnected graph with vertex set V_2. The subgraph $G_{\triangle}$ is the cycle $G_{\triangle}:\ u_0,\ u_1, \ldots,\ u_{2a-1}$. The set $P(G) =\ P_0,\ P_1, \ldots,\ P_{a-1}$ is the petal set of G. Consider the path $u_i u_{i+1} \ldots u_{i+k}, k \geq 1$ on the component $G_{\triangle_{\text{I}}}$ of $G_{\triangle}$. Let $v_j \in V_2$ be adjacent to u_i and u_{i+k}. Then the path $P_j = u_i v_j u_{i+k}$ is called a *petal* of G in the component $G_{\triangle_{\text{I}}}$. The path from u_i to u_{i+k} of length $p_j = \min\{k,\ 2a-k\}$ is called the *base* of P_j. We call p_j to be the *size* of the petal P_j. The *petal size* of G is given by $p(G) = \max\{p_j, j = 0, 1, \ldots, a-1\}$. The vertex v_j is called the *center* of the petal P_j. The vertices u_i and u_{i+k} are called the *base points* of P_j.

A petal graph G of size n with petal sequence $\{P_j\}$ is said to be a *p-petal graph* denoted $G =\ P_{n,p}$, if every petal in G is of size p and $l(P_i,\ P_{i+1}) = 2, i = 0, 1, 2, \ldots, a-1$ where the suffices are taken modulo a. In a p-petal graph the petal size p is always odd because, otherwise $l(P_i, P_{i+1})$ will not be 2 for some i. Figure 1 in page 2 shows a 5-petal graph.

For basic definitions, terminologies and theorems on petal graphs, see [2].

H.S.M. Coxeter introduced a family of graphs generalizing the Petersen graph in 1950 (see [3]). Mark Watkins [4] denoted these graphs as $G(n,k)$ and named them the generalized Petersen graphs.

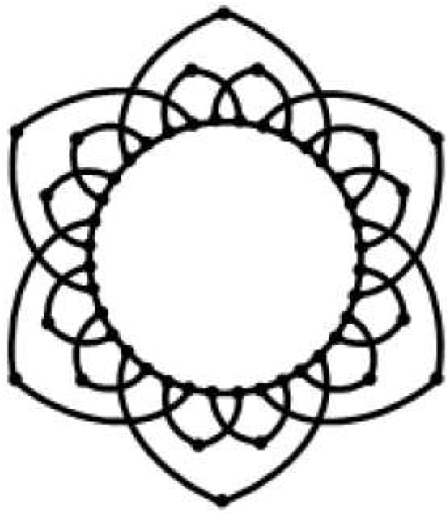

Figure 2: A $(3, 9)$-petal graph

A generalized Petersen graph $P(n, k)$ with parameters n and k, $1 \leq k \leq n-1$, $k \leq n/2$, is a graph on $2n$ vertices a_i, $0 \leq i \leq 1$ and b_j, $0 \leq j \leq n-1$, with $3n$ edges $a_i a_{(i\pm1)}$, $b_j b_{(j\pm k)}$ and $a_i b_i$, where all calculations have to be performed modulo n. These edges are called ring edges, chordal edges and spokes respectively. $P(5, 2)$ is the Petersen graph.

In Section 2, we define $(p_1, p_2, \ldots, p_r)$-petal graph and Petersen petal graph. Further, we identify the petal graphs that are Petersen petal graphs. In Section 3, we prove that Petersen petal graphs are unit distance graphs. We also derive some results on the domination number of the Petersen petal graphs.

2. Petersen Petal Graph

Definition 1. A petal graph G is called a $(p_1, p_2, \ldots, p_r)$-petal graph if a_i petals of G are of size p_i, $i = 1, 2, \ldots, r$ and is denoted by $G = P_{n,(p_1,p_2,\ldots,p_r)}$. Refer Figure 2 in page 3 for a $(3, 9)$-petal graph $G = P_{54,(3,9)}$ with 12 petals of size 3 and 6 petals of size 9.

Definition 2. A $(p_1, p_2, \ldots, p_r)$-petal graph G is said to be a Petersen petal graph, denoted $G = {P^*}_{n,(p_1,p_2,\ldots,p_r)}$ if G is isomorphic to a graph that can be obtained from the generalized Petersen graph $P(n, k)$ either by subdivision of some of its edges or deletion of some of its vertices.

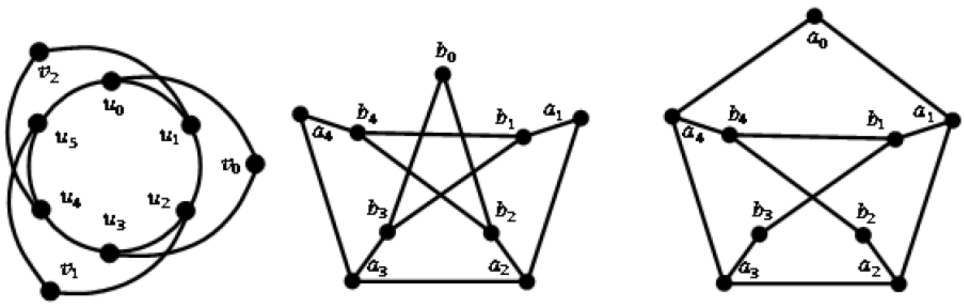

Figure 3: The graphs $P_{9,3}$, $P(5,2) \setminus \{u_0\}$ and $P(5,2) \setminus \{v_0\}$

Theorem 3. *The petal graph $P_{9,3}$ is a Petersen petal graph.*

Proof. The Petersen graph $P(5,2)$ is a 3-regular graph with ten vertices $a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3,\ b_4$ and fifteen edges. When any vertex is deleted from $P(5,2)$, it will have nine vertices and twelve edges. Out of the nine vertices, three will be of degree two and six will be of degree three.

Case 1: Deleting a vertex on the outer polygon

Without loss of generality, let us assume that the vertex a_0 is deleted. Properties 1 to 3 in the definition of petal graphs are satisfied because the vertices a_2, a_3, b_1, b_2, b_3, and b_4 are of degree three that induce the 6-cycle $G_\triangle$: $b_1, b_3, a_3, a_2, b_2, b_4$ and the vertices a_1, a_4 and b_0 are of degree two that induce a totally disconnected graph. Property 4 follows from the fact that the vertices a_1, a_4 and b_0 are adjacent to the vertices on $G_\triangle$. Thus $P(n,2) \setminus a_0$, is a petal graph.

Let $G = P_{9,3}$ be a 3-petal graph with vertices $u_0, u_1, \ldots, u_5, v_0, v_1$ and v_2. We define a mapping $\psi : P(5,2) \setminus a_0 \longrightarrow P_{9,3}$, such that $\psi(b_1) = u_0$, $\psi(b_3) = u_1$, $\psi(a_3) = u_2$, $\psi(a_2) = u_3$, $\psi(b_2) = u_4$, $\psi(b_4) = u_5$ and $\psi(a_1) = v_0$, $\psi(a_4) = v_1$, $\psi(b_0) = v_2$. From the adjacency matrices of the graphs $P(5,2) \setminus a_0$ and $P_{9,3}$, we can say that they are isomorphic to each other.

Case 2: Deleting a vertex on the inner polygon

Without loss of generality, let us assume that the vertex b_0 is deleted. Properties 1 to 3 in the definition of petal graphs are satisfied because the vertices a_1, a_2, a_3, a_4, b_1, and b_4 are of degree three that induce the 6-cycle $G_\triangle$: $b_1, a_1, a_2, a_3, a_4, b_4$ and the vertices b_3, b_2 and a_0 are of degree two that induce a totally disconnected graph. Property 4 follows from the fact that the vertices b_3, b_2 and a_0 are adjacent to the vertices on $G_\triangle$. Thus $P(5,2) \setminus b_0$, is also a petal graph.

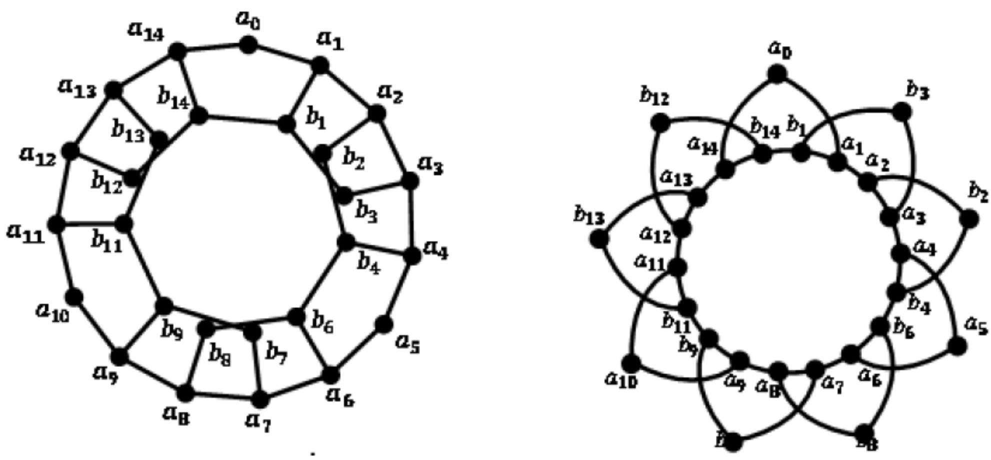

Figure 4: The graphs $P(15,2) \setminus \{b_0, b_5, b_{10}\}$ and $P_{27,3}$

Let $G = P_{9,3}$ be a 3-petal graph with vertices $u_0, u_1, \ldots, u_5, v_0, v_1$ and v_2. We define a mapping $\psi : P(5,2) \setminus b_0 \longrightarrow P_{9,3}$, such that $\psi(b_1) = u_0$, $\psi(a_1) = u_1$, $\psi(a_2) = u_2$, $\psi(a_3) = u_3$, $\psi(a_4) = u_4$, $\psi(b_4) = u_5$ and $\psi(b_3) = v_0$, $\psi(b_2) = v_1$, $\psi(a_0) = v_2$. From the adjacency matrices of the graphs $P(5,2) \setminus b_0$ and $P_{9,3}$, we can say that they are isomorphic to each other. Figure 3 in page 4 shows $P(5,2) \setminus a_0$, $P(5,2) \setminus b_0$ and $P_{9,3}$. □

Theorem 4. *The* 3*-petal graph* $P_{9l,3}$ *is a Petersen petal graph.*

Proof. When we delete the l vertices b_{i-1}, $i = 1, 6, 11, 16, \ldots, 5l-4$ from the inner polygon of $P(n,2)$, $n = 5l$, we obtain the graph $P(n,2) \setminus b_{i-1}$ with $9l$ vertices and $12l$ edges. The $3l$ neighbors a_{i-1}, b_{i+1} and b_{i+2}, will be of degree two each and induce a totally disconnected graph. The remaining $6l$ vertices each of degree three induce the $6l$-cycle $G_\triangle$: b_i, a_i, a_{i+1}, a_{i+2}, a_{i+3}, b_{i+3} and hence the first three properties in the definition of petal graphs are satisfied. The vertices a_{i-1}, b_{i+2} and b_{i+1} are adjacent to the vertices $\{a_{i-2}, a_i\}$, $\{b_i, a_{i+2}\}$ and $\{a_{i+1}, b_{i+3}\}$ respectively. Property 4 is satisfied by these $6l$ edges together with the $6l$ edges of the cycle, making $P(n,2)\setminus\{b_{i-1}\}$,$n = 5l$, $l \geq 3$, $i = 1, 6, \ldots, 5l-4$, a petal graph.

We consider the 3-petal graph $P_{9l,3}$ with vertices u_0, u_1, …, u_{6l-1}, v_0, v_1, …, v_{3l-1}. We define a mapping $\psi : P(n,2) \setminus \{b_{i-1}\} \longrightarrow P_{9l,3}$ such that $\psi(b_i) = u_j$, $\psi(a_i) = u_{j+1}$, $\psi(a_{i+1}) = u_{j+2}$, $\psi(a_{i+2}) = u_{j+3}$, $\psi(a_{i+3}) = u_{j+4}$, $\psi(b_{i+3}) = u_{j+5}$, and $\psi(a_{i-1}) = v_j$, $\psi(b_{i+2}) = v_{j+1}$, $\psi(b_{i+1}) = v_{j+2}$, where

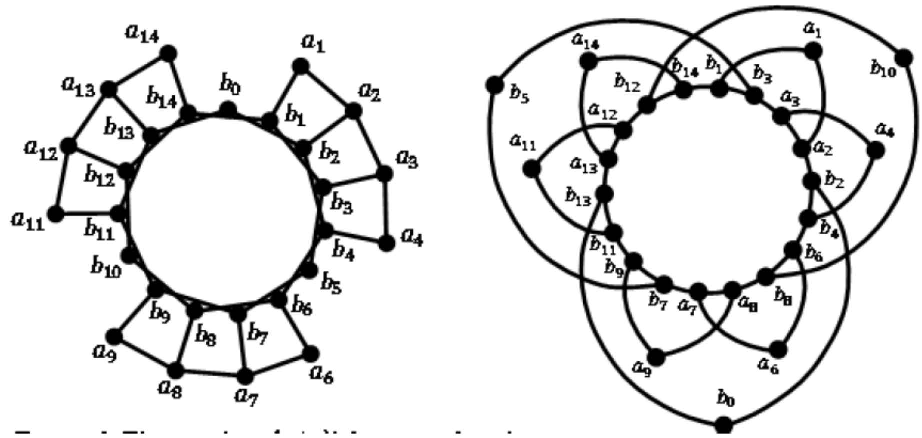

Figure 5: The graphs $P(15, 2) \setminus \{a_0, a_5, a_{10}\}$ and $P_{27,(3,9)}$

$i = 1, 6, 11, 16, \ldots, 5l - 4$, $j = 0, 6, \ldots, 6(l - 1)$ and all calculations are taken modulo $5l$. The adjacency matrices of $P(n, 2) \setminus \{b_{i-1}\}$ and $P_{9l,3}$ prove that these graphs are isomorphic to each other. In the petal graph in Figure 4 in page 5, the vertices are labeled with the vertices of $P(n, 2) \setminus \{b_{i-1}\}$. □

Theorem 5. *The petal graph $P_{9l,(3,9)}$, $l \geq 3$ is a Petersen petal graph.*

Proof. Consider the generalized Petersen graph $P(n, 2)$ where $n = 5l$, $l \geq 3$, which is a 3-regular graph with $2n$ vertices and $3n$ edges.

We delete from the outer polygon of $P(n, 2)$, $n = 5l$, $l \geq 3$, the l vertices a_{i-1}, $i = 1, 6, 11, 16, \ldots, 5l - 4$ to obtain the graph $P(n, 2) \setminus \{a_{i-1}\}$ with $9l$ vertices and $12l$ edges. The $3l$ vertices a_i, a_{i+3}, b_{i+9}, each of degree two induce a totally disconnected graph and the remaining $6l$ vertices each of degree three induce the $6l$-cycle $G_\triangle$: b_i, b_{i+2}, a_{i+2}, a_{i+1}, b_{i+1}, b_{i+3}. The vertices a_i, a_{i+3} and b_{i+9} are adjacent to the vertices $\{b_i, a_{i+1}\}, \{a_{i+2}, b_{i+3}\}$ and $\{b_{i-4}, b_{i+7}\}$ respectively. Thus all the four Properties are satisfied and hence $P(n, 2) \setminus \{a_{i-1}\}$, $n = 5l$, $l > 2$, $i = 1, 6, \ldots, 5l - 4$, a petal graph.

We consider the 3-petal graph $P_{9l,(3,9)}$ with vertices u_0, u_1, $\ldots$, u_{6l-1}, v_0, v_1, $\ldots$, v_{3l-1}. We define a mapping $\psi : P(n, 2) \setminus \{a_{i-1}\} \longrightarrow P_{9l,(3,9)}$ such that $\psi(b_i) = u_j$, $\psi(b_{i+2}) = u_{j+1}$, $\psi(a_{i+2}) = u_{j+2}$, $\psi(a_{i+1}) = u_{j+3}$, $\psi(b_{i+1}) = u_{j+4}$, $\psi(b_{i+3}) = u_{j+5}$, and $\psi(a_i) = v_j$, $\psi(a_{i+3}) = v_{j+1}$, $\psi(b_{i+9}) = v_{j+2}$, where $i = 1, 6, 11, 16, \ldots, 5l - 4$, $j = 0, 6, \ldots, 6(l - 1)$ all calculations are taken modulo $5l$. The adjacency matrices of $P(n, 2) \setminus \{a_{i-1}\}$ and $P_{9l,(3,9)}$ show that

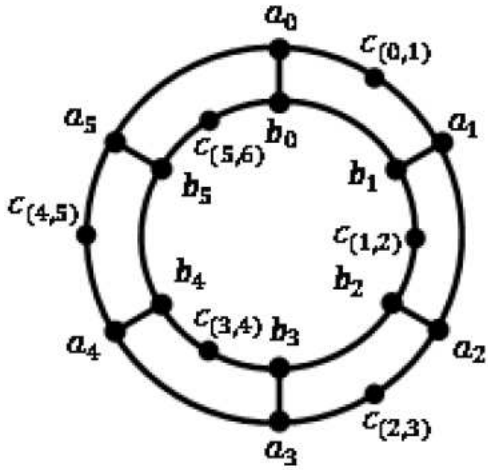

Figure 6: A subdivision of $P(6,1)$ to get $P^*_{18,3}$

$P(n,2) \setminus \{a_{i-1}\}$ is isomorphic to the $(3,9)$-petal graph $G = P_{9l,(3,9)}$ under the mapping ψ. Refer Figure 5 in page 6. □

Theorem 6. *Any planar 3-petal graph is a Petersen petal graph.*

Proof. We consider the generalized Petersen graph $P(n,1)$, n is even. Subdivide the $n/2$ edges (a_i, a_{i+1}) of the outer polygon with vertices $c_{i,i+1}$, $i = 0, 2, 4, \ldots, n-2$. Subdivide the $n/2$ edges (b_j, b_{j+1}) of the inner polygon with vertices $c_{j,j+1}$, $j = 1, 3, 5, \ldots, n-1$ where all calculations are taken modulo n. The resultant graph, denoted $P(n_1, 1)$ has $3n$ vertices, $2n$ of which have degree three and the remaining n have degree two. Refer Figure 6 in page 7. Also, the number of edges in $P(n_1,1)$ is $4n$. This graph has the $2n$-cycle, named $G_\triangle$: a_0, b_0, b_1, a_1, a_2, b_2, …, b_{n-1}, a_{n-1} of vertices of degree three. Each of the vertices $c_{i,i+1}$ and $c_{j,j+1}$ of degree two is adjacent to exactly two vertices of $G_\triangle$. Thus $P(n_1,1)$ satisfies all the four properties in the definition of petal graphs and hence is a petal graph.

Let $P_{n',3}$, $n' = 3a$ is even, be a planar 3-petal graph. We define a mapping $\phi : P(n_1,1) \longrightarrow P_(n',3)$ such that $\phi(a_j) = u_{2j}$, $\phi(b_j) = u_{2j+1}$, $\phi(c_{j,j+1}) = v_j$, $j = 0, 2, 4, \ldots, (n-2)$, $\phi(b_j) = u_{2j}$, $\phi(a_j) = u_{2j+1}$, $\phi(c_{j,j+1}) = v_j$, $j = 1, 3, 5, \ldots, (n-1)$. It is evident from the adjacency matrices of the two graphs $P(n_1,1)$ and $P_{n',3}$ that ϕ is an isomorphism. □

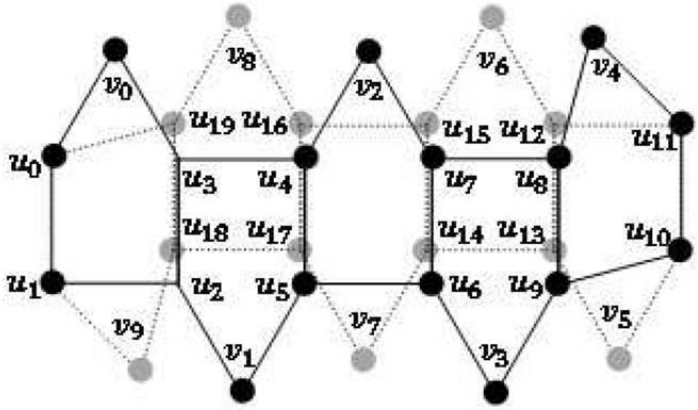

Figure 7: The unit distance representation of $P^*_{30,3}$

3. Some Results on Petersen Petal Graphs

Results on Unit distance, domination number, crossing number, number of Hamiltonian cycles, etc. of certain generalized Petersen graphs are known. In this section we attempt to explore some of these results on Petersen petal graphs.

Unit distance: A graph G is called a unit distance graph if it can be drawn in such a way that all the edges of G are of the same length. Such a representation of G is called a unit distance representation of G.

Theorem 7. *The Petersen petal graphs are unit distance graphs.*

Proof. Petersen petal graphs are obtained by either deletion of vertices from or subdivision of edges of generalized Petersen graphs, the unit distance graphs. Since deletion of vertices does not influence the distance between other vertices, the Petersen petal graphs $P^*_{9,3}$, $P_{9l,3}$ and $P_{9l,(3,9)}$, $l \geq 3$ that are obtained by deletion of vertices are also unit distance graphs.

Consider the planar Petersen petal graph $G = P^*_{n,3}$ with a petals, $n = 3a$ vertices and $m = 4a$ edges, obtained by subdivision of edges of generalized Petersen graph $P(n, 1)$. The number of petals a is even and the petal size $p = 3$. Let $S : u_0, u_1, \ldots, u_{2a-1}$ be the vertex sequence of G. Let $v_0, v_1, \ldots, v_{a-1}$ be the centers of the petals $P_0, P_1, \ldots, P_{a-1}$ of G. Each petal P_i together with its base forms a cycle of length 5. Let $C_0, C_1, \ldots, C_{a-1}$ be the cycles of length 5 each containing the vertices $v_0, v_1, \ldots, v_{a-1}$ respectively. We observe that the cycles C_i and C_{i+1}, $i = 0$ to $a - 1$, $i + 1 \equiv 0(mod n)$

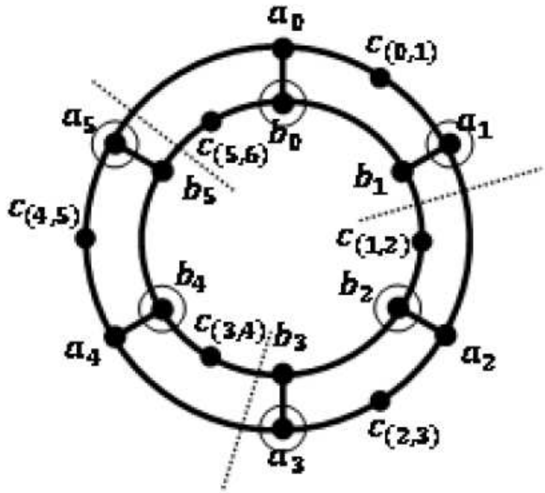

Figure 8: The domination number of $P^*_{18,3}$ is 6

share a common edge. We represent each cycle C_i as a polygon ABCDE with unit sides (an equilateral triangle ABE placed on either a square or a rhombus BCDE without their common side BE). Now we embed the cycles C_0, C_1, ..., C_{a-1} on a virtual parallelogram PQRS with sides PQ and RS of length $a-1$ and sides QR and SP of length 1. Refer Figure 7 in page 8. □

Domination Number: A set D of vertices of a graph G is a dominating set if each vertex in $V-D$ is adjacent to at least one vertex in D. The domination number of G denoted by $\gamma(G)$, is the cardinality of a minimum dominating set of G. For further definitions and results on domination number, one can refer to [5].

Theorem 8. *The domination number of the planar Petersen petal graph $G = P^*_{n,3}$ is n.*

Proof. The Petersen petal graph $G = P^*_{n,3}$ is isomorphic to a subdivision of $P(n,1)$, n is even. Let $n = 2l$, $l \geq 1$. $P(2l,1)$ is 3-regular and has $4l$ vertices. The graph $G = P^*_{2l,3}$ adds $2l$ vertices to $V(P(2l,1))$ by subdividing the l edges (a_i, a_{i+1}) of the outer polygon with vertices $c_{(i,i+1)}$, $i = 0, 2, 4, \ldots, 2l-2$ and subdividing the l edges (b_j, b_{j+1}) of the inner polygon with vertices $c_{j,j+1}$, $j = 1, 3, 5, \ldots, 2l-1$. Each of the l vertices b_{j-1} dominates its three neighbors $c_{j-2,j-1}, a_{j-1}$ and b_j. Each of the l vertices a_{i+1} dominates its neighbor $c_{i,i+1}$.

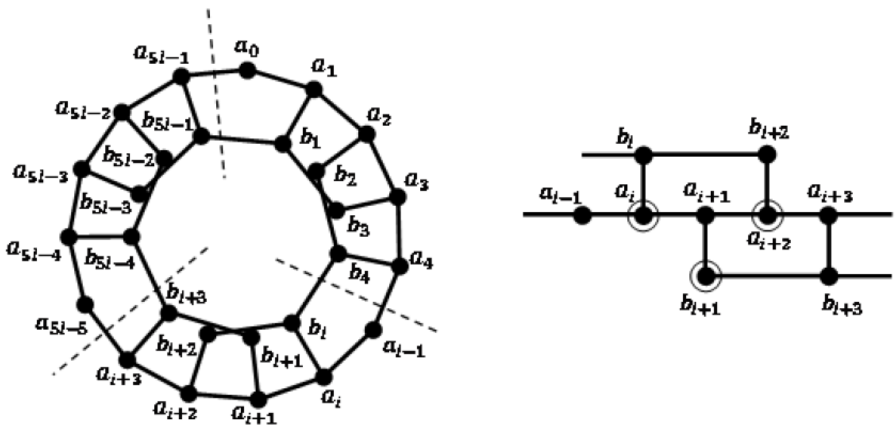

Figure 9: The domination number of $P(n, 2) \setminus \{b_{i-1}\}$ is $3l$

Refer Figure 8 in page 9. Hence, the domination number of the planar Petersen petal graph $G = P^*_{2l,3}$ is $2l$. □

Theorem 9. *The domination number of the Petersen petal graphs $G = P^*_{9l,3}$ and $G = P^*_{9l,(3,9)}$ is $3l$.*

Proof. The Petersen petal graphs $G = P^*_{9l,3}$ and $G = P^*_{9l,(3,9)}$ are isomorphic to the graphs $P(n, 2) \setminus \{b_{i-1}\}$ and $P(n, 2) \setminus \{a_{i-1}\}$, $i = 1, 6, 11, 16, \ldots, 5l - 4$, respectively.

Case 1: Consider the graph $P(n, 2) \setminus \{b_{i-1}\}$.

Case 1a: Let G be the union of the l subgraphs G_1, G_2, …, G_l with vertex sets $\{a_{i-1}, a_i, a_{i+1}, a_{i+2}, a_{i+3}\} \bigcup \{b_i, b_{i+1}, b_{i+2}, b_{i+3}\}$, $i = 1, 6, 11, 16, \ldots, 5l-4$. In G_i, let a_i dominate a_{i-1}, b_i and a_{i+1}. The vertex a_{i+2} can dominate b_{i+2} and a_{i+3}. The vertex b_{i+1} (or b_{i+3}) can dominate b_{i+3} (or b_{i+1}).

Case 1b: Let G be the union of the l subgraphs G_1, G_2, …, G_l as in case $1a$. In G_i, let a_i dominate a_{i-1} and b_i. The vertex a_{i+2} can dominate b_{i+2}, a_{i+1} and a_{i+3}. Then the vertex b_{i+1} can dominate b_{i+3}. (Otherwise, a_{i+2} can dominate b_{i+2} and a_{i+3}, and b_{i+1} can dominate a_{i+1} and b_{i+3}. Or else, a_{i+2} can dominate a_{i+1} and b_{i+2}, and b_{i+3} can dominate b_{i+1} and a_{i+3}). Refer Figure 9 in page 10.

Even if we consider G as a union of $l/2$ subgraphs or less, the result could not be improved.

Case 2: Consider the graph $P(n, 2) \setminus \{a_{i-1}\}$.

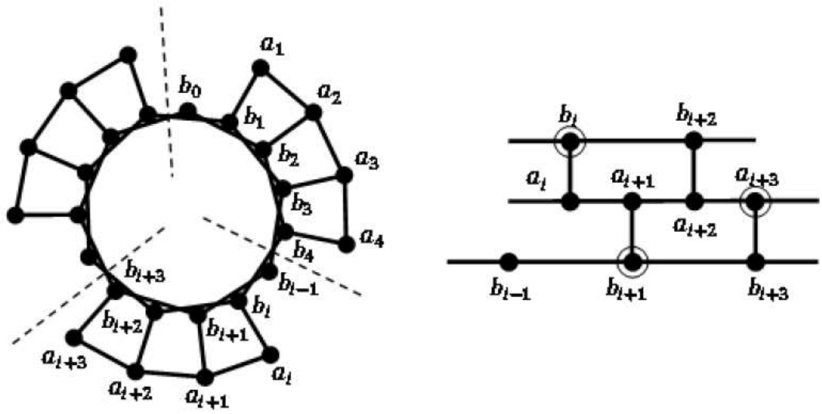

Figure 10: The domination number of $P(n,2) \setminus \{a_{i-1}\}$ is $3l$

Case 2a: Let G be the union of the l subgraphs G_1, G_2, ..., G_l with vertex sets $\{a_i, a_{i+1}, a_{i+2}, a_{i+3}\} \bigcup \{b_{i-1}, b_i, b_{i+1}, b_{i+2}, b_{i+3}\}$, $i = 1, 6, 11, 16, \ldots, 5l-4$. In G_i, let b_{i+1} dominate b_{i-1}, a_{i+1} and b_{i+3}. The vertex b_i can dominate a_i and b_{i+2}. The vertex a_{i+2} (or a_{i+3}) can dominate a_{i+3} (or a_{i+2}).

Case 2b: Let G be the union of the l subgraphs G_1, G_2, ..., G_l as in case 2a. In G_i, let b_{i+1} dominate b_{i-1} and a_{i+1}. The vertex a_{i+3} can dominate a_{i+2} and b_{i+3}, and the vertex b_i can dominate a_i and b_{i+2}. Refer Figure 10 in page 11.

The three more cases with u_3 dominating $\{u_2, v_3, u_4\}$ or $\{v_3, u_4\}$ or $\{u_2, v_3\}$ could not improve the result. □

4. Conclusion

The petal graph is a very interesting class of graphs whose properties and characteristics are yet to be explored. Researchers can throw light on more variants of petal graphs and prove important results, theorems and conjectures with reference to petal graphs.

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