International Journal of Pure and Applied Mathematics
Volume 75 No. 3 2012, 269-278
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

PETAL GRAPHS

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Abstract: In this paper we introduce p-petal graphs. We prove that the necessary and sufficient condition for a planar p-petal graph G is that G has even number of petals each of size three. We also characterize the planar partial p-petal graphs.

AMS Subject Classification: 05C10
Key Words: petal graph, p-petal graph, partial p-petal graph, planarity

1. Introduction

A *petal graph* G is a simple connected (possibly infinite) graph with maximum degree three, minimum degree two, and such that the set of vertices of degree three induces a 2-regular graph $G_{\triangle}$ (possibly disconnected) and the set of vertices of degree two induces a totally disconnected graph G_{δ}. (see [1]). If $G_{\triangle}$ is disconnected, then each of its components is a cycle. In this paper, we consider petal graphs with a petals, $P_0,\ P_1,\ \ldots, P_{a-1}$ and r components, $G_{\triangle_0},\ G_{\triangle_1}, \ldots, G_{\triangle_{r-1}}$.

The vertex set of G is given by $V = V_1 \cup V_2$, where $V_1 = \{u_i\},\ i = 0, 1, \ldots, 2a-1$ is the set of vertices of degree three, and $V_2 = \{v_j\},\ j = 0, 1, \ldots, a-1$ is the set of vertices of degree two. The subgraph G_{δ} is the totally disconnected graph with vertex set V_2. The subgraph $G_{\triangle}$ is the cycle $G_{\triangle} :\ u_0,\ u_1, \ldots,\ u_{2a-1}$. The set $P(G) =\ P_0,\ P_1, \ldots,\ P_{a-1}$ is the petal set

Received: July 7, 2011

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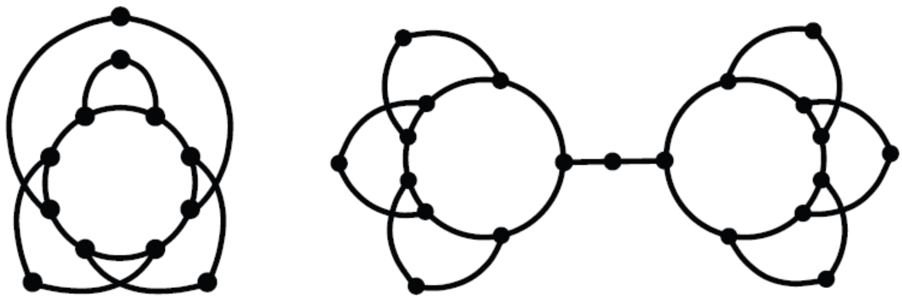

Figure 1: A finite and an infinite petal graph

of G. Consider the path $u_i u_{i+1} \dots u_{i+k}, k \geq 1$ on the component $G_{\triangle_l}$ of $G_\triangle$. Let $v_j \in V_2$ be adjacent to u_i and u_{i+k}. Then the path $P_j = u_i v_j u_{i+k}$ is called a *petal* of G in the component $G_{\triangle_l}$. The path from u_i to u_{i+k} of length $p_j = \min\{k,\ 2a-k\}$ is called the *base* of P_j. We call p_j the *size* of the petal P_j. The *petal size* of G is given by $p(G) = \max\{p_j, j = 0, 1, \dots, a-1\}$. The vertex v_j is called the *center* of the petal P_j. The vertices u_i and u_{i+k} are called the *base points* of P_j. We refer a petal of finite size to be a finite petal. We denote the number of finite petals in G as $a_f = \sum_{t=0}^{r-1} a_t$.

The petal $P_j = u_k v_j u_l$, where u_k and u_l are in distinct components of $G_\triangle$ is called an *infinite petal*. The size of an infinite petal is *infinity*. We call the vertices u_k and u_l as the *base points* of P_j. An infinite petal *connects* two components of G. The set of infinite petals connecting two components $G_{\triangle_k}$ and $G_{\triangle_l}$ is denoted by $P\left(G_{\triangle_k} \cup G_{\triangle_l}\right)$. The number of infinite petals in $P\left(G_{\triangle_k} \cup G_{\triangle_l}\right)$ is denoted by a_{kl}. The number of infinite petals in a petal graph is given by $a_\infty = \sum_{0 \leq k < l \leq r-1} a_{kl}$. The number of petals a in a petal graph G is given by $a = a_f + a_\infty$. Figure 1 presents a finite and an infinite petal graph.

We give basic definitions and prove preliminary theorems in Section 2. Our main results are given in Theorem 10 and Corollary 11 in Section 3.

2. Basic Definitions and Results

Definition 1. The sequence $\{u_i\}$ of vertices $u_0, u_1, \dots, u_{2a-1}$ of $G_\triangle$ is called the *vertex sequence* of G. The sequence $\{v_j\}$ of vertices $v_0, v_1, \dots, v_{a-1}$

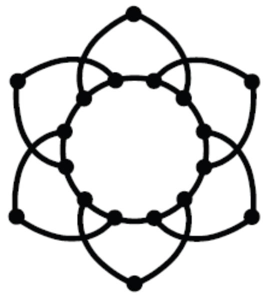

Figure 2: A 3-petal graph

of G_δ is called the *vertex sequence* of the petals of G. The sequence $\{P_j\}$ of petals $P_0,\ P_1,\ \ldots, P_{a-1}$ is called the *petal sequence* of G. Note that the suffices are written in the ascending order.

Definition 2. The *distance* $l(P_i,\ P_j)$ between two petals $P_i = u_k v_i u_l$ and $P_j = u_{k'} v_j u_{l'}$ in $G_{\triangle_l}$ is the length of the shortest path between their base points u_k and $u_{k'}$ or u_l and $u_{l'}$ in $G_{\triangle_l}$.

Let $P_i = u_k v_i u_l$ and $P_j = u_{k'} v_j u_{l'}$ be two petals of G. If $P_i \in P(G_{\triangle_s})$ and $P_j \in P(G_{\triangle_s} \cup G_{\triangle_t})$, then $l(P_i,\ P_j)$ in $G_{\triangle_s}$ is the length of the shortest path between u_k and $u_{k'}$ in $G_{\triangle_s}$. If P_i and $P_j \in P(G_{\triangle_s} \cup G_{\triangle_t})$, then $l(P_i,\ P_j)$ in $G_{\triangle_s}$ is the length of the shortest path between their base points u_k and $u_{k'}$ in $G_{\triangle_s}$; $l(P_i,\ P_j)$ in $G_{\triangle_t}$ is the length of the shortest path between their base points in u_l and $u_{l'}$ in $G_{\triangle_t}$. When each base point of P_i and P_j is in distinct components of $G_\triangle$, then $l\left(P_i,\ P_j\right) = \infty$. In this paper we consider petal graphs whose infinite petals do not cross one another.

Definition 3. A petal graph G of size n with petal sequence $\{P_j\}$ is said to be a *p-petal graph* denoted $G = P_{n,p}$, if every petal in G is of size p and $l(P_i,\ P_{i+1}) = 2, i = 0, 1, 2, \ldots, a-1$ where the suffices are taken modulo a. In a p-petal graph the petal size p is always odd because, otherwise $l(P_i, P_{i+1})$ will not be 2 for some i. Figure 2 shows a 3-petal graph with 6 petals.

Definition 4. A petal graph G is said to be a *partial petal graph* if $G_\triangle$ is disconnected. The partial petal graph G is called a *partial p-petal graph* if every finite petal in G is of size p and $l(P_i, P_{i+1}) = 2$ for any petal P_i in any component $G_{\triangle_l}$.

Definition 5. Two infinite petals P_i and P_j of $P\left(G_{\triangle_k} \cup G_{\triangle_l}\right)$ form an *infinite petal pair* if their base points lie on the bases of two successive finite

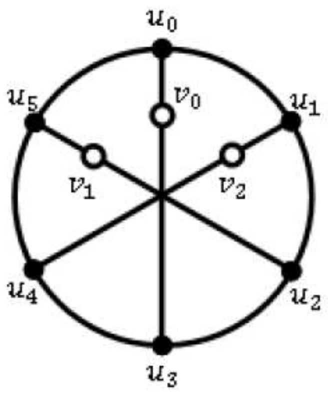

Figure 3: Non-planar representation of $G = P_{9,3}$

petals in both $G_{\triangle_k}$ and $G_{\triangle_l}$.

Definition 6. A component $G_{\triangle_k}$ is called a *pendant component* if it is connected to only one component of G by way of infinite petals.

Theorem 7. *If G is a p-petal graph of order n and size m, then $n = 3a$ and $m = 4a$ such that $m < 3n - 6$, where $a > 1$.*

Proof. Let G be a p-petal graph of order n and size m. Now $|V(G_\delta)| = a$ implies that $|V(G_\triangle)| = 2a$. Hence $|V(G)| = 3a$ or $n = 3a$. The contribution to the edge set of G by each petal is 2 and by $G_\triangle$ is $2a$. Thus, $m = 4a$. Now, $3n - 6 = 9a - 6 > m$, when $a > 1$. □

Theorem 8. *If G is a p-petal graph, then there are at least p petals in G.*

Proof. Consider a p-petal graph G with a petals. There are $p - 1$ vertices on the base of any petal $P_j = u_i v_j u_k$ other than u_i and u_k. This implies that there are $p - 1$ petals in G other than P_j with one of their base points lying in the base. □

3. Planarity of p-Petal Graphs

It can be easily verified that a p-petal graph $G = P_{n,p}$ is planar when $p(G) = 1$ for any value of n as well as a. The graph P^* obtained from the Petersen graph by removing one of the vertices is a 3-petal graph $P_{9,\ 3}$. The petal graph P^* is a subdivision of $K_{3,\ 3}$ and hence not a planar graph.

Note 9. The suffices of u_i are always taken modulo $2a$ and that of v_j are taken modulo a.

Note 10. For any petal $P_j = u_i v_j u_k$, $j = even(i, j) \div 2$.

Theorem 11. *A p-petal graph* $G = P_{n,p} (p \neq 1)$ *is planar if and only if the following conditions are satisfied.*

(i) $p = 3$.

(ii) a is an even integer.

Proof. Consider the p-petal graph $G = P_{n,p}$, where p is an odd number. Let $S: u_0, u_1, u_2, \ldots, u_{2a-1}$ be the vertex sequence of G. Assume that at least one of the above conditions is not satisfied. The following cases arise:

Case 1: $p = 3$ and a is odd.

Case 1a: $a = 3$. Let $G = P_{9,3}$ be the given p-petal graph with the vertex sequence $S: u_0,\ u_1,\ u_2,\ u_3,\ u_4,\ u_5$.

Partition the vertex set $V_1(G)$ into $V_1^1(G) = \{u_0,\ u_2, u_4\}$ and $V_1^2(G) = \{u_1,\ u_3,\ u_5\}$. Represent G using the following steps so that the adjacency of the vertices of G is preserved:

Take the cycle $u_0u_1u_2u_3u_4u_5$ containing the vertices of $V_1^1(G) \cup V_1^2(G)$; Subdivide the edges (u_{2i}, u_{2i+2}), with the vertices v_i, $i = 0$ to 2 respectively. Refer Figure 3 in page 5.

This representation of $G = P_{9,3}$ is isomorphic to a subdivision of $K_{3,3}$.

Case 1b: $a > 3$.

Partition the vertex set $V_1(G)$ into three sets $V_1^1(G)$, $V_1^2(G)$ and $V_1^3(G)$ such that $V_1^1(G) = \{u_0,\ u_2,\ u_{2a-2}\}$ and $V_1^2(G) = \{u_1,\ u_3,\ u_{2a-1}\}$ and $V_1^3(G)$ has the remaining vertices. Represent G using the following steps so that the adjacency of the vertices of G is preserved:

Take the cycle $u_0, u_1, u_2, u_3, u_{2a-2}, u_{2a-1}$ containing the vertices of $V_1^1(G) \cup V_1^2(G)$; Connect the pairs of vertices (u_0, u_3), (u_1, u_{2a-2}), (u_2, u_{2a-1}); Subdivide the edges (u_3, u_{2a-2}) with the vertices $u_4,\ u_7, \ldots, u_i,\ u_{i+3},\ u_{i+4},\ u_{i+7}, \ldots,$ $u_{2a-6},\ u_{2a-3}$ and the edge (u_2, u_{2a-1}) with the vertices $u_5,\ u_6, \ldots, u_i,\ u_{i+1},\ u_{i+4},$ $u_{i+5}, \ldots, u_{2a-5},\ u_{2a-4}$; Connect the pairs of vertices $(u_4, u_5), (u_6, u_7), \ldots, (u_{2a-4},$ $u_{2a-3})$; Subdivide the edges (u_{2i}, u_{2i+3}), with the vertices v_i, $i = 0$ to $a-1$ respectively. Refer Figure 4 in page 6.

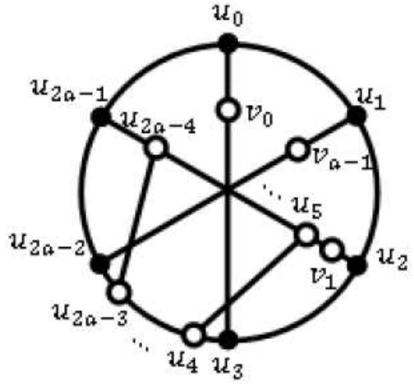

Figure 4: $G = P_{n,3}$ is non planar when $a > 3$ is odd

This representation of $G = P_{n,3}$ is isomorphic to a graph having a subdivision of $K_{3,3}$ as its subgraph.

Case 2: $p > 3$ and a is odd.

Case 2a: $a = p$.

Partition the vertex set $V_1(G)$ into three sets $V_1^1(G)$, $V_1^2(G)$ and $V_1^3(G)$ such that $V_1^1(G) = \{u_0,\ u_{a-1},\ u_{a+1}\}$ and $V_1^2(G) = \{u_1,\ u_a,\ u_{2a-1}\}$ and $V_1^3(G)$ has the remaining vertices. Represent G using the following steps so that the adjacency of the vertices of G is preserved:

Take the cycle $u_0, u_1, u_{a-1}, u_a, u_{a+1}, u_{2a-1}$ containing the vertices of $V_1^1(G)$ $\cup V_1^2(G)$; Connect the pairs of vertices (u_0, u_a), (u_1, u_{a+1}), (u_{a-1}, u_{2a-1}); Subdivide the edge (u_1, u_{a-1}) with the vertices $u_2,\ u_3,\ \ldots, u_{a-2}$ and (u_{a+1}, u_{2a-1}) with the vertices $u_{a+2},\ u_{a+3},\ \ldots, u_{2a-2}$; Connect the vertex pairs (u_2, u_{a+2}), (u_3, u_{a+3}), $\ldots, (u_i, u_{a+i})$, $\ldots, (u_{a-2}, u_{2a-2})$; Subdivide the edges (u_{2i}, u_{2i+a}), with the vertices v_i, $i = 0$ to $a-1$ respectively. Refer Figure 5 in page 7.

This representation of $G = P_{n,p}$ is isomorphic to a graph having a subdivision of $K_{3,3}$ as its subgraph.

Case 2b:$a > p$.

Partition the vertex set $V_1(G)$ into three sets $V_1^1(G)$, $V_1^2(G)$and $V_1^3(G)$ such that $V_1^1(G) = \{u_0, u_2, u_{2a-2}\}$ and $V_1^2(G) = \{u_1, u_{p-2}, u_{2a-1}\}$ and $V_1^3(G)$ has the remaining vertices. Represent G using the following steps so that the adjacency of the vertices of G is preserved:

Take the cycle $u_0, u_1, u_2, u_{p-2}, u_{2a-2}, u_{2a-1}$ containing the vertices of $V_1^1(G)$ $\cup V_1^2(G)$; Connect the pairs of vertices $(u_0,\ u_{p-2})$, $(u_1,\ u_{2a-2})$, $(u_2,\ u_{2a-1})$;

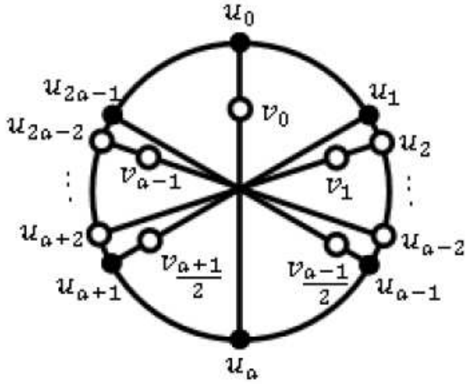

Figure 5: $G = P_{n,p}$ is not planar when $p > 3$ and $a = p$

Subdivide the edge (u_2, u_{p-2}) with the vertices u_i, $i = 3top-3$, the edge (u_{p-2}, u_0) with the vertices u_{p-1} and u_p, the edge (u_2, u_{2a-1}) with the vertices u_{p+2}, u_{p+3}, $\ldots, u_{2-p-1}$; the edge (u_1, u_{2a-2}) with the vertices u_{2a-p+1}, u_{2a-p+2}, $\ldots, u_{2a-3}$; Connect the vertices u_p, u_{p+2} and subdivide the edge with u_{p+1}, connect the vertices u_{2a-p-1}, u_{2a-p+1} and subdivide it with u_{2a-p}; Connect the vertices (u_{i-1}, u_{p+i-1}) for the remaining values of i, and Subdivide the edges (u_{2i}, u_{2i+p}), with the vertices v_i, $i = 0$ to $a-1$ respectively. Refer Figure 6 in page 8.

This representation of $G = P_{n,p}$ is isomorphic to a graph having a subdivision of $K_{3,3}$ as its subgraph.

Case 3:$p > 3$ and a is even.

Case 3a: $a = p + 1$.

Partition the vertex set $V_1(G)$ into three sets $V_1^1(G)$, $V_1^2(G)$and $V_1^3(G)$ such that $V_1^1(G) = \{u_0, u_2, u_{2a-2}\}$ and $V_1^2(G) = \{u_1, u_{p-2}, u_{2a-1}\}$ and $V_1^3(G)$ has the remaining vertices. Represent G using the following steps so that the adjacency of the vertices of G is preserved:

Take the cycle u_0, u_1, u_2, u_{p-2}, u_{2a-2}, u_{2a-1} containing the vertices of $V_1^1(G) \cup V_1^2(G)$; Connect the pairs of vertices (u_0, u_{p-2}), (u_1, u_{2a-2}), (u_2, u_{2a-1}); Subdivide the edge (u_{p-2}, u_0) with the vertices u_{p-1}, u_p, the edge (u_{2a-1}, u_2) with the vertices u_{p+1}, u_{p+2}, the edge(u_1, u_{2a-2}) with the vertices u_{p+3}, u_{p+4}, $\ldots u_{2a-3}$; Connect the pairs of vertices (u_{p-1}, u_{2a-3}), (u_p, u_{p+1}) and $\big(u_{p+2}, u_{p+3)}\big)$; Connect the vertices (u_{2i}, u_{p+2i}) for the remaining values of i (calculations on suffices taken modulo 2a), and Subdivide the edges (u_{2i}, u_{2i+p}), with the vertices

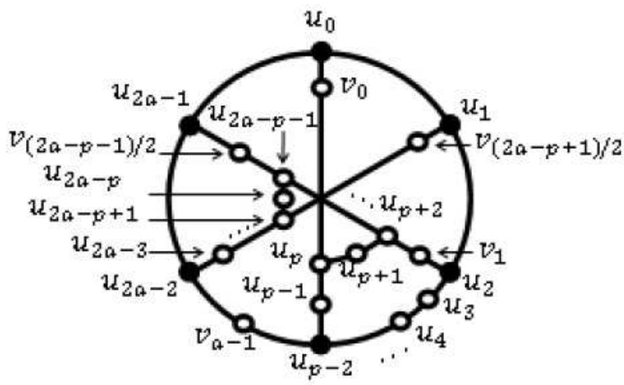

Figure 6: $G = P_{n,p}$ is not planar when $p > 3$ and $a > p$ is odd

v_i, $i = 0$ to $a - 1$ respectively. . Refer Figure 7 in page 9.

This representation of $G = P_{n,p}$ is isomorphic to a graph having a subdivision of $K_{3,3}$ as its subgraph.

Case 3b: $a > p + 1$.

Same as Case 2b

Conversely, let $G = P_{n,3}$ be a 3-petal graph with petal sequence $\{P_i\}$, $i = 0, 1, 2, \ldots, a-1$, where a is even. The cycle $G_\triangle$ divides the plane into two regions, the inner and the outer region. It is possible to draw the $\frac{a}{2}$ petals P_0, P_2, $\ldots$, P_{a-2} of G in the inner (or outer) region so that they do not cross the remaining $\frac{a}{2}$ petals P_1, P_3, $\ldots$, P_{a-1} that are in the outer (or inner) region.

This representation of the 3-petal graph is obviously planar. □

Corollary 12. *Let G be a partial p- petal graph with a petals. Let $G_{\triangle_1}$, $G_{\triangle_2}, \ldots, G_{\triangle_r}$ be the components of G with $a_1, a_2, \ldots, a_r$ finite petals respectively. Let a_{kl} denote the number of infinite petals in $P(G_{\triangle_k} \cup G_{\triangle_l})$ for arbitrary k and l. If the following conditions are satisfied, then the graph G is planar.*

(i) $p = 3$ *and* a *is even;*

(ii) The number of finite petals $P_{i-1}, P_{i+1}, \ldots, P_{j-1}$ in $G_{\triangle_k}$ where $P_i \in P(G_{\triangle_k} \cup G_{\triangle_l})$ and $P_j \in P(G_{\triangle_k} \cup G_{\triangle_q})$, possibly $G_{\triangle_l} = G_{\triangle_q}$, is either zero or odd.

Proof. Let G be a partial p-petal graph as given. From Theorem 3 any p-petal graph is planar if and only if $p = 3$ and a is an even integer. Hence it is sufficient to prove that G is planar if condition (iii) is satisfied. Let us assume that

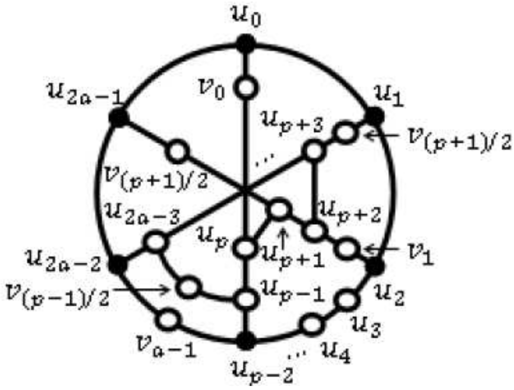

Figure 7: $G = P_{n,p}$ is not planar when $p > 3$ and a is odd such that $a = p + 1$

condition (iii) holds true. Consider the a_k^1 finite petals $P_{i-1}, P_{i+1}, P_{i+2}, \ldots, P_{j-1}$ and the infinite petals $P_i, P_j (i < j)$ such that $P_i \in P(G_{\triangle_k} \cup G_{\triangle_l})$ and $P_j \in P(G_{\triangle_k} \cup G_{\triangle_q})$. From condition (iii), if $a_k^1 > 0$, then a_k^1 is odd. Now, draw the $\left\lceil \frac{a_k^1}{2} \right\rceil$ finite petals $P_{i-1}, P_{i+2}, \ldots, P_{j-1}$ in the inner region of $G_{\triangle_k}$ and the remaining $\left\lfloor \frac{a_k^1}{2} \right\rfloor$ petals in the outer region of $G_{\triangle_k}$. This representation of $G_{\triangle_k}$ is obviously planar. Since $G_{\triangle_k}$ is an arbitrary component of G, we conclude that G is planar. Refer Figure 8 in page 10. □

The converse part is not true. Let the partial p-petal graph G be planar. Therefore, each component $G_{\triangle_k},\ k = 0, 1, 2, \ldots, r-1$ is also planar. From given conditions, $p = 3$ and each a_i is even. Assume that $G_{\triangle_k}$ is pendant, connected to $G_{\triangle_l}$. Let $P_i = u_s v_i u'_s$ and $P_j = u_t v_j u'_t$ be the infinite petal pair between $G_{\triangle_k}$ and $G_{\triangle_l}$ where u_s and u_t are in $G_{\triangle_k}$. Let $G'_{\triangle_k}$ and $G'_{\triangle_l}$ be the components obtained by identifying v_i and v_j to get a new vertex v_{ij}. The paths $u_s v_{ij} u_t$ and $u'_s v_{ij} u'_t$ can act as finite petals in $G'_{\triangle_k}$ and $G'_{\triangle_l}$ respectively. Clearly, each of these components is planar. Hence, the number of finite petals other than $u_s v_{ij} u_t$ in $G'_{\triangle_k}$ is odd. Similarly, the number of finite petals other than $u'_s v_{ij} u'_t$ in $G'_{\triangle_l}$ is also odd. This idea is applicable to any pendant component connected to $G_{\triangle_k}$. When $G_{\triangle_k}$ is connected to $G_{\triangle_l}$ and the pendant component $G_{\triangle_q}$, even if condition (iii) is not satisfied, planarity can be preserved by drawing the plane graph of $G_{\triangle_q}$ in the inner region of $G_{\triangle_k}$.

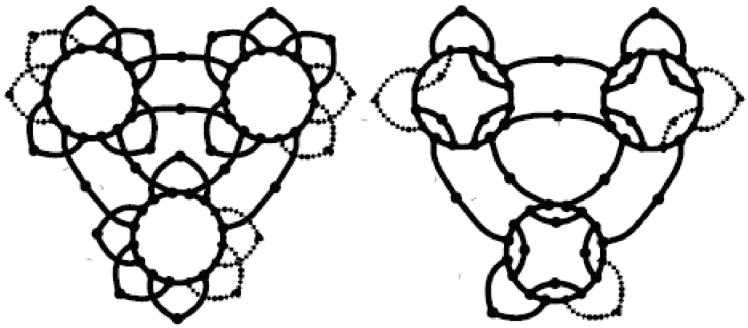

Figure 8: A partial p-petal graph that satisfies condition (iii)

4. Conclusion

The petal graph is a very interesting class of graphs whose properties and characteristics are yet to be explored and many conjectures in graph theory can be solved with reference to petal graphs. Further, the authors wish to acknowledge the contribution of Dr. Neela in the preparation of the final draft of this paper.

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