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ON THE INVERSE PROBLEM FOR MULTIWEIGHTED SHAPLEY VALUES OF COOPERATIVE TU GAMES

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Abstract: In this paper, we solve the inverse problem for the class of Values of Cooperative TU games, called the Multiweighted Shapley Values, introduced in an earlier work of the author (see [2]): find out the set of games for which the MWSV equals an apriori given n-vector. The inverse problem for other values has been solved in previous works ([1] and [3]), but the approach is different here.

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1. Introduction

In an earlier work, (Dragan, [2]), we introduced the Multiweighted Shapley Values as a family of linear operators on the space G^N of TU games with the set of players N satisfying the efficiency and the dummy player axioms. In this family, we find well known values, like the Shapley Value, the Weighted Shapley Value, the Random order values, the Harsanyi payoff vectors, etc. In another work, (Dragan [1]), we solved what we called the inverse problem for the Shapley Value and the Weighted Shapley Value. In the first case, the problem can be stated as follows: given an n- vector of payoffs, say L, find out the set of all

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games in G^N, such that for these TU games the Shapley Value equals L. Then, given a different system of weights, the similar problem was solved for the more general case of the Weighted Shapley Values, based upon some results from Linear Algebra and the potential of these values due to Hart and Mas Colell ([5] and [6]). In this case, the solution was successful because we discovered a new basis for the null space of the linear operator, that was called the potential basis. A similar basis was used in solving the inverse problem for semivalues, (Dragan [4]). However, in the case of Multiweighted Shapley Values such a basis is missing, so that in the present paper, we had to use another approach to solve the inverse problem for these values. The main tool is the pair of bases: the standard basis of linear algebra and the unanimity basis, used previously by L. S. Shapley in the axiomatic definition of the Shapley Value. In the first section, we give our earlier results, the formulas for the Multiweighted Shapley Values associated to a matrix of weights satisfying some algebraic conditions, when the game is given either in the coalitional form, or in the dividend form. In the second section, a basis for the null space of this operator is obtained by using an expression of an arbitrary game in the unanimity basis and proving that some terms form a basis for the null space; the solution of the inverse problem follows easily. Then, an example for a three person TU game is illustrating the results, in the last section.

Let N be a fixed finite set, with $|N| = n$, a set of n players. Any cooperative n- person transferable utilities game (TU game) in coalitional form is a function $\nu : P(N) \to R$ where $P(N)$ is the set of subsets of N, and $\nu(\emptyset) = 0$. The number $\nu(S)$ for $S \subseteq N$, $S \neq \emptyset$, is the worth of coalition S in the game. The sum and the scalar multiplication of games is defined for any pair of games u and w in G^N, and any real numbers α and β by

$$\nu = \alpha u + \beta w \quad \leftrightarrow \quad \nu(S) = \alpha u(S) + \beta w(S), \forall\, S \subseteq N. \tag{1.1}$$

It is easy to check that the set of TU games in G^N form a vector space. In this vector space, beside the standard basis of linear algebra, a popular basis is the unanimity basis, used by L. S. Shapley in [7] for deriving the formula for the Shapley Value from a group of axioms imposed to this solution. Let the standard basis be $D = \{D_S \in G^N : S \subseteq N, S \neq \emptyset\}$, where we have $D_S(S) = 1$, and $D_S(T) = 0$, for $T \neq S$, and let the unanimity basis be denoted by $U = \{U_S \in G^N : S \subseteq N, S \neq \emptyset\}$, where we have $U_S(T) = 1$, $\forall\, T \supseteq S$, and $U_S(T) = 0$, otherwise. Any game $\nu \in G^N$ is getting an expansion in each basis

$$\nu = \sum_{S \subseteq N} \nu(S) D_S = DV, \quad \nu = \sum_{S \subseteq N} \Delta_\nu(S) U_S = U\Delta, \tag{1.2}$$

where D is the matrix of the basic vectors in the standard basis and $V_s = \nu(S)$, is the worth of S, and U is matrix of the basic vectors in the unanimity basis and $\Delta_S = \Delta_\nu(S)$ is the dividend of S. If the game is given by the dividends, then it is said to be in dividend form.
Any functional $\Phi : G^N \to R^n$ is a value, and $\Phi_i(\nu)$ is the outcome of the game offered to the player $i \in N$. The value Φ is linear, if for any pair of games u and w, and for any real numbers α and β, we have

$$\nu = \alpha u + \beta w \quad \to \Phi(\nu) = \alpha\Phi(u) + \beta\Phi(w). \tag{1.3}$$

From (1.3), which will hold for any number of games, and the two expansions (1.2), by linearity, we shall get in the two bases

$$\Phi(\nu) = \sum_{S\subseteq N} \nu(S)\Phi(D_S), \quad \Phi(\nu) = \sum_{S\subseteq N} \Delta_\nu \Phi(U_S), \forall\, \nu \in G^N. \tag{1.4}$$

The Shapley Value may be defined as the value which satisfies the axioms of linearity, symmetry, efficiency and dummy player, as shown later by R. J. Weber (1988). Shapley has imposed an equivalent system of axioms to the basic unanimity games and the value has been obtained by linearity, while the same type of procedure has been used by R. J. Weber, but the axioms have been imposed to the basic vectors of the standard basis. In the following we shall use Shapleys procedure, working with the unanimity basis, but we shall exclude from the axioms the symmetry. The results obtained working with games in the dividend form will be translated into results on the coalitional form, by using the relationships between the coordinates in the two bases, namely

$$\nu(S) = \sum_{T\subseteq S} \Delta_\nu(T), \quad \Delta_\nu(S) = \sum_{T\subseteq S} (-1)^{s-t}\nu(T), \forall S \subseteq N. \tag{1.5}$$

If we assume that the operator we are about to define is linear, then (1.4) shows that in the two bases we can write respectively

$$\Phi(\nu) = \Gamma V, \quad \text{and} \quad \Phi(\nu) = \Lambda\Delta, \tag{1.6}$$

where V is the vector containing the values of the characteristic function and Δ is the vector of dividends; the matrices are $\Gamma = (\gamma_i^S)$ and $\Lambda = (\lambda_i^S)$, with $\gamma_i^S = \Phi_i(D_S)$ and $\lambda_i^S = \Phi_i(U_S)$, for all $i \in N$, and all $S \subseteq N$, $S \neq \emptyset$. In other words, the operator may be defined by one of the matrices Γ or Λ. What we have done in our earlier paper, [2], was to impose to the unanimity basic games, beside the linearity, the other two axioms, in order to get conditions for the weight matrix Λ in a formula for a new value of a game in dividend form, then these conditions were translated into conditions for the weight matrix Γ in a formula for this new value for a game in coalitional form. The results without proofs will be reported below.

Definition 1.1. A linear value $\Phi : G^N \to R^n$ is a Multiweighted Shapley Value, (MWSV), if the value satisfies efficiency and dummy player axioms.

In our earlier paper [2], we determined the linear values satisfying the two axioms. We started with the linearity assumption and imposed the other two axioms shown in Definition 1.1, independently one of the other, on the basic unanimity games. Then the results were reunited to get the necessary and sufficient conditions for the value to be a $MWSV$ for a game in dividend form. Further the conditions were translated into conditions for the value to be a $MWSV$ for a game in coalitional form. Weber worked with the coalitional form and imposed sequentially the linearity, dummy, monotonicity and efficiency axioms, to get necessary and sufficient conditions for a linear value to be a Random Order Value for a game in coalitional form. Obviously, this will be a $MWSV$. The result obtained earlier for a game in dividend form can be stated as follows:

Theorem 1.2. (see [2], Theorem 1.5) *A linear operator $\Phi : G^N \to R^n$ is a Multiweighted Shapley Value if and only if its matrix representation Λ relative to the unanimity basis will satisfy for all coalitions $S \subseteq N$, $S \neq \emptyset$, the equalities*

$$\lambda_i^S = 0, \forall i \notin S, \quad \sum_{i \in S} \lambda_i^S = 1. \tag{1.7}$$

In this case, the value can be represented by

$$\Phi_i(\nu) = \Delta_\nu(\{i\}) + \sum_{S: i \in S, |S| \geq 2} \lambda_i^S \Delta_\nu(S), \forall i \in N, \tag{1.8}$$

where the coefficients should satisfy the second conditions 1.7.

A similar theorem was obtained by translating Theorem 1.2 into a result about the matrix representation relative to the standard basis, by using formulas (1.5) given above. We obtained:

Theorem 1.3. (see [2], Theorem 2.4) *A linear operator $\Phi : G^N \to R^n$ is a Multiweighted Shapley Value if and only if its matrix representation Γ relative to the standard basis will satisfy for all coalitions $S \subseteq N$, $S \neq \emptyset$, the equalities*

$$\gamma_i^{S-\{i\}} = -\gamma_i^S, \forall\, i \in S, \quad \sum_{T: T \subseteq S} (\sum_{j \in S} \gamma_j^S) = 1, \tag{1.9}$$

where for $S = \{i\}$ it is understood that $\gamma_i^{\emptyset} = -\gamma_i^{\{i\}}$, $\forall\, i \in N$. In this case, the value can be represented by

$$\Phi_i(\nu) = \nu(\{i\}) + \sum_{S: i\in S, |S|\geq 2} \gamma_i^S[\nu(S) - \nu(S - \{i\})], \; \forall\, i \in N, \tag{1.10}$$

where the coefficients should satisfy the second conditions (1.9).

The Harsanyi payoff vectors are given for games in dividend form by the second formulas (1.6), where the matrix Λ satisfies, beside (1.7) , the conditions $\lambda_i^S \geq 0$, for all $i \in N$, and all $S \subseteq N$, $S \neq \emptyset$ as shown by Vasiliev in [9]. In our work, [2], we have shown that monotonicity implies, beside 1.9, the conditions $\gamma_i^S \geq 0$, $\forall i \in N$ and all $S \subseteq N$ containing player i. Therefore, the Random Order Values, due to R.J. Weber (see [10]), are given for games in coalitional form by the first formulas (1.6), where some entries satisfy the conditions just mentioned. It follows that the $MWSV$s are more general values than the Random Order Values and the Harsanyi payoff vectors. The following example will show that there exist $MWSV$s which are neither Random Order Values, nor Harsanyi payoff vectors.

Example 1.4. Consider a game $\nu \in G^{\{1,2,3\}}$ and the linear operator Φ given in terms of the coalitional form by

$$\Phi_1(\nu) = \frac{1}{4}\nu(1) + \frac{5}{4}[\nu(1,2) - \nu(2)] - \frac{3}{4}[\nu(1,3) - \nu(3)] + \frac{1}{4}[\nu(1,2,3) - \nu(2,3)],$$

$$\Phi_2(\nu) = \frac{1}{4}\nu(2) - \frac{3}{4}[\nu(1,2) - \nu(1)] + \frac{5}{4}[\nu(2,3) - \nu(3)] + \frac{1}{4}[\nu(1,2,3) - \nu(1,3)],$$

$$\Phi_3(\nu) = \frac{1}{2}\nu(3) + [\nu(1,3) - \nu(1)] - [\nu(2,3) - \nu(2)] + \frac{1}{2}[\nu(1,2,3) - \nu(1,2)].$$

It is clear that we have $\gamma_1^{\{1,3\}}$, $\gamma_2^{\{1,2\}}$, $\gamma_3^{\{2,3\}}$ negative, hence this is not a Random Order Value. Now, we can compute the operator for the simple monotonic game

$$\nu(1) = \nu(2) = \nu(3) = -\frac{1}{2}, \quad \nu(1,2) = \nu(1,3) = \nu(2,3) = \nu(1,2,3) = 1.$$

We obtain $\Phi(\nu) = (\frac{5}{8}, \frac{5}{8}, -\frac{1}{4})^T$, which proves again that Φ is not a Random Order Value, as the Weber conditions for a monotone value do not hold, (see [10]). On the other hand, it is easy to check that Φ shown here above satisfies the conditions (1.9) of Theorem 1.3, hence it is a $MWSV$. Further, from the same expression of the operator for games in coalitional form we may collect the entries of the matrix Γ, then we obtain the matrix Λ from (1.5), or better from (1.5) written in matrix form as $\Lambda = \Gamma M$ with M the Mobius matrix of vectors U_S; of course, in all matrices the coalitions are taken in the same order. For what follows, it is interesting to show only the matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{3}{2} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

We notice by inspection that there are negative entries in Λ, hence Φ is not a Harsanyi payoff vector, it contradicts the Vasiliev definition (see [9], p. 142). On the other hand, again by inspection we see that Φ satisfies the conditions (1.7) of Theorem 1.2, hence it is a $MWSV$. Obviously, for the simple monotonic game given above, we can compute the dividend form by the second formulas (1.5) and obtain $\Delta = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 2, 2, 2, -\frac{7}{2})^T$; further we get the same Φ computed above, by multiplying $\Phi = \Lambda\Delta$.

2. The Null Space of a Multiweighted Shapley Value

Let Λ be a fixed $n \times (2^n - 1)$ matrix satisfying the conditions (1.7) of the previous section. Let $L \in R^n$ be a given outcome vector. What we call the inverse problem can be stated as follows: find out the set of games $\nu \in G^N$ such that the Multiweighted Shapley Value defined by Λ, say Φ_Λ, satisfies $\Phi_\Lambda(\nu) = L$. Further, the index will be omitted and understand that Φ was defined by Λ. The problem has been solved for the Shapley Value, the Weighted Shapley Value, in [2], and the Semivalues, in [4]. In the present section, we solve the problem based upon a well known result in linear algebra (see A. Tucker, [8] , Th.3,page 422): for a linear operator in a vector space, the dimension of the null space plus the dimension of the range makes the dimension of the space. This result would help us in determining the null space. Indeed, the dimension of the range is n and the dimension of G^N is $2^n - 1$. It follows that the dimension of null space is $2^n - n - 1$. Now, we intend to find a basis for the null space, by showing $2^n - n - 1$ linear independent vectors in the null space of Φ.

Theorem 2.1. *Let $\{U_S \in G^N : S \subseteq N, S \neq \emptyset\}$ be the unanimity basis of G^N. Let $\lambda_i^S = \Phi_i(U_S), \forall S \neq \emptyset$ where $\Phi : G^N \to R^n$ is a Multiweighted Shapley Value associated with Λ; consider the set of games $W = \{W_S \in G^N : S \subseteq N, S \neq \emptyset\}$ where*

$$W_{\{i\}} = U_{\{i\}}, \forall i \in N, \tag{2.1}$$

$$W_S = U_S - \sum_{j \in S} \lambda_j^S U_{\{j\}}, \text{ for all coalitions } S \text{ with } |S| \geq 2. \tag{2.2}$$

Then W is a basis for G^N, and $W^* = \{W_S \in G^N : S \subseteq N, S \neq \emptyset\}$ is basis for the null space.

Proof. Clearly, W is a basis for G^N, as it is derived from the unanimity basis for G^N, and the linear transformation (2.1), (2.2), is proper. On the other hand, the games W_S with $|S| \geq 2$ form a set of $2^n - n - 1$ linear independents vectors. Hence, it remains to be shown that each of these games has a null $MWSV$. By using the linearity in (2.2), we obtain

$$\Phi_i(W_S) = \Phi_i(U_S) - \sum_{j \in S} \lambda_j^S \Phi_i(U_{\{j\}}), \forall\, i \in N, \tag{2.3}$$

so that by taking into account that by (1.7) we have $\Phi_i(U_{\{j\}}) = \lambda_i^{\{j\}} = 0$, for all $j \neq i$ and $\Phi_i(U_{\{j\}}) = \lambda_i^{\{j\}} = 1$. The sum has only one term equal to λ_i^S, and the right hand side vanishes. In consequence, the games W_S with $|S| \geq 2$ form a basis of the null space of the $MWSW$.

Now, starting from the expansion (1.2) in terms of the dividend form, and using the formulas

$$U_{\{i\}} = W_{\{i\}}, \forall\, i \in N, \tag{2.4}$$

$$U_S = W_S + \sum_{j \in S} \lambda_j^S W_{\{j\}}, \quad \forall\, S \subseteq N,\, |S| \geq 2, \tag{2.5}$$

obtained from (2.1) and (2.2), we get the expansion of the game in basis W; our formulas (2.4) and (2.5) have been used in the second formula (1.2):

$$\nu = \sum_{j \in N} \Delta_\nu(\{j\}) W_{\{j\}} + \sum_{S : S \subseteq N, |S| \geq 2} \Delta_\nu(S)[W_S + \sum_{j \in S} \lambda_j^S W_{\{j\}}] \tag{2.6}$$

We obtain

$$\nu = \sum_{j \in N} [\Delta_\nu(\{j\}) + \sum_{S : S \subseteq N, |S| \geq 2} \lambda_j^S \Delta_\nu(S)] W_{\{j\}} \tag{2.7}$$

$$+ \sum_{S : S \subseteq N, |S| \geq 2} \Delta_\nu(S) W_S.$$

Now, as $\Phi_i(W_S) = 0$, for all $i \in N$, and $S \subseteq N$, $|S| \geq 2$, and $\Phi_i(W_{\{j\}}) = 0$, for all $j \neq i$, and $\Phi_i(W_{\{i\}}) = 1$, by linearity we get from (2.7) the expression

$$\Phi_i(\nu) = \Delta_\nu(\{i\}) + \sum_{S : i \in S, |S| \geq 2} \lambda_i^S \Delta_\nu(S), \tag{2.8}$$

and in this way, from (2.8) and (2.7) we obtain

$$\nu = \sum_{j\in N} \Phi_j(\nu) W_{\{j\}} + \sum_{S:S\subseteq N, |S|\geq 2} \Delta_\nu(S) W_S. \tag{2.9}$$

□

This proves

Theorem 2.2. *For a given vector $L \in R^N$, and an $n \times (2^n - 1)$ weight matrix Λ, the Multiweighted Shapley Value defined by the matrix Λ, gives the outcome L for the games*

$$\nu = \sum_{j\in N} L_j W_{\{j\}} + \sum_{S:S\subseteq N, |S|\geq 2} a_S W_S, \tag{2.10}$$

where a_S for all coalitions S with cardinality at least 2 are arbitrary constants.

Proof. (2.10) of the Theorem 2.2 shows that we solved the inverse problem. □

Example 2.3. To illustrate the results, return to Example 1.4, to solve the inverse problem for that linear operator. Consider the $MWSV$ defined by the matrix Λ shown in that example 1.4, and let a given n- vector be $L = (\frac{5}{8}, \frac{5}{8}, -\frac{1}{4})^T$, for $n = 3$. We want to show all TU games for which the $MWSV$ defined by the matrix Λ equals L. Computing the basis of the null space formed by the vectors $W_S \in W^\star$, with $S \subseteq N$, $|S| \geq 2$, and using formula (2.10), we obtain the solution of the inverse problem

$$\begin{aligned}
\nu(\{1\}) &= \frac{5}{8} - \frac{3}{2}a_{\{1,2\}} + \frac{1}{2}a_{\{1,3\}} - \frac{1}{4}a_{\{1,2,3\}}, \\
\nu(\{2\}) &= \frac{5}{8} + \frac{1}{2}a_{\{1,2\}} - \frac{3}{2}a_{\{2,3\}} - \frac{1}{4}a_{\{1,2,3\}}, \\
\nu(\{3\}) &= -\frac{1}{4} - \frac{3}{2}a_{\{1,3\}} + \frac{1}{2}a_{\{2,3\}} - \frac{1}{2}a_{\{1,2,3\}}, \\
\nu(\{1,2\}) &= \frac{5}{4} + \frac{1}{2}a_{\{1,3\}} - \frac{3}{2}a_{\{2,3\}} - \frac{1}{2}a_{\{1,2,3\}}, \\
\nu(\{1,3\}) &= \frac{3}{8} - \frac{3}{2}a_{\{1,2\}} + \frac{1}{2}a_{\{2,3\}} - \frac{3}{4}a_{\{1,2,3\}}, \\
\nu(\{2,3\}) &= \frac{3}{8} + \frac{1}{2}a_{\{1,2\}} - \frac{3}{2}a_{\{1,3\}} - \frac{3}{4}a_{\{1,2,3\}},
\end{aligned}$$

and $\nu(\{1,2,3\}) = 1$; the set depends on four parameters, and in general of $2^n - n - 1$ parameters; one may check that our game is obtained for the parameter values $a_{\{1,2\}} = a_{\{1,3\}} = a_{\{2,3\}} = 2$, $a_{\{1,2,3\}} = -\frac{7}{2}$.

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