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# ON NONUNIFORM POLYNOMIAL DICHOTOMY OF EVOLUTION OPERATORS IN BANACH SPACES

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**Abstract:** In this paper we study two nonuniform polynomial dichotomy concepts for evolution operators in Banach spaces. Our main objective is to give integral characterizations for nonuniform polynomial dichotomies. As for applications we obtain characterizations of these concepts in terms of Lyapunov functions. Some illustrating examples are given.

**AMS Subject Classification:** 34D05, 34E05
**Key Words:** evolution operator, nonuniform polynomial dichotomy

## 1. Introduction

The notion of exponential dichotomy for linear differential equations introduced in 1930 by O. Perron [15] plays a central role in the theory of dinamical systems.

The exponential dichotomy property has gained prominence since the appearance of two fundamental monographs due to J.L. Massera and J.J. Schäffer [9], respectively J.L. Daleckii and M.G. Krein [7]. Diverse and important concepts of exponential dichotomy were studied by C. Chicone and Y. Latushkin in [5], S.N. Chow and H. Leiva in [6], R.S. Sacher and G.R. Sell in [21].

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In their notable contributions [1], [2] L. Barreira and C. Valls were obtained relevant results in the case of a notion of nonuniform exponential dichotomy motivated by ergodic theory.

For other papers about exponential dichotomies we refer to [12], [13], [14], [16], [17], [18], [19] and the references therein.

The existence of exponential dichotomies is a strong requirement and hence it is of considerable interest to look for more general types of dichotomic behaviors.

Recently, a notion of nonuniform polynomial dichotomy was introduced independently by L. Barreira and C. Valls in [3] and A. Bento and C. Silva in [4] in somewhat distinct forms, respectively in the case of continous and discrete-time. In this case the rates of contraction and expansion vary polynomially.

In this paper we investigate two nonuniform polynomial dichotomy concepts for the general case of evolution operators in Banach spaces. The case of uniform polynomial dichotomy is considered in [20]. The particular cases of polynomial stability concepts has been discussed in [10] and [11]. Our approach is based on the extension of techniques for exponential dichotomy to the case of polynomial dichotomy.

The main objective is to give integral characterizations for nonuniform polynomial dichotomies. As applications we obtain characterizations of these concepts in terms of Lyapunov functions. Some simple examples are included to illustrate the connections between the dichotomy concepts considered in this paper.

## 2. Notations and Preliminary Results

Let $X$ be a real or complex Banach space and let $I$ be the identity operator on $X$. The norm on $X$ and on $\mathcal{B}(X)$ the algebra Banach of all bounded linear operators acting on $X$, will be denote by $\| \cdot \|$.

Let $\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}$ and $T = \{(t,s,r) \in \mathbb{R}_+^3 : t \geq s \geq r\}$

**Definition 1.** A mapping $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is said *evolution operator* on $X$ if the following relations hold:

$e_1$) $\Phi(t,t) = I$, for all $t \geq 0$;

$e_2$) $\Phi(t,s)\Phi(s,r) = \Phi(t,r)$, for all $(t,s,r) \in T$.

**Definition 2.** An evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is said *strongly measurable*, if for all $(s,x) \in \mathbb{R}_+ \times X$ the mapping defined by $t \longmapsto \| \Phi(t,s)x \|$ is measurable on $[s,\infty)$.

**Definition 3.** A strongly measurable application $P : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ is said to be *a family of projections on* $X$ if

$$P^2(t) = P(t), \text{ for all } t \geq 0.$$

**Remark 1.** If $P : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ is a family of projections on $X$, then the mapping $Q : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$, $Q(t) = I - P(t)$ is also a family of projections on $X$, which is called the *complementary projection of* $P$.

One can easily see that

$$P(t)Q(t) = Q(t)P(t) = 0, \text{ for each } t \geq 0.$$

**Definition 4.** A family of projection $P : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ is *compatible with an evolution operator* $\Phi$ if

$$\Phi(t,r)P(r) = P(t)\Phi(t,r), \text{ for all } (t,r) \in \Delta.$$

**Remark 2.** If the family of projections $P : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ is compatible with the evolution operator $\Phi$ then its complementary projection $Q : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ is also compatible with the evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$.

In what follows, for a family of projections $P : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ compatible with the evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$, we shall denote

$$\Phi_P(t,r) = \Phi(t,r)P(r) \text{ and } \Phi_Q(t,r) = \Phi(t,r)Q(r)$$

We deduced that

$$\Phi_P(t,s)\Phi_P(s,r) = \Phi_P(t,r) \text{ and } \Phi_Q(t,s)\Phi_Q(s,r) = \Phi_P(t,r)$$

for all $(t,s,r) \in T$.

## 3. Exponential Dichotomy Concepts

Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be an evolution operator and let $P : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ be a family of projections compatible with $\Phi$.

**Definition 5.** The evolution operator $\Phi$ is said:

(*i*) *P-uniformly exponentially dichotomic* (and denote *P.u.e.d*) if there are some constants $\alpha > 0$ and $N \geq 1$ such that

$$e^{\alpha(t-s)}\left(\|\Phi_P(t,s)x\| + \|Q(s)x\|\right) \leq N\left(\|P(s)x\| + \|\Phi_Q(t,s)x\|\right)$$

for all $(t,s,x) \in \Delta \times X$;

(*ii*) *P-nonuniformly exponentially dichotomic* (and denote *P.n.e.d*) if there exist a constant $\alpha > 0$ and a nondecreasing function $N : \mathbb{R}_+ \longrightarrow [1,\infty)$ such that

$$e^{\alpha(t-s)}\left(\|\Phi_P(t,s)x\| + \|Q(s)x\|\right) \leq N(s)\|P(s)x\| + N(t)\|\Phi_Q(t,s)x\|$$

for all $(t, s, x) \in \Delta \times X$;

(*iii*) *P-exponentially dichotomic* (and denote *P.e.d*) if there are some constants $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$e^{\alpha(t-s)} \left(\|\Phi_P(t,s)x\| + \|Q(s)x\|\right) \leq N \left(e^{\beta s}\|P(s)x\| + e^{\beta t}\|\Phi_Q(t,s)x\|\right)$$

for all $(t, s, x) \in \Delta \times X$.

The constant $\alpha > 0$ is called *the exponential dichotomy constant* of $\Phi$.

**Remark 3.** An evolution operator $\Phi$ is :

(*i*) P-uniformly exponentially dichotomic if and only if there are some constants $\alpha > 0$ and $N \geq 1$ such that

$$e^{\alpha(t-s)} \left(\|\Phi_P(t,r)x\| + \|\Phi_Q(s,r)x\|\right) \leq N \left(\|\Phi_P(s,r)x\| + \|\Phi_Q(t,r)x\|\right)$$

for all $(t, s, r, x) \in T \times X$;

(*ii*) P-nonuniformly exponentially dichotomic if and only if there exist a constant $\alpha > 0$ and a nondecreasing function $N : \mathbb{R}_+ \longrightarrow [1, \infty)$ such that

$$e^{\alpha(t-s)} \left(\|\Phi_P(t,r)x\| + \|\Phi_Q(s,r)x\|\right) \\ \leq N(s)\|\Phi_P(s,r)x\| + N(t)\|\Phi_Q(t,r)x\|$$

for all $(t, s, r, x) \in T \times X$;

(*iii*) P-exponentially dichotomic if and only if there are some constants $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$e^{\alpha(t-s)} \left(\|\Phi_P(t,r)x\| + \|\Phi_Q(s,r)x\|\right) \\ \leq N \left(e^{\beta s}\|\Phi_P(s,r)x\| + e^{\beta t}\|\Phi_Q(t,r)x\|\right)$$

for all $(t, s, r, x) \in T \times X$.

**Remark 4.** Is obvious that

$$P.u.e.d \Longrightarrow P.e.d \Longrightarrow P.n.e.d$$

**Remark 5.** The converse implications are not valid (see [11], [14]).

## 4. Polynomial Dichotomy Concepts

Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be an evolution operator and let $P : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ be a family of projections compatible with $\Phi$.

**Definition 6.** The evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is said to be *P-nonuniformly polynomially dichotomic* (and denote *P.n.p.d*) if there are a nondecreasing function $N : \mathbb{R}_+ \longrightarrow [1, \infty)$ and a constant $\alpha > 1$ such that:

$$\left(\frac{t+1}{s+1}\right)^{\alpha} (\|\Phi_P(t,s)x\| + \|Q(s)x\|) \leq N(s)\|P(s)x\| + N(t)\|\Phi_Q(t,s)x\|$$

for all $(t, s, x) \in \Delta \times X$.

The constant $\alpha > 1$ is called *the polynomial dichotomy constant* of $\Phi$.

In particular case when the function $N$ is constant, we say that $\Phi$ is *P-uniformly polynomially dichotomic* (and denote *P.u.p.d.*)

**Remark 6.** The evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is P-nonuniformly polynomially dichotomic if and only if there are a nondecreasing function $N : \mathbb{R}_+ \longrightarrow [1, \infty)$ and a constant $\alpha > 1$ such that:

$$\left(\frac{t+1}{s+1}\right)^{\alpha} (\|\Phi_P(t,r)x\| + \|\Phi_Q(s,r)x\|) \\ \leq N(s)\|\Phi_P(s,r)x\| + N(t)\|\Phi_Q(t,r)x\|,$$

for all $(t, s, r, x) \in T \times X$.

**Remark 7.** It is obvious that if the evolution operator $\Phi$ is P-nonuniformly exponentially dichotomic then it is P-nonuniformly polynomially dichotomic. The converse implication is not true, which is illustrated by

**Proposition 1.** *If the evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is P.n.e.d with the exponenetial dichotomy constant $\alpha > 1$, then it is P.n.p.d.*

*Proof.* According to the hypothesis there exist $\alpha > 1$ and a nondecreasing function $N : \mathbb{R}_+ \longrightarrow [1, \infty)$ such that

$$\left(\frac{t+1}{s+1}\right)^{\alpha} (\|\Phi_P(t,s)x\| + \|Q(s)x\|) \leq \left(\frac{e^t}{e^s}\right)^{\alpha} (\|\Phi_P(t,s)x\| + \|Q(s)x\|) \\ \leq N(s)\|P(s)x\| + N(t)\|\Phi_Q(t,s)x\|.$$

Hence $\Phi$ is *P.n.p.d.* □

We present an example of evolution operator which is nonuniformly polynomially dichotomic, but is not nonuniformly exponentially dichotomic.

**Example 1.** Let $X = \mathbb{R}^2$, $P(s)x = (x_1, 0)$ and $Q(s)x = (0, x_2)$, for $x = (x_1, x_2) \in X$.

We consider $u : [1, \infty) \longrightarrow \mathbb{R}_+^{*}$, $u(t) = (t+2)^2$. Then

$$\Phi : \Delta \longrightarrow \mathcal{B}(X),\ \Phi(t,s)x = \left(\frac{u(s)}{u(t)}x_1, \frac{u(t)}{u(s)}x_2\right)$$

is an evolution operator on $X$ with

$$\left(\frac{t+1}{s+1}\right)^2 \|\Phi_P(t,s)x\| = \frac{(t+1)^2(s+2)^2}{(s+1)^2(t+2)^2}\|P(s)x\| \leq 4(s+1)^3\|P(s)x\| = N(s)\|P(s)x\|$$

and

$$N(t)\|\Phi_Q(t,s)x\| = \frac{4(t+1)^3(t+2)^2}{(s+2)^2}\|Q(s)x\| \geq \left(\frac{t+1}{s+1}\right)^2 \|Q(s)x\|,$$

for all $(t, s, x) \in \Delta \times \mathbb{R}^2$. It results that $\Phi$ is *P.n.p.d.*

If we suppose that $\Phi$ is *P.n.e.d*, then there exist a nondecreasing function $N : \mathbb{R}_+ \longrightarrow [1, \infty)$ and a constant $\alpha > 0$ such that

$$e^{\alpha(t-s)}\left(\frac{s+2}{t+2}\right)^2 \leq N(s)$$

for all $(t, s) \in \Delta$. If we consider $s = 0$ and $t \longrightarrow \infty$, we obtain a contradiction, which shows that $\Phi$ is not *P.n.e.d.*

**Definition 7.** A family of projection $P : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ is *polynomial compatible with the evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ if

$c_1$) $\Phi(t,r)P(r) = P(t)\Phi(t,r)$, for all $(t,r) \in \Delta$;

$c_2$) there exist $M \geq 1$ and $\omega > 0$ such that

$$\|\Phi_P(t,r)x\| \leq M\left(\frac{t+1}{s+1}\right)^{\omega}\|\Phi_P(s,r)x\|$$

and

$$\|\Phi_Q(s,r)x\| \leq M\left(\frac{t+1}{s+1}\right)^{\omega}\|\Phi_Q(t,r)x\|$$

for all $(t,s,r,x) \in T \times X$.

**Theorem 1.** *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a strongly measurable evolution operator on the Banach space $X$ and let $P : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ be a family of projections polynomial compatible with $\Phi$.*

*The evolution operator $\Phi$ is $P$-nonuniformly polynomially dichotomic if and only if there exist a nondecreasing function $D : \mathbb{R}_+ \longrightarrow [1,\infty)$ and a constant $d > 0$ such that*

$$\int_s^{\infty} \frac{(\tau+1)^d}{(s+1)^{d+1}} \|\Phi_P(\tau,s)x\| d\tau + \int_s^t \frac{(t+1)^{d+1}}{(\tau+1)^d} \|\Phi_Q(\tau,s)x\| d\tau \leq D(s)\|P(s)x\| + D(t)\|\Phi_Q(t,s)x\|,$$

*for all $(t,s,x) \in \Delta \times X$.*

*Proof.* Necessity. Let $(t,s,x) \in \Delta \times X$ and $d \in (0,\alpha-1)$. We have

$$\int_s^{\infty} \frac{(\tau+1)^d}{(s+1)^{d+1}} \|\Phi_P(\tau,s)x\| d\tau + \int_s^t \frac{(t+1)^{d+1}}{(\tau+1)^d} \|\Phi_Q(\tau,s)x\| d\tau$$

$$\leq N(s)(s+1)^{\alpha-d-1} \|P(s)x\| \int_s^{\infty} (\tau+1)^{d-\alpha} d\tau$$

$$+N(t)(t+1)^{d+1-\alpha} \|\Phi_Q(t,s)x\| \int_s^t (\tau+1)^{\alpha-d} d\tau$$

$$\leq D(s) \| P(s)x \| + D(t) \| \Phi_Q(t,s)x \|,$$

where

$$D(t) = \frac{(t+1)^2 N(t)}{\alpha-d-1}.$$

Sufficiency. Let $(t,s,x) \in \Delta \times X$. If $t \geq 2s+1$ we have

$$\left(\frac{t+1}{s+1}\right)^{d+1} (\|\Phi_P(t,s)x\| + \|Q(s)x\|)$$

$$= \frac{2}{t+1} \int_{\frac{t-1}{2}}^t \frac{(\tau+1)^d}{(s+1)^{d+1}} \frac{(t+1)^{d+1}}{(\tau+1)^d} \|\Phi_P(t,s)x\| d\tau$$

$$+ \frac{1}{s+1} \int_s^{2s+1} \frac{(t+1)^{d+1}}{(\tau+1)^d} \frac{(\tau+1)^d}{(s+1)^{d+1}} \|Q(s)x\| d\tau$$

$$\leq 2^{d+\omega+1} M \int_s^{\infty} \frac{(\tau+1)^d}{(s+1)^{d+1}} \|\Phi_P(\tau,s)x\| d\tau$$

$$+2^{d+\omega}M\int_s^t \frac{(t+1)^{d+1}}{(\tau+1)^d}\|\Phi_Q(\tau,s)x\|d\tau$$
$$\leq\ 2^{d+\omega+1}M[D(s)\|P(s)x\|+D(t)\|\Phi_Q(t,s)x\|]$$
$$=N(s)\|P(s)x\|+N(t)\|\Phi(t,s)x\|.$$

For $t\in[s,2s+1)$, we have

$$\left(\frac{t+1}{s+1}\right)^{d+1}(\|\Phi_P(t,s)x\|+\|Q(s)x\|)\ \leq 2^{d+\omega+1}M\,(\|P(s)x\|+\|\Phi_Q(t,s)x\|)$$
$$\leq N(s)\|P(s)x\|+N(t)\|\Phi_Q(t,s)x\|.$$

So,

$$\left(\frac{t+1}{s+1}\right)^{d+1}(\|\Phi_P(t,s)x\|+\|Q(s)x\|)\leq N(s)\|P(s)x\|+N(t)\|\Phi_Q(t,s)x\|$$

for all $(t,s,x)\in\Delta\times X$, which shows that $\Phi$ is *P.n.p.d.* □

**Definition 8.** An application $L:\Delta\times\ X\longrightarrow\mathbb{R}_+$ is called to be a *nonuniform polynomial Lyapunov function for the evolution operator* $\Phi:\Delta\longrightarrow\mathcal{B}(X)$ if there is a nondecreasing function $K:\mathbb{R}_+\longrightarrow[1,\infty)$ and a constant $a>0$ such that:

($l_1$) $L(t,r,P(r)x)+\int_s^t\frac{(\tau+1)^{a}}{(s+1)^{a+1}}\|\Phi_P(\tau,r)x\|d\tau\leq\ L(s,r,P(r)x)$

($l_2$) $L(s,r,Q(r)x)+\int_s^t\frac{(t+1)^{a+1}}{(\tau+1)^{a}}\|\Phi_Q(\tau,r)x\|d\tau\leq\ L(t,r,Q(r)x)$

($l_3$) $L(t,r,x)\leq\ K(r)\|P(r)x\|+K(t)\|\Phi_Q(t,r)x\|$

for all $(t,s,r,x)\in T\times\ X$.

**Theorem 2.** *Let $\Phi:\Delta\longrightarrow\mathcal{B}(X)$ be a strongly measurable evolution operator and let $P:\mathbb{R}_+\longrightarrow\mathcal{B}(X)$ be a family of projections polynomial compatible with $\Phi$.*

*The evolution operator $\Phi$ is P-nonuniformly polynomially dichotomic if and only if there exist a nonuniform polynomial Lyapunov function for the evolution operator $\Phi$.*

*Proof.* Necessity. Let $d>0$ given by Theorem 1. We consider the application $L:\Delta\times\ X\longrightarrow\mathbb{R}_+$

$$L(t,r,x)=\int_t^{\infty}\frac{(\tau+1)^d}{(t+1)^{d+1}}\|\Phi_P(\tau,r)x\|d\tau+\int_r^t\frac{(t+1)^{d+1}}{(\tau+1)^d}\|\Phi_Q(\tau,r)x\|d\tau$$

Then for $t \geq s \geq r \geq 0$ and $x \in X$ we have

$$L(t,r,P(r)x) + \int_s^t \frac{(\tau+1)^d}{(s+1)^{d+1}} \|\Phi_P(\tau,r)x\| d\tau \leq \int_s^{\infty} \frac{(\tau+1)^d}{(s+1)^{d+1}} \|\Phi_P(\tau,r)x\| d\tau = L(s,r,P(r)x)$$

and

$$L(s,r,Q(r)x) + \int_s^t \frac{(t+1)^{d+1}}{(\tau+1)^d} \|\Phi_Q(\tau,r)x\| d\tau \leq \int_r^t \frac{(t+1)^{d+1}}{(\tau+1)^d} \|\Phi_Q(\tau,r)x\| d\tau = L(t,r,Q(r)x).$$

Now, using Theorem 1 we have

$$\begin{aligned} L(t,r,x) &= \int_t^{\infty} \frac{(\tau+1)^d}{(t+1)^{d+1}} \|\Phi_P(\tau,r)x\| d\tau + \int_r^t \frac{(t+1)^{d+1}}{(\tau+1)^d} \|\Phi_Q(\tau,r)x\| d\tau \\ &\leq \int_r^{\infty} \frac{(\tau+1)^d}{(r+1)^d} \|\Phi_P(\tau,r)x\| d\tau + \int_r^t \frac{(t+1)^{d+1}}{(\tau+1)^d} \|\Phi_Q(\tau,r)x\| d\tau \\ &\leq D(r)\|P(r)x\| + D(t)\|\Phi_Q(t,r)x\|. \end{aligned}$$

We obtain that there exist a nondecreasing function $K : \mathbb{R}_+ \longrightarrow [1,\infty)$ and a constant $a > 0$ such that the conditions of Definition 9 hold.

Sufficiency. Let $(t,s,r,x) \in T \times X$. If $L : \Delta \times X \longrightarrow \mathbb{R}_+$ is a nonuniform polynomial Lyapunov function for $\Phi$ then there exist $K : \mathbb{R}_+ \longrightarrow [1,\infty)$ and $a > 0$ such that the relations $(l_1)$, $(l_2)$ and $(l_3)$ hold. We have that

$$\int_s^{\infty} \frac{(\tau+1)^a}{(s+1)^{a+1}} \|\Phi_P(\tau,s)x\| d\tau + \int_s^t \frac{(t+1)^{a+1}}{(\tau+1)^a} \|\Phi_Q(\tau,s)x\| d\tau \leq L(s,s,P(s)x) + L(t,s,Q(s)x) \leq K(s)\|P(s)x\| + K(t)\|\Phi_Q(t,s)x\|,$$

for all $(t,s,x) \in \Delta \times X$. Using the Theorem 1 we obtain that $\Phi$ is *P.n.p.d.* □

A particular case of nonuniform polynomial dichotomy is given by

**Definition 9.** The evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is said to be *P-polynomially dichotomic* (and denote *P.p.d.*) if there are some constants $N \geq 1$, $\alpha > 1$ and $\beta \geq 0$ such that:

$$\left(\frac{t+1}{s+1}\right)^{\alpha}(\parallel \Phi_P(t,s)x \parallel + \parallel Q(s)x \parallel)$$
$$\leq N[(s+1)^{\beta} \parallel P(s)x \parallel + (t+1)^{\beta} \parallel \Phi_Q(t,r)x \parallel],$$

for all $(t,s,x) \in \Delta \times X$.

**Remark 8.** The evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is P-polynomially dichotomic if and only if there are some constants $N \geq 1$, $\alpha > 1$ and $\beta \geq 0$ such that:

$$\left(\frac{t+1}{s+1}\right)^{\alpha}(\parallel \Phi_P(t,r)x \parallel + \parallel \Phi_Q(s,r)x \parallel)$$
$$\leq N[(s+1)^{\beta} \parallel \Phi_P(s,r)x \parallel + (t+1)^{\beta} \parallel \Phi_Q(t,r)x \parallel],$$

for all $(t,s,r,x) \in T \times X$.

**Remark 9.** If the evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is P-exponentially dichotomic then $\Phi$ is P-polynomially dichotomic. The converse statement is not true, as shown in what follows.

**Example 2.** Let $X = \mathbb{R}^2$, $P(s)x = (x_1, 0)$ and $Q(s)x = (0, x_2)$, $u : [1,\infty) \longrightarrow \mathbb{R}_+^{*}$, $u(t) = (t+1)^3 + 1$. Then

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \Phi(t,s)x = \left(\frac{u(s)}{u(t)}x_1, \frac{u(t)}{u(s)}x_2\right)$$

has the property

$$\left(\frac{t+1}{s+1}\right)^{2}\|\Phi_P(t,s)x\| = \frac{(t+1)^2}{(s+1)^2}\frac{(s+1)^3+1}{(t+1)^3+1}\|P(s)x\|$$
$$\leq \frac{(s+1)^3+1}{(s+1)^2}\|P(s)x\| \leq 2(s+1)^4\|P(s)x\|$$

and

$$2(t+1)^4\|\Phi_Q(t,s)x\| = 2(t+1)^4\frac{(t+1)^3+1}{(s+1)^3+1}\|Q(s)x\| \geq \left(\frac{t+1}{s+1}\right)^{2}\|Q(s)x\|,$$

for all $(t,s,x) \in \Delta \times \mathbb{R}^2$, which shows that $\Phi$ is *P.p.d.*

We suppose that $\Phi$ is *P.e.d*, then there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$e^{\alpha(t-s)}\frac{(s+1)^3+1}{(t+1)^3+1} \leq Ne^{\beta s}$$

for all $(t,s) \in \Delta$. From here, for $s = 0$ and $t \longrightarrow \infty$, we obtain a contradiction, which shows that $\Phi$ is not *P.e.d.*

**Remark 10.** If the evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is P-polynomially dichotomic, then $\Phi$ is P-nonuniformly polynomially dichotomic. The following example shows that the converse implication is not true.

**Example 3.** Let $X = \mathbb{R}^2$, $P(s)x = (x_1, 0)$ and $Q(s)x = (0, x_2)$.
Consider $N : [0, \infty) \longrightarrow [1, \infty)$, $N(t) = e^{t^2}$ and
$v : [0, \infty) \longrightarrow [1, \infty)$

$$v(t) = \begin{cases} e^{n^2} & \text{if} \quad t = n \\ e^4 & \text{if} \quad t = n + \frac{1}{n^2} \\ 1 & \text{if} \quad t \neq n,\, t \neq n + \frac{1}{n^2} \end{cases}$$

for every $n \in \mathbb{N}^{*}$. Then

$$\Phi : \Delta \longrightarrow \mathcal{B}(\mathbb{R}^2), \Phi(t,s)(x_1, x_2) = \left( \frac{(s+1)^2 v(s)}{(t+1)^2 v(t)} x_1, \frac{(t+1)^2 v(t)}{(s+1)^2 v(s)} x_2 \right)$$

has the property

$$\left(\frac{t+1}{s+1}\right)^2 \|\Phi_P(t,s)x\| = \frac{(t+1)^2}{(s+1)^2} \frac{(s+1)^2 v(s)}{(t+1)^2 v(t)} \|P(s)x\| \\ \leq v(s)\|P(s)x\| \leq N(s)\|P(s)x\|$$

and

$$N(t)\|\Phi_Q(t,s)x\| = N(t) \frac{(t+1)^2 v(t)}{(s+1)^2 v(s)} \|Q(s)x\| \geq \left(\frac{t+1}{s+1}\right)^2 \|Q(s)x\|$$

for all $(t, s, x) \in \Delta \times X$. This show that $\Phi$ is *P.n.p.d.*

If we suppose that $\Phi$ is *P.p.d.* then there are $N \geq 1$, $\alpha > 1$ and $\beta \geq 0$ such that

$$\left(\frac{t+1}{s+1}\right)^{\alpha} \frac{(s+2)^2 v(s)}{(t+2)^2 v(t)} \leq (s+1)^{\beta}$$

for all $(t, s) \in \Delta$.

Then for $s = n$ and $t = n + \frac{1}{n^2}$ we obtain

$$\left(\frac{1 + 1/n^3 + 1/n}{1 + 1/n}\right)^{\alpha} \left(\frac{1 + 1/n}{1 + 1/n^3 + 1/n}\right)^2 \leq N \frac{n^{\beta}}{e^{n^2}} (1 + 1/n)^{\beta} e^4$$

which for $n \longrightarrow \infty$ gives a contradiction and hence $\Phi$ is not *P.p.d.*

**Corollary 1.** *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a strongly measurable evolution operator and let $P : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ be a family of projections polynomial compatible with $\Phi$.*

*The evolution operator $\Phi$ is P-polynomially dichotomic if and only if there exist the constants $D \geq 1$, $\xi \geq 0$ and $d > 0$ such that*

$$\int_s^\infty \frac{(\tau+1)^d}{(s+1)^{d+1}} \|\Phi_P(\tau,s)x\| d\tau + \int_s^t \frac{(t+1)^{d+1}}{(\tau+1)^d} \|\Phi_Q(\tau,s)x\| d\tau \leq D[(s+1)^\xi \parallel P(s)x \parallel + (t+1)^\xi \parallel \Phi_Q(t,s)x \parallel],$$

*for all $(t,s,x) \in \Delta \times X$.*

*Proof.* Necessity it results for $D = \frac{N}{\alpha-d-1}$, $d \in (0, \alpha-1)$ and $\xi = \beta + 1$.
Sufficiency. It results from the proof of Theorem 1 for $D(t) = (t+1)^\xi$. □

**Definition 10.** An application $L : \Delta \times X \longrightarrow \mathbb{R}_+$ is called to be a *polynomial Lyapunov function for the evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ if there are $K \geq 1$, $a > 0$ and $b \geq 0$ such that:

$$(l_1)\ \ L(t,r,P(r)x) + \int_s^t \frac{(\tau+1)^a}{(s+1)^{a+1}} \|\Phi_P(\tau,r)x\| d\tau \leq L(s,r,P(r)x)$$

$$(l_2)\ \ L(s,r,Q(r)x) + \int_s^t \frac{(t+1)^{a+1}}{(\tau+1)^a} \|\Phi_Q(\tau,r)x\| d\tau \leq L(t,r,Q(r)x)$$

$$(l_3)\ \ L(t,r,x) \leq K[(r+1)^b \|P(r)x\| + (t+1)^b \|\Phi_Q(t,r)x\|]$$

for all $(t,s,r,x) \in T \times X$.

**Corollary 2.** *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a strongly measurable evolution operator and let $P : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ be a family of projections polynomial compatible with $\Phi$.*

*The evolution operator $\Phi$ is P-polynomially dichotomic if and only if there exist a polynomial Lyapunov function for the evolution operator $\Phi$.*

*Proof.* It results from the proof of Theorem 2 for $K(t) = (t+1)^b$. □

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