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UPPER MINUS TOTAL DOMINATION NUMBER OF 6-REGULAR GRAPH

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Abstract: Let $\Gamma_t^-(G)$ be upper minus total domination number of G. In this paper, We establish an upper bound of the upper minus total domination number of a 6-regular graph G and characterize the extremal graphs attaining the bound. Thus, we partially answer an open problem by Yan, Yang and Shan.

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1. Introduction

All graphs under consideration are undirected, finite and simple. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The order of G denotes the number of vertices of G. We write $d(x)$ for the degree of x in G, that is, the number of edges of G incident with x. A graph G=(V, E) is called k-regular if $d(v) = k$ for all $v \in V$. The open neighborhood of a vertex v is the set of vertices adjacent to v, i. e. , $N(v) = \{u \in V|\ uv \in E\}$. For a subset A of V, set $d_A(v) = |\{u \in A|uv \in E\}|$. If A and B are disjoint subsets of V, we write $e(A, B)$ for the number of edges between A and B. A minus total dominating function (MTDF) of a graph is defined in [1] as a function

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$f : V \to \{-1, 0, +1\}$ such that the sum of its function values over any open neighborhood is at least one. An MTDF f is minimal if there does not exist an MTDF $g : V \to \{-1, 0, +1\}, f \neq g$, for which $g(v) \leq f(v)$ for every $v \in V$. The weight of f is $\omega(f) = \sum_{v \in V} f(v)$, for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V)$ For a vertex v in V, we denote $f(N(v))$ by $f[v]$ for notational convenience and if $f[v] = 1$, v is called a critical vertex under f. The upper minus total domination number $\Gamma_t^-(G)$ is the maximum cardinality of a minimal minus total dominating set in G.

In [2] the authors asked for the upper bound on $\Gamma_t^-(G)$ for k-regular graph, $k \geq 5$. Wang and Shan gave a sharp bound for 5-regular graph[3]. Wu Miao and Lu gave a sharp bound for k-regular graph where k is odd [4]. In Section 2 of this paper, we establish an upper bound of the minus total domination number of a 6-regular graph G and characterize the extremal graphs attaining the bound.

2. 6-Regular Graph

In this section, we establish an upper bound on the upper minus total domination number of a 6-regular graph in terms of its order and characterize the 6-regular graphs attaining this bound.

For this purpose, we define a family $\mathcal{F} = \{G_{k,l} |\ k \geq 2l \geq 0, k \geq 2\}$ of 6-regular graphs. For two integers $k \geq 2l \geq 0$, $k \geq 2$, let $G_{k,l}$ be a 6-regular graph with vertex set $\bigcup_{i=1}^{8} A_i$ with $|A_i| = a_i$, for $1 \leq i \leq 8$, where all $a_i^{'}$s are integers satisfying $a_1 = 2a_3 = 4l$, $a_2 = 2a_4 = 2k$, $a_5 = 2a_6 = 12l$, $a_7 = 6k - 12l$ and $a_8 = 12k + 30l$, where A_2, A_4, A_5 and A_6 are independent sets. The edge set of $G_{k,l}$ is constructed as follows.

Add 4l edges joining vertices of A_1 so that A_1 induces a 2-regular graph. Add l (resp. 6k-12l) edges joining vertices of A_3 (resp. A_7) so that A_3 (resp. A_7) induces a 1-regular graph. Add 30k+75l edges joining vertices of A_8 so that A_8 induces a 5-regular graph. Add 4l edges between A_1 and A_3 so that each vertex in A_1 is adjacent to precisely one vertex of A_3 and each vertex in A_3 is adjacent to precisely two vertices of A_1, so each vertex of $A_1 \cup A_3$ has degree 3. Add 12l (resp. 6l) edges between A_1 (resp. A_3) and A_5 (resp. A_6) so that each vertex in A_1 (resp. A_3) is adjacent to three vertices of A_5 (resp. A_6) and each vertex of A_5 (resp. A_6) is adjacent to one vertex of A_1 (resp. A_3), so each vertex in $A_1 \cup A_3$ has degree 6 while each vertex in $A_5 \cup A_6$ has degree 1. Add 12k (resp. 6k)edges between A_2 (resp. A_4) and $A_5 \cup A_6 \cup A_7$ (resp. $A_5 \cup A_7$) so that each vertex of $A_2 \cup A_4$ has degree 6 while each vertex of A_5

is , respectively, adjacent to a vertex of A_2 and A_4 and each vertex of A_6 is adjacent to two vertices of A_2 and each vertex of A_7 is adjacent to a vertex of A_4 and two vertices of A_2. Then each vertex in $A_5 \cup A_6$ has degree 3 while each vertex in A_7 has degree 4. Finally, add 24k+60l edges between $A_5 \cup A_6 \cup A_7$ and A_8 in such a way that each vertex $A_5 \cup A_6$ is adjacent to precisely three vertices of A_8, and each vertex of A_7 is adjacent to two vertices of A_8 while each vertex of A_8 is adjacent to precisely one vertex of $A_5 \cup A_6 \cup A_7$.

Lemma 1. *An MTDF on a graph G is minimal if and only if for every vertex $v \in V$ with $f(v) \geq 0$, there exists a vertex $u \in N(v)$ with $f[u] = 1$*

Theorem 2. *If G is a 6-regular graph of order n, then*

$$\Gamma_t^-(G) \leq \frac{16}{21} n$$

the equality if and only if $G \in \mathcal{F}$

Proof. Let f be a minus total dominating function satisfying $\Gamma_t^-(G) = f(V)$. We let P, Q and M denote the sets of those vertices in G which are assigned under f the value +1, 0 and -1, respectively. And we define:

$$P_{ij} = \{v \in P |\ d_Q(v) = i, d_M(v) = j, 0 \leq j \leq 2, 0 \leq i \leq 5 - 2j\}$$

$$Q_{ij} = \{v \in Q |\ d_P(v) = i, d_M(v) = j, 0 \leq j \leq 2, j + 1 \leq i \leq 6 - j\}$$

$$M_{ij} = \{v \in M |\ d_P(v) = i, d_Q(v) = j, 0 \leq j \leq 5, \lceil \frac{7 - j}{2} \rceil \leq i \leq 6 - j\}$$

and let $|P| = p$, $|Q| = q$, $|M| = m$, $|P_{ij}| = p_{ij}$, $|Q_{ij}| = q_{ij}$ and $|M_{ij}| = m_{ij}$. Then $n = p+q+m$, $\Gamma_t^-(G) = p-m$. Furthermore, we write $P' = P_{50} \cup P_{31} \cup P_{12}$, $Q' = Q_{10} \cup Q_{21} \cup Q_{32}$, $M' = M_{31} \cup M_{23} \cup M_{15}$ Clearly, each vertex v in $P' \cup Q' \cup M'$ is a critical vertex of G under f, i. e. $f[v] = 1$, while for each vertex $v \in V - (P' \cup Q' \cup M')$, $f[v] \geq 2$. By counting the edge number $e(P, Q)$, $e(P, M)$, $e(Q, M)$, we immediately get the following equalities:

$$e(P, Q) = p_{10} + 2p_{20} + 3p_{30} + 4p_{40} + 5p_{50} + p_{11} + 2p_{21} + 3p_{31} + p_{12} = 6q - (5q_{10} + 4q_{20} + 3q_{30} + 2q_{40} + q_{50} + 4q_{21} + 3q_{31} + 2q_{41} + q_{51} + 3q_{32} + 2q_{42}) \quad (1)$$

$$e(P, M) = p_{01} + p_{11} + p_{21} + p_{31} + 2p_{02} + 2p_{12} = 6m - (2m_{40} + m_{50} + 2m_{31} + m_{41} + m_{23} + q_{21} + q_{31} + q_{41} + q_{51} + 2q_{32} + 2q_{42}) \quad (2)$$

$$e(P,Q) = q_{21} + q_{31} + q_{41} + q_{51} + 2q_{32} + 2q_{42} = m_{31} + m_{41} + m_{51}$$
$$+2m_{32} + 2m_{42} + 3m_{23} + 3m_{33} + 4m_{24} + 5m_{15} \quad (3)$$

By Lemma 1, for every vertex $v \in P - P^{'} = P_{00} \cup P_{10} \cup P_{20} \cup P_{30} \cup P_{40} \cup P_{01} \cup P_{11} \cup P_{21} \cup P_{02}$, there exists a vertex $u \in N(v)$ such that $f[u] = 1$. It follows that for every vertex $v \in P - P'$, there must exist a neighbor of v that belongs to $P^{'} \cup Q^{'} \cup M^{'}$, hence we have

$$\begin{aligned} & p_{00} + p_{10} + p_{20} + p_{30} + p_{40} + p_{01} + p_{11} + p_{21} + p_{02} \\ \leq \; & e(P - P^{'}, P^{'} \cup Q^{'} \cup M^{'}) = e(P - P^{'}, P^{'}) + e(P - P^{'}, Q^{'} \cup M^{'}) \\ = \; & e(P - P^{'}, P_{50}) + e(P - P^{'}, P_{31}) + e(P - P^{'}, P_{12}) \\ & + e(P - P^{'}, Q^{'} \cup M^{'}) \end{aligned} \quad (4)$$

Furthermore, we note that for every vertex $v \in P_{12}$, there must exist a neighbor u of v satisfying $f[u] = 1$, that is, $u \in Q^{'} \cup M^{'}$. If $u \in P^{'}$, then v is adjacent to at most two vertices of $P - P^{'}$, while if $u \in M^{'}$, then v is adjacent to at most three vertices of $P - P^{'}$. Hence we can write P_{12} as the disjoint union of two sets $P_{12}^{'}$ and $P_{12}^{''}$ where $P_{12}^{'} = \{v \in P_{12} |\, d_{P-P'}(v) = 3\}$ and $P_{12}^{''} = P_{12} - P_{12}^{'}$. Let $|P_{12}^{'}| = p_{12}^{'}$, and so $|P_{12}^{''}| = p_{12}^{''} = p_{12} - p_{12}^{'}$. Since each vertex $v \in P_{12}^{'}$ is adjacent to at least one vertex of $M^{'}$, it follows that $p_{12}^{'} \leq e(p_{12}^{'}, M^{'})$. So we get

$$\begin{aligned} e(P - P^{'}, P_{12}) &= e(P - P^{'}, p_{12}^{'} \cup p_{12}^{''}) \leq 3p_{12}^{'} + 2(p_{12} - p_{12}^{'}) \\ &= 2p_{12} + p_{12}^{'} \leq 2p_{12} + e(P_{12}^{'}, M^{'}) \end{aligned} \quad (5)$$

Similarly, it follows that for every vertex $v \in P_{31}$, there must exist a neighbor u of v that belongs to $P^{'} \cup Q^{'} \cup M^{'}$. If $u \in P^{'}$, then v is adjacent to at most a vertex of $P_{P}^{'}$, while if $u \in Q^{'} \cup M^{'}$, then v is adjacent to at most two vertices of $P - P^{'}$. Therefore we can partition P_{31} into two subsets $P_{31}^{'} = \{v \in P_{31} |\, d_{P-P'}(v) = 1\}$ and $P_{31}^{''} = P_{31} - P_{31}^{'}$. Let $|\, P_{31}^{'}| = p_{31}^{'}$, and so $|P_{31}^{''}| = p_{31}^{''} = p_{31} - p_{31}^{'}$. Because each vertex $v \in P_{31}^{'}$ is adjacent to at least one vertex of $Q^{'} \cup M^{'}$, we have $p_{31}^{'} \leq e(P_{31}^{'}, Q^{'} \cup M^{'})$. Hence we obtain

$$\begin{aligned} e(P - P^{'}, P_{31}) &= e(P - P^{'}, P_{31}^{'} \cup P_{31}^{''}) \leq 2p_{31}^{'} + (p_{31} - p_{31}^{'}) \\ &= p_{31} + p_{31}^{'} \leq p_{31} + e(P_{31}^{'}, Q^{'} \cup M^{'}) \end{aligned} \quad (6)$$

By the minimality of f, for each vertex $v \in P_{50}$, there must exist a critical neighbor u of v. If $u \in P^{'}$, then v is adjacent to no vertex of $P - P^{'}$, while if $u \in Q^{'}$, then v is adjacent to at most one vertex of $P - P^{'}$. So, we can write P_{50}

as the disjoint union of two sets $P_{50}^{'}$ and $P_{50}^{''}$ where $P_{50}^{'} = \{v \in P_{50} | d_{P-P'} = 1\}$ and $P_{50}^{''} = P_{50} - P_{50}^{'}$. Let $|P_{50}^{'}| = p_{50}^{'}$, and so $|P_{50}^{''}| = p_{50}^{''} = p_{50} - p_{50}^{'}$. Since each vertex $v \in P_{50}^{'}$ is adjacent to at least one vertex of $Q^{'}$, it follows that

$$\begin{aligned} e(P-P^{'}, P_{50}) &= e(P-P^{'}, P_{50}^{'} \cup P_{50}^{''}) = e(P-P^{'}, P_{50}^{'}) \\ &= p_{50}^{'} \leq e(P_{50}^{'}, Q^{'}) \end{aligned} \tag{7}$$

Thus, by(4)(5)(6)and(7) we get

$$\begin{aligned} & p_{00}+p_{10}+p_{20}+p_{30}+p_{40}+p_{01}+p_{11}+p_{21}+p_{02} \\ \leq\ & 2p_{12}+p_{31}+e(P_{12}^{'}, M^{'})+e(P_{31}^{'}, Q^{'} \cup M^{'})+e(P_{50}^{'}, Q^{'}) \\ & +e(P-P^{'}, Q^{'} \cup M^{'}) \leq 2p_{12}+p_{31}+e(P, Q^{'} \cup M^{'}) \\ =\ & 2p_{12}+p_{31}+q_{10}+2q_{21}+3q_{32}+3m_{31}+2m_{23}+m_{15} \end{aligned} \tag{8}$$

Next, we start to establish the upper bound on $\Gamma_t^-(G)$. We obtain

$$\begin{aligned} & n = q+m+p = q+m+(p_{00}+p_{10}+p_{20}+p_{30}+p_{40}+p_{01}+p_{11} \\ & +p_{21}+p_{02})+(p_{50}+p_{31}+p_{12}) \leq q+m+p_{50}+2p_{31}+3p_{12}+q_{10} \\ & +2q_{21}+3q_{32}+3m_{31}+2m_{23}+m_{15} \leq 7(q+m)-a \end{aligned}$$

where $a = p_{10}+2p_{20}+3p_{30}+4p_{40}+4p_{50}+p_{01}+2p_{11}+3p_{21}+2p_{31}+2p_{02}+4q_{10}+4q_{20}+3q_{30}+2q_{40}+q_{50}+3q_{21}+4q_{31}+3q_{41}+2q_{51}+2q_{32}+4q_{42}+2m_{40}+m_{50}+m_{41}-m_{31}-m_{23}-m_{15}$ Hence, it follows that

$$q+m \geq \frac{1}{7}n+\frac{1}{7}a$$

so

$$p = n-(q+m) \leq \frac{6}{7}n-\frac{1}{7}a \tag{9}$$

On the other hand, we have

$$\begin{aligned} & p = (p_{00}+p_{10}+p_{20}+p_{30}+p_{40}+p_{01}+p_{11}+p_{21}+p_{02})+(p_{50}+ \\ & p_{31}+p_{12}) \leq 2p_{21}+p_{31}+q_{10}+2q_{21}+3q_{32}+3m_{31}+2m_{23}+m_{15} \\ & = 9m-\frac{1}{2}b \end{aligned} \tag{10}$$

where $b = 3p_{01}+3p_{11}+3p_{21}+6p_{02}-2p_{50}-p_{31}+3q_{31}+3q_{41}+3q_{51}+6q_{42}-2q_{10}-q_{21}+6m_{40}+3m_{50}+3m_{41}-m_{23}-2m_{15}$ So, we obtain

$$m \geq \frac{1}{9}p+\frac{1}{18}b$$

Combining(2)(9)and(10), we immediately get

$$\begin{aligned}\Gamma_t^-(G) &= p-m \leq \frac{8}{9}p-\frac{1}{18}b \leq \frac{16}{21}n-\frac{1}{126}(16+7b)\\ &= \frac{16}{21}n-\frac{1}{126}c \leq \frac{16}{21}n\end{aligned}$$

where $c = 16p_{10}+32p_{20}+48p_{30}+64p_{40}+50p_{50}+37p_{01}+53p_{11}+69p_{21}+25p_{31}+84p_{02}+50q_{10}+64q_{20}+48q_{30}+32q_{40}+16q_{50}+25q_{21}+69q_{31}+53q_{41}+37q_{51}+74q_{42}+74m_{40}+37m_{50}+37m_{41}+16m_{51}+9m_{32}+32m_{42}+48m_{23}+48m_{33}+64m_{24}+50m_{15}$

For a 6-regular graph G of order n, we next show that if $\Gamma_t^-(G) = 16n/21$, then $G \in \mathcal{F}$. Suppose that$\Gamma_t^-(G) = 16n/21$, then equalities hold for the above inequalities. By $c = 0$, we immediately have

$$p_{10} = p_{20} = p_{30} = p_{40} = p_{50} = p_{01} = p_{11} = p_{21} = p_{31} = p_{02} = 0$$

$$q_{10} = q_{20} = q_{30} = q_{40} = q_{50} = q_{21} = q_{31} = q_{41} = q_{51} = q_{32} = q_{42} = 0$$

$$m_{40} = m_{50} = m_{41} = m_{51} = m_{32} = m_{42} = m_{23} = m_{33} = m_{24} = m_{15} = 0$$

Consequently, we obtain $V = P_{00} \cup P_{12} \cup Q_{60} \cup Q_{32} \cup M_{60} \cup M_{31}$. Applying the equality (1), (2), (3) we get

$$p_{12} = 6q_{60} + 3q_{32}, \qquad 2p_{12} = 6m_{60} + 3m_{31}$$

$$2q_{32} = m_{31}, \qquad p_{00} = 2p_{12} + 3q_{32} + 3m_{31}$$

Write $m_{31} = 2q_{32} = 4l$, $m_{60} = 2q_{60} = 2k$. Thus $p_{12} = 6k + 6l$, $p_{00} = 12k + 30l$. and the equality from (8), we get

$$p_{00} = e(P_{00}, P_{02}) = 3p_{12}' + 2p_{12}'' = 2p_{12} + p_{12}' = 2p_{12} + e(P_{12}', M_{30})$$

So each vertex in P_{00} is adjacent to precisely one vertex of $P_{12} = P_{12}' \cup P_{12}''$, and each vertex in P_{12}' is adjacent to exactly three vertices of P_{00} while each vertex in P_{12}'' is adjacent to exactly two vertices of P_{00}. Obviously, $G[M_{60}]$, $G[Q_{60}]$ and $G[P_{12}']$ are independent sets, and $G[M_{31}]$, $G[Q_{32}]$ and $G[P_{00}]$ are 2-regular, 1-regular and 4-regular, respectively. By Lemma 1, every vertex v of P_{12}' has at least a neighbor u of v such that $f[u] = 1$. Thus if v has a neighbor that belongs to Q_{60}, then v must have a neighbor that belongs to the critical vertex set M_{31}. Similarly, v must have a neighbor that belongs to the critical vertex set Q_{32}. So we have $e(P_{12}', M_{60}) = e(P_{12}', Q_{32}) = e(P_{12}', Q_{60}) = 12l$ and $e(P_{12}', M_{31}) = 2e(P_{12}', Q_{32}) = 12l$. Write $M_{31} = A_1$, $M_{60} = A_2$, $Q_{32} = A_3$, $Q_{60} = A_4$, $P_{00} = A_8$, $P_{12}'' = A_7$ and$P_{12}' = A_5 \cup A_6$ where each vertex of A_5

has a neighbor that belong to Q_{60} while A_5 has a neighbor that belong to Q_{32}. Therefore, $G \in \mathcal{F}$.

Conversely, suppose that $G \in \mathcal{F}$. Thus, there exist two integers $k \geq 2l \geq 0$, $k \geq 2$ such that $G = G_{k,l}$ is 6-regular graph of order 21(k+2l). Let f be a function on $G_{k,l}$ which assigns to every vertex of $A_1 \cup A_2$ and $A_3 \cup A_4 \cup A_5$ the value -1 and +1, respectively. Then the set $A_1 \cup A_3 \cup A_4$is critical set of $G_{k,l}$ under f, which implies that for every vertex $v \in V$, there exists a vertex $u \in N(v)$ such that $f[u] = 1$. So, f is a minimal minus total dominating function with weight $\omega(f) = \Sigma_{i=5}^{8} a_i - (a_1 + a_2) = 16(k + 2l)$. Consequently, $\Gamma_t^-(G) = 16n/21$. □

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