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A NOTE ON TWO POINT TAYLOR EXPANSION

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Abstract: Let f be a piecewise polynomial continuous function such that f is a polynomial p on the interval $[0,\infty)$ and f is a polynomial q on the interval $(-\infty,0]$. Then, we show that f is expressed as the two point Taylor expansion about $-1,1$ on the interval $(-\sqrt{2},\sqrt{2})$.

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1. Introduction

It is well known that polynomial approximation has a long history and lays the foundation of approximation theory. Especially, interpolation by polynomials takes a front seat of polynomial approximation and has been furnishing many important topics.

Before stating the purpose of this note, it is of use to review briefly Hermite interpolation by polynomials.

Let $[a,b]$ $(a<b)$ be a compact interval of $\mathbf{R}$ and let f be a real-valued function f on $[a,b]$. For any given $(n+1)$ distinct points $X: x_0,\dots,x_n$ in $[a,b]$ and for any sequence of positive integers $k_0,\dots,k_n$, if f is sufficiently differentiable at $x_0,\dots,x_n$, then there exists a unique approximating polynomial $p_{f,X(k_0,\dots,k_n)}(x)$ to f which is of degree at most $m(=k_0+\cdots+k_n-1)$ and

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satisfy that

$$p^{(j)}_{f,X(k_0,\dots,k_n)}(x_i) = f^{(j)}(x_i), \quad 0 \leqq j \leqq k_i - 1,\ 0 \leqq i \leqq n.$$

The points $x_0, \dots, x_n$ and the polynomial $p_{f,X(k_0,\dots,k_n)}$ are called *nodes* and the *Hermite interpolating polynomial to f at $x_0, \dots, x_n$ with mutiplicities* $k_0, \dots, k_n$, respectively. As is well known, for one node X : x_0 with multiplicity n, the Hermite interpolating polynomial $p_{f,X(n)}$ to f is the Taylor polynomial of f about x_0

$$p_{f,X(n)}(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1}.$$

Furthermore, if f is infinitely differentiable at x_0 and if

$$f(x) = \lim_{n\to\infty} p_{f,X(n)}(x) \quad \text{for all } x \in (x_0 - \rho, x_0 + \rho) \ \ (\rho \text{ is some positive number})$$

then f has the Taylor expansion of f about x_0 on $(x_0 - \rho, x_0 + \rho)$. By this fact, we make the following definition.

Definition 1. Let f be a real-valued function on an interval I. If there exists a list X consisting of m nodes $x_0, \dots, x_{m-1}$ in I such that f is infinitely differentiable at $x_0, \dots, x_{m-1}$ and

$$\lim_{n\to\infty} p_{f,X(n,\dots,n)}(x) = f(x) \quad \text{for all } x \in I,$$

then it is said that f has *m point Taylor expansion about $x_0, \dots, x_{m-1}$ on I*. And the set of all functions to have m point Taylor expansion on I is denoted by $\mathcal{T}_m(I)$ or $\mathcal{T}_m$ for short.

The notion of two point or m point Taylor expansion is not new. One can see the theory of m point Taylor expansion in the complex plane in chap. 3 of [6]. [3, 4] explained how m point Taylor expansion in the complex plane can be used in deriving uniform asymptotic expansions of integrals. Let f be an analytic function on $[-1, 1]$ and suppose that the radiuses of convergence of the Taylor series about $-1, 1$ for f are more than 2. By the result stated in Remark 2 in [2], we easily show f belongs to $\mathcal{T}_n([-1, 1])$ for all $n \in \mathbf{N}$. In this note, we are concerned with showing functions which are note analytic but belong to $\mathcal{T}_2(I)$. Concretely speaking, the purpose of this note is to prove the following theorem.

Theorem. *Let f be a function on* $\mathbf{R}$, *which is expressed as*

$$f(x) = \begin{cases} p(x) & x \in [0, \infty) \\ q(x) & x \in (-\infty, 0) \end{cases},$$

where p and q are polynomials of degree at most m. Let $P_n, n \in \mathbf{N}$ be the Hermite interpolating polynomials to f at $-1, 1$ with mutiplicities n, n. Then, the following assertions hold:

(1) $$\lim_{n\to\infty} P_n(x) = f(x) \ , \quad \textit{for all } x \in (-\sqrt{2}, 0) \cup (0, \sqrt{2}).$$

(2) *Moreover, if $p(0) = q(0)$, then f has two point Taylor expansion about $-1, 1$ on $(-\sqrt{2}, \sqrt{2})$, that is,*

$$\lim_{n\to\infty} P_n(x) = f(x) \ , \quad \textit{for all } x \in (-\sqrt{2}, \sqrt{2}).$$

2. n Point Taylor Polynomials

First we begin with a proposition which states the existence of Hermite interpolating polynomials.

Proposition 1. (see p. 365 in [1]) *Let $x_0 \leqq x_1 \leqq \cdots \leqq x_n$ be a list of nodes. In the list of nodes, only distinct nodes $z_0, \ldots, z_p$ appear and each node $z_i, i = 0, \ldots, p$ is just appeared k_i times. Let f be sufficiently differentiable at $z_0, \ldots, z_p$. Then, there exists a unique polynomial p of degree at most n satisfying that*

$$p^{(j)}(z_i) = f^{(j)}(z_i) \ , \qquad j = 0, \ldots, k_i - 1, \ i = 0, \ldots, p. \tag{$*$}$$

In Proposition 1, we call each positive integer $k_i, i = 0, \ldots, p$ *the multiplicity at x_i*. Divided differences of functions can be defined by this proposition.

Definition 2. Let $x_0 \leqq x_1 \leqq \cdots \leqq x_n$ be a list of nodes and let f be sufficiently differentiable at $x_0, \ldots, x_n$. Then the coefficient of x^n in the polynomial p with the property $(*)$ stated above is called *the n-th order divided difference of f at $x_0, \ldots, x_n$* and is denoted by $f[x_0, \ldots, x_n]$.

By Definition 2, it is easily seen that the divided difference $f[x_0]$ of a function f at a point x_0 is equal to $f(x_0)$. The following proposition of a recursive formula and a divided difference table are of much use to calculate divided differences of functions.

Proposition 2. (see p. 372 in [1]) *Let $x_0 \leqq \cdots \leqq x_n$ be a list of nodes and let f be sufficiently differentiable at $x_0, \ldots, x_n$. Then the divided differences* *obey this recursive formula:*

$$f[x_0, \dots, x_n] = \begin{cases} \dfrac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0} & (x_0 \neq x_n) \\ \dfrac{f^{(n)}(x_0)}{n!} & (x_0 = x_n) \end{cases}$$

If data points $(x_i, f(x_i)), i = 0, \dots, n$ are given, then we can construct the following divided difference table from them. By Proposition 2, $(i+1)$-th order divided differences in the table are calculated from i-th order divided differences.

$$\begin{array}{ccccc}
x_0 & f[x_0] & & & \\
 & & f[x_0, x_1] & & \\
x_1 & f[x_1] & & & \\
 & & f[x_1, x_2] & & \\
x_2 & f[x_2] & & & \\
\vdots & \vdots & \vdots & \cdots & f[x_0, x_1, \cdots, x_{n-1}, x_n] \\
x_{n-2} & f[x_{n-2}] & & & \\
 & & f[x_{n-2}, x_{n-1}] & & \\
x_{n-1} & f[x_{n-1}] & & & \\
 & & f[x_{n-1}, x_n] & & \\
x_n & f[x_n] & & &
\end{array}$$

Divided Difference Table

In the divided difference table stated above, we call the column consisting of the i-th order divided differences *the i-th order column* for convenience.

Notation. Let $x_0 \leqq x_1 \leqq \cdots \leqq x_n$ be a list of nodes and let f be sufficiently differentiable at $x_0, \dots, x_n$. In the list of nodes, only distinct points $z_0, \dots, z_p$ appear and each point $z_i, i = 0, \dots, p$ is just appeared k_i times. To make sure of multiplicities, we write

$$f[z_0, \dots, z_p; k_0, \dots, k_p]$$

for the divided difference $f[x_0, \dots, x_n]$.

Let $x_0 \leqq x_1 \leqq \cdots \leqq x_n$ be a list of nodes and let f be sufficiently differentiable at $x_0, \dots, x_n$. It is well known that the Hermite interpolating polynomial

p of f at $x_0, \ldots, x_n$, that is p satisfies $(*)$, is expressed as

$$p(x) = \sum_{j=0}^{n} f[x_0, \ldots, x_j]\Pi_{i=0}^{j-1}(x - x_i),$$

where $\Pi_{i=0}^{-1}(x - x_i) = 1$ (see p.370 in [1]).

Definition 3. Let $x_0 < x_1 < \cdots < x_n$ be a list of distinct nodes and let f be sufficiently differentiable at $x_0, \ldots, x_n$. For a positive integer k, the Hermite interpolating polynomial p of f at $x_0, \ldots, x_n$ with multiplicities $k_i = k, i = 0, \ldots, n$ is called *the k-th order $(n+1)$ point Taylor polynomial about* $x_0, \ldots, x_n$

The following propostion is a basic statement, but it is a key result to prove our theorem.

Propostion 3. *Let $x_0 \leqq x_1 \leqq \cdots \leqq x_n$ be a list of nodes and let f be a real-valued function on an interval $[a, b]$ which is sufficiently differentiable at $x_0, \ldots, x_n$. If p is the the Hermite interpolating polynomial of f at $x_0, \ldots, x_n$, then*

$$f(x) - p(x) = f[x, x_0, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_n), \qquad x \in [a, b].$$

3. Proof of Theorem

Proof of Theorem. For each positive integer ℓ, let h_ℓ be the ℓ-th order 2 point Taylor polynomial of f about $-1, 1$. By Proposition 3, we have for each $x \in \mathbf{R}$

$$|f(x) - h_\ell(x)| = |(x-1)^\ell (x+1)^\ell f[-1, x, 1; \ell, 1, \ell]| = |x^2 - 1|^\ell |f[-1, x, 1; \ell, 1, \ell]|.$$

(1) Since $|x^2 - 1| < 1$ for each $x \in (-\sqrt{2}, 0) \cup (0, -\sqrt{2})$, it is sufficient to show

$$\sup_{\ell \geqq 1} |f[-1, x, 1; \ell, 1, \ell]| < +\infty.$$

Without loss of generality, we assume that x is a point in $(0, \sqrt{2}) - \{1\}$. To evaluate $f[-1, x, 1; \ell, 1, \ell]$, we make the divided difference table of f with the nodes $-1, x, 1$. The 0-th and the first columns of the the divided difference table are

$$
\begin{array}{cc}
q(-1) & \\
 & \dfrac{q'(-1)}{1!} \\
q(-1) & \\
 & \vdots \\
\vdots & \\
 & \dfrac{q'(-1)}{1!} \\
q(-1) & \\
 & \dfrac{p(x)-q(-1)}{x+1} \\
p(x) & \\
 & \dfrac{p(x)-p(1)}{x-1} \\
p(1) & \\
 & \dfrac{p'(1)}{1!} \\
p(1) & \\
 & \dfrac{p'(1)}{1!} \\
\vdots & \\
 & \vdots \\
 & \dfrac{p'(1)}{1!} \\
p(1) &
\end{array}
$$

Furthermore, the $k(\geqq 2)$-th column of the divided difference table is

$$\frac{q^{(k)}(-1)}{k!}$$
$$\frac{q^{(k)}(-1)}{k!}$$
$$\vdots$$
$$\frac{q^{(k)}(-1)}{k!}$$
$$\frac{1}{(x+1)^k}\left\{p(x)-\left(q(-1)+\frac{q'(-1)}{1!}(x+1)+\cdots+\frac{q^{(k-1)}(-1)}{(k-1)!}(x+1)^{k-1}\right)\right\}$$
$$\frac{1}{2}\{f[-1,x,1;k-2,1,1]-f[-1,x;k-1,1]\}=f[-1,x,1;k-1,1,1],$$
$$\frac{1}{2}\{f[-1,x,1;k-3,1,2]-f[-1,x,1;k-2,1,1]\}=f[-1,x,1;k-2,1,2],$$
$$\vdots$$
$$\frac{1}{2}\{f[x,1;1,k-1]-f[-1,x,1;1,1,k-2]\}=f[-1,x,1;1,1,k-1]$$
$$\frac{1}{(x-1)^k}\left\{p(x)-\left(p(1)+\frac{p'(1)}{1!}(x-1)+\cdots+\frac{p^{(k-1)}(1)}{(k-1)!}(x-1)^{k-1}\right)\right\}$$
$$\frac{p^{(k)}(1)}{k!}$$
$$\vdots$$
$$\frac{p^{(k)}(1)}{k!}.$$

Hence, the $m+1$-th column of the divided difference table is

$$\begin{gathered}
\frac{q^{(m+1)}(-1)}{(m+1)!} = 0 \\
\vdots \\
\frac{q^{(m+1)}(-1)}{(m+1)!} = 0 \\
\frac{1}{(x+1)^{m+1}}\{p(x) - q(x)\} \\
\frac{1}{2}\{f[-1,x,1;m-1,1,1] - f[-1,x;m,1]\} = f[-1,x,1;m,1,1] \\
\frac{1}{2}\{f[-1,x,1;m-2,1,2] - f[-1,x,1;m-1,1,1]\} = f[-1,x,1;m-1,1,2], \\
\vdots \\
\frac{1}{2}\{f[x,1;1,m] - f[-1,x,1;1,1,m-1]\} = f[-1,x,1;1,1,m], \\
\frac{1}{(x-1)^{m+1}}\{p(x) - p(x)\} = 0 \\
\frac{p^{(m+1)}(1)}{(m+1)!} = 0 \\
\vdots \\
\frac{p^{(m+1)}(1)}{(m+1)!} = 0.
\end{gathered}$$

We set M as the maximum of the absolute values of elements of the $m+1$-th column of the divided difference table. From Proposition 2 and the fact that the sequence $\dfrac{1}{(x+1)^n}|p(x) - q(x)|, n \in \mathbf{N}$ is monotone decreasing, we easily observe that the absolute value of any element of the $n(\geqq m+1)$-th column of the divided difference table is equal to or less than M, which proves (1).

(2) Without loss of generality, we assume that $p(0) = q(0) = 0$. Suppose that ℓ is much larger than m. By (1), the $(m+1)$-th column of the divided difference table is

$$\frac{q^{(m+1)}(-1)}{(m+1)!} = 0$$
$$\vdots$$
$$\frac{q^{(m+1)}(-1)}{(m+1)!} = 0$$
$$\frac{1}{(x+1)^{m+1}}\{p(0) - q(0)\} = 0$$
$$\frac{1}{2}\{f[-1,0,1;m-1,1,1] - f[-1,0;m,1]\} = f[-1,0,1;m,1,1]$$
$$\frac{1}{2}\{f[-1,0,1;m-2,1,2] - f[-1,0,1;m-1,1,1]\} = f[-1,0,1;m-1,1,2],$$
$$\vdots$$
$$\frac{1}{2}\{f[0,1;1,m] - f[-1,0,1;1,1,m-1]\} = f[-1,0,1;1,1,m],$$
$$\frac{1}{(-1)^{m+1}}\{p(0) - p(0)\} = 0$$
$$\frac{p^{(m+1)}(1)}{(m+1)!} = 0$$
$$\vdots$$
$$\frac{p^{(m+1)}(1)}{(m+1)!} = 0.$$

It is sufficient to show that $\lim_{\ell\to\infty} f[-1,0,1;\ell,1,\ell] = 0$.

To estimate $f[-1,0,1;\ell,1,\ell]$, we introduce a recursive relation by which a column vector $\mathbf{a}' = (a'_i)_{1\le i\le n}$ is obtained from a column vector $\mathbf{a} = (a_i)_{1\le i\le n+1}$ such that

$$a'_i = \frac{a_i + a_{i+1}}{2}\ , \qquad i = 1,2,\ldots,n. \tag{$**$}$$

Let $\mathbf{a}^{(k)}, 1 \le k \le 2\ell - m$ be the column consisting of the absolute values of the elements of the $m+k$-th column of the divided difference table. Putting

$$L := \max\{|f[-1,0,1;m,1,1]|,\ldots,|f[-1,0,1;1,1,m]|\},$$

to do estimations easily, we define $\mathbf{b}^{(1)} = (b_i^{(1)})_{1\le i\le 2\ell-m}$ such that

$$b_i^{(1)} = \begin{cases} 0\ , & i = 1,\ldots,\ell-m,\ell+1,\ldots,2\ell-m \\ \begin{pmatrix} m-1 \\ j-1 \end{pmatrix} L\ , & i = \ell-m+j,\ j = 1,\ldots,m, \end{cases}$$

where $\begin{pmatrix} m-1 \\ j-1 \end{pmatrix}$ denote binomial coefficients. Let $\mathbf{b}^{(k)}, 1 \le k \le 2\ell - m$ be the k-th column which is obtained by the recursive relation $(**)$ with the first column $\mathbf{b}^{(1)}$. We easily have $\mathbf{a}^{(1)} \leqq \mathbf{b}^{(1)}$, that is, $a_i^{(1)} \leqq b_i^{(1)}, i = 1, \ldots, 2\ell - m$. Furthemore, by the recursive relations of divided differences and $(**)$ we get

$$\mathbf{a}^{(j)} \leqq \mathbf{b}^{(j)}, \qquad j = 1, \ldots, 2\ell - m. \tag{$***$}$$

On the other hand, using Pascal's triangle, we see that $\mathbf{b}^{(\ell-m+1)}$ is expressed as

$$\begin{array}{c} \dfrac{\binom{\ell-1}{0} L}{2^{\ell-m}} \\ \dfrac{\binom{\ell-1}{1} L}{2^{\ell-m}} \\ \vdots \\ \dfrac{\binom{\ell-1}{\ell-1} L}{2^{\ell-m}} \end{array}$$

Since every element of $\mathbf{b}^{(\ell-m+1)}$ is positive, if we set T as the maximum of the elements of $\mathbf{b}^{(\ell-m+1)}$, then by the recursive relation $(**)$, we observe that every element of $\mathbf{b}^{(k)}, \ell - m + 1 \leq k \leq 2\ell - m$ is equal to or less than T. So we have

$$|f[-1, 0, 1; \ell, 1, \ell]| = \mathbf{a}^{(2\ell-m)} \leq \mathbf{b}^{(2\ell-m)} \leq T.$$

Hence, it is sufficient to show that $T \to 0$ as $\ell \to \infty$. Suppose that $\ell - 1$ is even, say, we put $\ell - 1 = 2r$. Then we have

$$T = \frac{\binom{2r}{r} L}{2^{2r+1-m}} = \frac{\binom{2r}{r}}{2^{2r}} \cdot 2^{m-1} L = \frac{(2r-1)!!}{(2r)!!} \cdot 2^{m-1} L.$$

Since $2^{m-1}L$ is a constant and $r \to \infty$ as $\ell \to \infty$, we obtain

$$\lim_{\ell \to \infty} T = \lim_{r \to \infty} \frac{(2r-1)!!}{(2r)!!} \cdot 2^{m-1} L = 0.$$

In case $\ell - 1$ is odd, the same conclusion follows from an analogous way to the case that $\ell - 1$ is even. This completes the proof. $\square$

Remark 1. It is possible to evaluate how fast T tends to 0 as $\ell \to \infty$. In fact, by virtue of Stirling's formula, we are able to show that

$$T = \frac{C}{\sqrt{\ell}}(1 + o(1))$$

as $\ell \to \infty$ for some positive constant C.

Remark 2. Let $a = \xi_0, \ldots, \xi_n = b$ be $(n+1)$ distinct points of the real line and for a given positive integer k, let $S(k, \xi_0, \ldots, \xi_n)$ be the space of spline functions of degree k on $[a, b]$ that have the knots $\xi_1, \ldots, \xi_{n-1}$,i.e., real-valued continuous functions f on $[a, b]$ satisfying that

(i) f is a polynomial of degree at most k on each subintervals $[\xi_{i-1}, \xi_i], i = 1, \ldots, n$,

(ii) f is $k-1$ times continuously differentiable on $[a, b]$.

It is well known that each $h \in S(k, \xi_0, \ldots, \xi_n)$ is expressed as

$$h(x) = \sum_{i=0}^{k} c_i x^i + \sum_{i=0}^{n-1} d_i (x - \xi_i)_+^k \ , \qquad x \in [a, b]$$

where $(x - \xi_i)_+^k = \begin{cases} (x - \xi_i)^k & x \geq \xi_i \\ 0 & x < \xi_i \end{cases}$ (see p. 29 in [5]). From Theorem since every $d_i (x - \xi_i)_+^k, i = 0, \ldots, n-1$ has a two point Taylor expansion $s_i(x)$ about $\xi_i + \delta_i, \xi_i - \delta_i$ (δ_i is some positive number) on $[a, b]$, we see that $h(x)$ can be expressed as the sum of the polynomial $\sum_{i=0}^{k} c_i x^i$ and two point Taylor expansions $s_i(x), i = 0, \ldots, n-1$.

In this note, we only show a first step to the problem "What functions does $\mathcal{T}_n$ consist of ?" Finally, we give problems which leads to a second step to this problem.

(1) Show that Theorem holds under the condition that p and q in Theorem are analytic functions havng appropriate properties.

(2) Consider cases corresponding to Theorem of $k(\geq 3)$ point Taylor expansions

(3) Find other types of functions which belong to $\mathcal{T}_n (n \geq 2)$.

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