International Journal of Pure and Applied Mathematics
Volume 76 No. 1 2012, 49-68
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

# EXACT SOLUTIONS FOR A FAMILY OF BOUSSINESQ EQUATION WITH NONLINEAR DISPERSION

Wei Li[1 §], Jinhua Zhang[2], Yunmei Zhao[3]
[1,2,3]Department of Mathematics
Honghe University
Mengzi, Yunnan, 661100, P.R. CHINA

**Abstract:** New exact solutions including the solitary wave solutions and periodic wave solutions for a family of Boussinesq equation with nonlinear dispersion are obtained using the Riccati equation mapping method. The results presented in this paper improve the previous results.

**AMS Subject Classification:** 35Q51, 35Q58, 36K65, 34A20
**Key Words:** Boussinesq equation, Riccati equation mapping method, exact solution, solitary wave solution

## 1. Introduction

Nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in various fields of science, especially in physics. Searching for exact soliton solutions of NPDEs plays an important and significant role in the study on the dynamics of those phenomena. Up to now, many effective ansatz methods have been presented, such as the tanh method [1], Jacobi elliptic function method [2], F-expansion method [3], the Exp-function method [4-6], auxiliary equation method [7,8], and so on. Here, it is worth to mention that the Riccati equation mapping method [9,10]. The solution procedure of this method, with the aid of Maple, is of utter simplicity and this method can easily extented to other kinds of nonlinear evolution equations.

Received: August 17, 2011



§Correspondence author

In this paper, by using the Riccati equation mapping method, we shall study the following Boussinesq equation with nonlinear dispersion [11]

$$u_{tt} = au_{xx} + (u^{m+1})_{xx} + b[u(u^m)_{xx}]_{xx}, \tag{1}$$

where $a$, $b$ and $m$ are arbitrary constants. In [11], using the classical Lie method of infinitesimals, Bruzon and Gandarias obtained some travelling wave solutions for equation (1). In this work, we will explore more types of exact solutions for equation (1) in the cases $m \in Z$ and $m \neq 0$.

## 2. The Solutions of Subsidiary Riccati Equation

The subsidiary Riccati equation is of the form as

$$\phi'(\xi) = \frac{d}{d\xi}\phi(\xi) = A + \gamma\phi^2(\xi), \tag{2}$$

where $A$ and $\gamma$ are arbitrary constants.

Seeking for the exact solutions of equation (2), we introducing a complex variable $\eta$, defined by

$$\eta = \mu\xi + \xi_0, \tag{3}$$

where $\mu$ is a constant to be determined later, $\xi_0$ is an arbitrary constant, Riccati equation (2) converts to

$$\mu\phi' - A - \gamma\phi^2 = 0, \tag{4}$$

where prime denotes the derivative with respect to $\eta$.

Case 1. By using the rational hyperbolic functions method and trigonometric functions method to solve equation (4), respectively, the new exact solutions of the Riccati equation (2) are derived as

$$\phi_1 = \frac{A[H\sinh(\sqrt{-A\gamma}\xi) + S\cosh\sqrt{-A\gamma}\xi)]}{\sqrt{-A\gamma}[S\sinh(\sqrt{-A\gamma}\xi) + H\cosh\sqrt{-A\gamma}\xi)]}, \tag{5}$$

where $H$ and $S$ are arbitrary constants.

$$\phi_2 = \frac{\sqrt{A\gamma}[H\sin(\sqrt{A\gamma}\xi) - S\cos\sqrt{A\gamma}\xi)]}{\gamma[S\sin(\sqrt{A\gamma}\xi) + H\cos\sqrt{A\gamma}\xi)]}, \tag{6}$$

where $H$ and $S$ are arbitrary constants.

$$\phi_3 = \sqrt{-\frac{A}{\gamma}} - \frac{2C\sqrt{-\frac{A}{\gamma}}}{H\cosh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) - H\sinh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) + C}, \tag{7}$$

where $H$ and $C$ are arbitrary constants.

$$\phi_4 = C - \frac{(A+C^2\gamma)\sqrt{-\frac{\gamma}{A}}\sinh(A\sqrt{-\frac{\gamma}{A}}\xi)}{\gamma\left[C\sqrt{-\frac{\gamma}{A}}\sinh(A\sqrt{-\frac{\gamma}{A}}\xi)+\cosh(A\sqrt{-\frac{\gamma}{A}}\xi)\right]}, \tag{8}$$

where $C$ is arbitrary constants.

$$\phi_5 = -\frac{\sqrt{-\frac{A}{\gamma}}\sinh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{\cosh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\pm 1}. \tag{9}$$

$$\phi_6 = \frac{\sqrt{\frac{A}{\gamma}}\sin(2\gamma\sqrt{\frac{A}{\gamma}}\xi)}{\cos(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\pm 1}. \tag{10}$$

Case 2. In Refs. [9,10], by using the Exp-function method to solve equation (4), the Riccati equation (2) admits the exact solutions which are

$$\phi_7 = \frac{-\sqrt{-\frac{A}{\gamma}}b_1\exp(\gamma\sqrt{-\frac{A}{\gamma}}\xi+\xi_0)+a_{-1}\exp(-\gamma\sqrt{-\frac{A}{\gamma}}\xi-\xi_0)}{b_1\exp(\gamma\sqrt{-\frac{A}{\gamma}}\xi+\xi_0)+\frac{a_{-1}}{\sqrt{-\frac{A}{\gamma}}}\exp(-\gamma\sqrt{-\frac{A}{\gamma}}\xi-\xi_0)}, \tag{11}$$

where $a_{-1}$ and $b_1$ are free parameters.

(i) When $\xi_0 = 0$, $b_1 = 1$, $a_{-1} = \pm\sqrt{-\frac{A}{\gamma}}$, $\frac{A}{\gamma} < 0$, the solution (11) becomes

$$\phi = -\sqrt{-\frac{A}{\gamma}}\tanh(\gamma\sqrt{-\frac{A}{\gamma}}\xi), \tag{12}$$

and

$$\phi = -\sqrt{-\frac{A}{\gamma}}\coth(\gamma\sqrt{-\frac{A}{\gamma}}\xi). \tag{13}$$

(ii) When $\xi_0 = 0$, $b_1 = i$, $a_{-1} = \mp\sqrt{\frac{A}{\gamma}}$, $\frac{A}{\gamma} > 0$, the solution (11) becomes

$$\phi = \sqrt{\frac{A}{\gamma}}\tan(\gamma\sqrt{\frac{A}{\gamma}}\xi), \tag{14}$$

and

$$\phi = -\sqrt{\frac{A}{\gamma}}\cot(\gamma\sqrt{\frac{A}{\gamma}}\xi). \tag{15}$$

$$\phi_8 =$$

$$\frac{\frac{(\gamma a_0^2+Ab_0^2)}{4\gamma\sqrt{-\frac{A}{\gamma}}b_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi+\xi_0)+a_0+\sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi-\xi_0)}{\frac{(\gamma a_0^2+Ab_0^2)}{4Ab_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi+\xi_0)+b_0+b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi-\xi_0)}, \tag{16}$$

where $a_0$, $b_0$ and $b_{-1}$ are free parameters.

(i) When $\xi_0 = 0$, $b_0 = 0$, $b_{-1} = 1$, $a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}$, $\frac{A}{\gamma} < 0$, the solution (16) becomes

$$\phi = -\sqrt{-\frac{A}{\gamma}}[\coth(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm \operatorname{csch}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)]. \tag{17}$$

(ii) When $\xi_0 = 0$, $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}$, $\frac{A}{\gamma} < 0$, the solution (16) becomes

$$\phi = -\sqrt{-\frac{A}{\gamma}}[\tanh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm i\operatorname{sech}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)]. \tag{18}$$

(iii) When $\xi_0 = 0$, $b_0 = 0$, $b_{-1} = 1$, $a_0 = \pm 2\sqrt{\frac{A}{\gamma}}$, $\frac{A}{\gamma} > 0$, the solution (16) becomes

$$\phi = \sqrt{\frac{A}{\gamma}}[\tan(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \pm \sec(2\gamma\sqrt{\frac{A}{\gamma}}\xi)]. \tag{19}$$

(iv) When $\xi_0 = 0$, $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{\frac{A}{\gamma}}$, $\frac{A}{\gamma} > 0$, the solution (16) becomes

$$\phi = -\sqrt{\frac{A}{\gamma}}[\cot(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \mp \csc(2\gamma\sqrt{\frac{A}{\gamma}}\xi)]. \tag{20}$$

For simplicity, in the rest of the paper, we consider $\xi_0 = 0$.

## 3. The Exact Solutions of Equation (1)

In this section, using the Riccati equation (2) as auxiliary equation and its the exact traveling wave solutions, we investigate equation (1) and derive the various exact solutions of equation (1). In order to obtain new exact travelling wave solutions for equation (1), we use

$$u(x,t) = u(\xi), \quad \xi = k(x - \omega t), \tag{21}$$

where $k$ and $w$ are constants, and substituting the (21) into equation (1), integrating two sides of the equation with respect to $\xi$ and setting the constants of integration equal to zero, we obtain

$$(a-\omega^2)u + u^{m+1} + bk^2u(u^m)'' = 0. \tag{22}$$

Balancing the order of the nonlinear term $u(u^m)''$ with the term $u$ in (22), we obtain

$$mP + 2 + P = P,$$

so that

$$P = -\frac{2}{m}. \tag{23}$$

To get a closed form solution, it is natural to use the transformation

$$u = v^{-\frac{1}{m}}, \tag{24}$$

we have

$$v^2 + (a-\omega^2)v^3 + 2bk^2(v')^2 - bk^2vv''. \tag{25}$$

Now, we assume that the solution of equation (25) can be expressed in the following form

$$v = v(\xi) = \sum_{j=0}^{N} \alpha_j \phi^j(\xi) + \sum_{j=1}^{N} \beta_j \phi^{-j}(\xi), \tag{26}$$

where N is positive integers which are given by the homogeneous balance principle, $\phi(\xi)$ is a solution of equation (2). Balancing $v^3$ term with $vv''$ term in (25) gives N = 2. Therefore, we obtain

$$v = \alpha_0 + \alpha_1\phi(\xi) + \alpha_2\phi^2(\xi) + \frac{\beta_1}{\phi(\xi)} + \frac{\beta_2}{\phi^2(\xi)}. \tag{27}$$

Substituting equation (27) into (25) and using the Riccati equation (2), collecting the coefficients of $\phi(\xi)$, we have

$$\frac{1}{D}\left[C_0 + C_1\phi(\xi) + C_2\phi^2(\xi) + \cdots + C_{11}\phi^{11}(\xi) + C_{12}\phi^{12}(\xi)\right] = 0. \tag{28}$$

Because the expresses to these coefficients $D,\ C_0 = 0,\ C_1 = 0,\ C_2 = 0,\ C_3 = 0,\ C_4 = 0,\ \cdots,\ C_{11} = 0,\ C_{12}$ of $\phi(\xi)$ in equation (28) are too lengthiness, so we omit them. But we can directly use the command "solve" in mathematical software *Maple* to solve the following set of algebraic equations

$$C_0 = 0,\quad C_1 = 0,\quad C_2 = 0,\quad C_3 = 0,\quad C_4 = 0,\quad \cdots,\quad C_{11} = 0,\quad C_{12} = 0. \tag{29}$$

Solved the above algebraic equations, we obtain the following six sets of solutions

**Case 1**

$$\alpha_0 = \frac{1}{2(\omega^2 - a)}, \quad \alpha_1 = \frac{1}{2(\omega^2 - a)}\sqrt{-\frac{\gamma}{A}}, \quad \alpha_2 = 0, \quad \beta_1 = 0,$$

$$\beta_2 = 0, \qquad k = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}. \tag{30}$$

**Case 2**

$$\alpha_0 = \frac{1}{2(\omega^2 - a)}, \quad \alpha_1 = 0, \quad \alpha_2 = 0, \quad \beta_1 = \frac{1}{2(\omega^2 - a)}\sqrt{-\frac{A}{\gamma}},$$

$$\beta_2 = 0, \qquad k = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}. \tag{31}$$

**Case 3**

$$\alpha_0 = \frac{1}{2(\omega^2 - a)}, \quad \alpha_1 = \frac{1}{4(\omega^2 - a)\sqrt{-\frac{A}{\gamma}}}, \quad \alpha_2 = 0,$$

$$\beta_1 = \frac{1}{4(\omega^2 - a)}\sqrt{-\frac{A}{\gamma}}, \quad \beta_2 = 0, \quad k = \frac{1}{4}\sqrt{\frac{1}{bA\gamma}}. \tag{32}$$

**Case 4**

$$\alpha_0 = \frac{1}{2(\omega^2 - a)}, \quad \alpha_1 = 0, \quad \alpha_2 = \frac{\gamma}{2A(\omega^2 - a)}, \quad \beta_1 = 0,$$

$$\beta_2 = 0, \qquad k = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}. \tag{33}$$

**Case 5**

$$\alpha_0 = \frac{1}{2(\omega^2 - a)}, \quad \alpha_1 = 0, \quad \alpha_2 = 0, \quad \beta_1 = 0,$$

$$\beta_2 = \frac{A}{2\gamma(\omega^2 - a)}, \qquad k = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}. \tag{34}$$

**Case 6**

$$\alpha_0 = \frac{1}{4(\omega^2 - a)}, \quad \alpha_1 = 0, \quad \alpha_2 = \frac{\gamma}{8A(\omega^2 - a)}, \quad \beta_1 = 0,$$

$$\beta_2 = \frac{A}{8\gamma(\omega^2 - a)}, \qquad k = \frac{1}{4}\sqrt{\frac{1}{bA\gamma}}. \tag{35}$$

Thus from equations (27), (30)-(35), we obtain families of exact solutions to equation (25) as follows,

$$v = \frac{1}{2(\omega^2 - a)} + \frac{1}{2(\omega^2 - a)}\sqrt{-\frac{\gamma}{A}}\phi(\xi), \tag{36}$$

$$v = \frac{1}{2(\omega^2 - a)} + \frac{1}{2(\omega^2 - a)}\sqrt{-\frac{A}{\gamma}}\frac{1}{\phi(\xi)}, \tag{37}$$

$$v = \frac{1}{2(\omega^2 - a)} + \frac{1}{4(\omega^2 - a)\sqrt{-\frac{A}{\gamma}}}\phi(\xi) + \frac{1}{4(\omega^2 - a)}\sqrt{-\frac{A}{\gamma}}\frac{1}{\phi(\xi)}, \tag{38}$$

$$v = \frac{1}{2(\omega^2 - a)} + \frac{\gamma}{2A(\omega^2 - a)}\phi^2(\xi), \tag{39}$$

$$v = \frac{1}{2(\omega^2 - a)} + \frac{A}{2\gamma(\omega^2 - a)}\frac{1}{\phi^2(\xi)}, \tag{40}$$

$$v = \frac{1}{4(\omega^2 - a)} + \frac{\gamma}{8A(\omega^2 - a)}\phi^2(\xi) + \frac{A}{8\gamma(\omega^2 - a)}\frac{1}{\phi^2(\xi)}, \tag{41}$$

where $\phi(\xi)$ is a solution of the Riccati equation (2).

Substituting solutions (5)-(20) of the Riccati equation (2) into (36)-(41), using the transformation (24), we have the following several families of solutions to equation (1).

**Family 1**

$$u_{1(1)}(x,t) = \left\{ R \pm R\left[\frac{H\sinh(\sqrt{-A\gamma}\xi) + S\cosh\sqrt{-A\gamma}\xi)}{S\sinh(\sqrt{-A\gamma}\xi) + H\cosh\sqrt{-A\gamma}\xi)}\right]\right\}^{-\frac{1}{m}}, \tag{42}$$

where $R = \frac{1}{2(\omega^2 - a)}$, $\quad \xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x - \omega t)$.

$$u_{1(2)}(x,t) = \left\{ R \pm iR\left[\frac{H\sin(\sqrt{A\gamma}\xi) - S\cos\sqrt{A\gamma}\xi)}{S\sin(\sqrt{A\gamma}\xi) + H\cos\sqrt{A\gamma}\xi)}\right]\right\}^{-\frac{1}{m}}, \tag{43}$$

where $R = \frac{1}{2(\omega^2 - a)}$, $\quad \xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x - \omega t)$.

$$u_{1(3)}(x,t) = \left\{\frac{1}{2(\omega^2-a)} + \frac{1}{2(\omega^2-a)} \times \left[1 - \frac{2C}{H\cosh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) - H\sinh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) + C}\right]\right\}^{-\frac{1}{m}}, \quad (44)$$

where $\xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

$$u_{1(4)}(x,t) = \left\{\frac{1}{2(\omega^2-a)} + \frac{\sqrt{-\frac{\gamma}{A}}}{2(\omega^2-a)} \times \left[C - \frac{(A+C^2\gamma)\sqrt{-\frac{\gamma}{A}}\sinh(A\sqrt{-\frac{\gamma}{A}}\xi)}{\gamma\left[C\sqrt{-\frac{\gamma}{A}}\sinh(A\sqrt{-\frac{\gamma}{A}}\xi) + \cosh(A\sqrt{-\frac{\gamma}{A}}\xi)\right]}\right]\right\}^{-\frac{1}{m}}, \quad (45)$$

where $\xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

$$u_{1(5)}(x,t) = \left\{\frac{1}{2(\omega^2-a)} + \frac{1}{2(\omega^2-a)}\left[\frac{\sinh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{\cosh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm 1}\right]\right\}^{-\frac{1}{m}}, \quad (46)$$

where $\xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

$$u_{1(6)}(x,t) = \left\{\frac{1}{2(\omega^2-a)} + i\frac{1}{2(\omega^2-a)}\left[\frac{\sin(2\gamma\sqrt{\frac{A}{\gamma}}\xi)}{\cos(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \pm 1}\right]\right\}^{-\frac{1}{m}}, \quad (47)$$

where $\xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

**Family 2**

$$u_{2(1)}(x,t) = \left\{R - R\left[\frac{H\sinh(\sqrt{-A\gamma}\xi) + S\cosh\sqrt{-A\gamma}\xi)}{S\sinh(\sqrt{-A\gamma}\xi) + H\cosh\sqrt{-A\gamma}\xi)}\right]^2\right\}^{-\frac{1}{m}}, \quad (48)$$

where $R = \frac{1}{2(\omega^2-a)}$, $\quad \xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

$$u_{2(2)}(x,t)=\left\{R+R\left[\frac{H\sin(\sqrt{A\gamma}\xi)-S\cos\sqrt{A\gamma}\xi)}{S\sin(\sqrt{A\gamma}\xi)+H\cos\sqrt{A\gamma}\xi)}\right]^2\right\}^{-\frac{1}{m}}, \qquad (49)$$

where $R=\frac{1}{2(\omega^2-a)}$, $\quad \xi=\frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

$$\begin{aligned}u_{2(3)}(x,t) &= \left\{\frac{1}{2(\omega^2-a)}-\frac{1}{2(\omega^2-a)}\right.\\ &\quad\left.\times\left[1-\frac{2C}{H\cosh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)-H\sinh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)+C}\right]^2\right\}^{-\frac{1}{m}},\end{aligned} \qquad (50)$$

where $\xi=\frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

$$\begin{aligned}u_{2(4)}(x,t) &= \left\{\frac{1}{2(\omega^2-a)}+\frac{\gamma}{2A(\omega^2-a)}\right.\\ &\quad\left.\times\left[C-\frac{(A+C^2\gamma)\sqrt{-\frac{\gamma}{A}}\sinh(A\sqrt{-\frac{\gamma}{A}}\xi)}{\gamma\left[C\sqrt{-\frac{\gamma}{A}}\sinh(A\sqrt{-\frac{\gamma}{A}}\xi)+\cosh(A\sqrt{-\frac{\gamma}{A}}\xi)\right]}\right]^2\right\}^{-\frac{1}{m}},\end{aligned} \qquad (51)$$

where $\xi=\frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

$$u_{2(5)}(x,t)=\left\{\frac{1}{2(\omega^2-a)}-\frac{1}{2(\omega^2-a)}\left[\frac{\sinh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{\cosh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\pm 1}\right]^2\right\}^{-\frac{1}{m}}, \qquad (52)$$

where $\xi=\frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

$$u_{2(6)}(x,t)=\left\{\frac{1}{2(\omega^2-a)}+\frac{1}{2(\omega^2-a)}\left[\frac{\sin(2\gamma\sqrt{\frac{A}{\gamma}}\xi)}{\cos(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\pm 1}\right]^2\right\}^{-\frac{1}{m}}, \qquad (53)$$

where $\xi=\frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

**Family 3**

$$u_3(x,t) = \left\{ \frac{1}{2(\omega^2-a)} + \frac{\sqrt{-\frac{\gamma}{A}}}{2(\omega^2-a)} \times \left[ \frac{-\sqrt{-\frac{A}{\gamma}}b_1\exp(\gamma\sqrt{-\frac{A}{\gamma}}\xi)+a_{-1}\exp(-\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{b_1\exp(\gamma\sqrt{-\frac{A}{\gamma}}\xi)+\frac{a_{-1}}{\sqrt{-\frac{A}{\gamma}}}\exp(-\gamma\sqrt{-\frac{A}{\gamma}}\xi)} \right] \right\}^{-\frac{1}{m}}, \tag{54}$$

where $\xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

If we set $b_1 = 1,\ a_{-1} = \pm\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (54), we obtain

$$u_{3(1)}(x,t) = \left\{ \frac{1}{2(\omega^2-a)} - \frac{1}{2(\omega^2-a)}\tanh(\gamma\sqrt{-\frac{A}{\gamma}}\xi) \right\}^{-\frac{1}{m}}, \tag{55}$$

and

$$u_{3(2)}(x,t) = \left\{ \frac{1}{2(\omega^2-a)} - \frac{1}{2(\omega^2-a)}\coth(\gamma\sqrt{-\frac{A}{\gamma}}\xi) \right\}^{-\frac{1}{m}}. \tag{56}$$

Setting $b_1 = i,\ a_{-1} = \mp\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (54), we get

$$u_{3(3)}(x,t) = \left\{ \frac{1}{2(\omega^2-a)} + i\frac{1}{2(\omega^2-a)}\tan(\gamma\sqrt{\frac{A}{\gamma}}\xi) \right\}^{-\frac{1}{m}}, \tag{57}$$

and

$$u_{3(4)}(x,t) = \left\{ \frac{1}{2(\omega^2-a)} - i\frac{1}{2(\omega^2-a)}\cot(\gamma\sqrt{\frac{A}{\gamma}}\xi) \right\}^{-\frac{1}{m}}. \tag{58}$$

**Family 4**

$$u_4(x,t) = \left\{ \frac{1}{2(\omega^2-a)} + \frac{\sqrt{-\frac{A}{\gamma}}}{2(\omega^2-a)} \times \left[ \frac{b_1\exp(\gamma\sqrt{-\frac{A}{\gamma}}\xi)+\frac{a_{-1}}{\sqrt{-\frac{A}{\gamma}}}\exp(-\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{-\sqrt{-\frac{A}{\gamma}}b_1\exp(\gamma\sqrt{-\frac{A}{\gamma}}\xi)+a_{-1}\exp(-\gamma\sqrt{-\frac{A}{\gamma}}\xi)} \right] \right\}^{-\frac{1}{m}}, \tag{59}$$

where $\xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

**Family 5**

$$\begin{aligned} u_5(x,t) &= \left\{ 2R_1 + \frac{R_1}{\sqrt{-\frac{A}{\gamma}}} \left[ \frac{-\sqrt{-\frac{A}{\gamma}}b_1\exp(\eta) + a_{-1}\exp(-\eta)}{b_1\exp(\eta) + \frac{a_{-1}}{\sqrt{-\frac{A}{\gamma}}}\exp(-\eta)} \right] \right. \\ &\left. + R_1\sqrt{-\frac{A}{\gamma}} \left[ \frac{b_1\exp(\eta) + \frac{a_{-1}}{\sqrt{-\frac{A}{\gamma}}}\exp(-\eta)}{-\sqrt{-\frac{A}{\gamma}}b_1\exp(\eta) + a_{-1}\exp(-\eta)} \right] \right\}^{-\frac{1}{m}}, \end{aligned} \tag{60}$$

where $\eta = \gamma\sqrt{-\frac{A}{\gamma}}\xi$, $\quad \xi = \frac{1}{4}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$, $\quad R_1 = \frac{1}{4(\omega^2-a)}$.

If we set $b_1 = 1,\ a_{-1} = \pm\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (60), we obtain

$$\begin{aligned} u_{5(1)}(x,t) &= \left\{ \frac{1}{2(\omega^2-a)} - \frac{1}{4(\omega^2-a)}\tanh(\gamma\sqrt{-\frac{A}{\gamma}}\xi) \right. \\ &\left. - \frac{1}{4(\omega^2-a)}\coth(\gamma\sqrt{-\frac{A}{\gamma}}\xi) \right\}^{-\frac{1}{m}}. \end{aligned} \tag{61}$$

Setting $b_1 = i,\ a_{-1} = \mp\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (60), we get

$$\begin{aligned} u_{5(2)}(x,t) &= \left\{ \frac{1}{2(\omega^2-a)} - i\frac{1}{4(\omega^2-a)}\tan(\gamma\sqrt{\frac{A}{\gamma}}\xi) + i\frac{1}{4(\omega^2-a)} \right. \\ &\left. \times \cot(\gamma\sqrt{\frac{A}{\gamma}}\xi) \right\}^{-\frac{1}{m}}. \end{aligned} \tag{62}$$

**Family 6**

$$\begin{aligned} u_6(x,t) &= \left\{ \frac{1}{2(\omega^2-a)} + \frac{\sqrt{-\frac{\gamma}{A}}}{2(\omega^2-a)} \right. \\ &\left. \times \left[ \frac{\frac{(\gamma a_0^2 + Ab_0^2)}{4\gamma\sqrt{-\frac{A}{\gamma}}b_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) + a_0 + \sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{\frac{(\gamma a_0^2 + Ab_0^2)}{4Ab_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) + b_0 + b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)} \right] \right\}^{-\frac{1}{m}}, \end{aligned} \tag{63}$$

where $\xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x-\omega t)$.

If we set $b_0 = 0$, $b_{-1} = 1$, $a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (63), we obtain

$$\begin{aligned} u_{6(1)}(x,t) &= \left\{ \frac{1}{2(\omega^2 - a)} - \frac{1}{2(\omega^2 - a)} \right. \\ &\quad \left. \times \left[ \coth(2\gamma\sqrt{-\tfrac{A}{\gamma}}\xi) \pm \operatorname{csch}(2\gamma\sqrt{-\tfrac{A}{\gamma}}\xi \right] \right\}^{-\frac{1}{m}}. \end{aligned} \tag{64}$$

Setting $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (63), we get

$$\begin{aligned} u_{6(2)}(x,t) &= \left\{ \frac{1}{2(\omega^2 - a)} - \frac{1}{2(\omega^2 - a)} \right. \\ &\quad \left. \times \left[ \tanh(2\gamma\sqrt{-\tfrac{A}{\gamma}}\xi) \pm i\operatorname{sech}(2\gamma\sqrt{-\tfrac{A}{\gamma}}\xi) \right] \right\}^{-\frac{1}{m}}. \end{aligned} \tag{65}$$

Setting $b_0 = 0$, $b_{-1} = 1$, $a_0 = \pm 2\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (63), we have

$$\begin{aligned} u_{6(3)}(x,t) &= \left\{ \frac{1}{2(\omega^2 - a)} + i\frac{1}{2(\omega^2 - a)} \right. \\ &\quad \left. \times \left[ \tan(2\gamma\sqrt{\tfrac{A}{\gamma}}\xi) \pm \sec(2\gamma\sqrt{\tfrac{A}{\gamma}}\xi) \right] \right\}^{-\frac{1}{m}}. \end{aligned} \tag{66}$$

Setting $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (63), we have

$$\begin{aligned} u_{6(4)}(x,t) &= \left\{ \frac{1}{2(\omega^2 - a)} - i\frac{1}{2(\omega^2 - a)} \right. \\ &\quad \left. \times \left[ \cot(2\gamma\sqrt{\tfrac{A}{\gamma}}\xi) \mp \csc(2\gamma\sqrt{\tfrac{A}{\gamma}}\xi) \right] \right\}^{-\frac{1}{m}}. \end{aligned} \tag{67}$$

**Family 7**

$$\begin{aligned} u_7(x,t) &= \left\{ \frac{1}{2(\omega^2 - a)} + \frac{\sqrt{-\frac{A}{\gamma}}}{2(\omega^2 - a)} \right. \\ &\quad \left. \times \left[ \frac{\frac{(\gamma a_0^2 + A b_0^2)}{4Ab_{-1}} \exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) + b_0 + b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{\frac{(\gamma a_0^2 + A b_0^2)}{4\gamma\sqrt{-\frac{A}{\gamma}}b_{-1}} \exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) + a_0 + \sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)} \right] \right\}^{-\frac{1}{m}}, \end{aligned} \tag{68}$$

where $\xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x - \omega t)$.

If we set $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (68), we obtain

$$u_{7(1)}(x,t) = \left\{ \frac{1}{2(\omega^2 - a)} - \frac{1}{2(\omega^2 - a)} \times \frac{1}{\left[\tanh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm i \operatorname{sech}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\right]} \right\}^{-\frac{1}{m}}. \tag{69}$$

Setting $b_0 = 0$, $b_{-1} = 1$, $a_0 = \pm 2\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (68), we get

$$u_{7(2)}(x,t) = \left\{ \frac{1}{2(\omega^2 - a)} + i \frac{1}{2(\omega^2 - a)} \times \frac{1}{\left[\tan(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \pm \sec(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\right]} \right\}^{-\frac{1}{m}}. \tag{70}$$

**Family 8**

$$u_8(x,t) = \left\{ 2R_1 + \frac{R_1}{\sqrt{-\frac{A}{\gamma}}} \left[ \frac{B\exp(\eta) + a_0 + \sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-\eta)}{Q\exp(\eta) + b_0 + b_{-1}\exp(-\eta)} \right] + R_1\sqrt{-\frac{A}{\gamma}} \left[ \frac{Q\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}{B\exp(\eta) + a_0 + \sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-\eta)} \right] \right\}^{-\frac{1}{m}}, \tag{71}$$

where $B = \frac{(\gamma a_0^2 + Ab_0^2)}{4\gamma\sqrt{-\frac{A}{\gamma}}b_{-1}}$, $\quad Q = \frac{(\gamma a_0^2 + Ab_0^2)}{4Ab_{-1}}$, $\quad \eta = 2\gamma\sqrt{-\frac{A}{\gamma}}\xi$,
$\xi = \frac{1}{4}\sqrt{\frac{1}{bA\gamma}}(x - \omega t)$, $\quad R_1 = \frac{1}{4(\omega^2 - a)}$.

If we set $b_0 = 0$, $b_{-1} = 1$, $a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (71), we obtain

$$u_{8(1)}(x,t) = \left\{ 2R_1 - R_1\left[\coth(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm \operatorname{csch}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\right] - R_1 \frac{1}{\left[\coth(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm \operatorname{csch}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\right]} \right\}^{-\frac{1}{m}}. \tag{72}$$

Setting $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (71), we get

$$u_{8(2)}(x,t) = \Bigg\{2R_1 - R_1\left[\tanh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm i\operatorname{sech}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\right] - R_1\frac{1}{\left[\tanh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\pm i\operatorname{sech}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\right]}\Bigg\}^{-\frac{1}{m}}. \tag{73}$$

Setting $b_0 = 0$, $b_{-1} = 1$, $a_0 = \pm 2\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (71), we have

$$u_{8(3)}(x,t) = \Bigg\{2R_1 - iR_1\left[\tan(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \pm \sec(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\right] + iR_1\frac{1}{\left[\tan(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\pm \sec(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\right]}\Bigg\}^{-\frac{1}{m}}. \tag{74}$$

Setting $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (71), we have

$$u_{8(4)}(x,t) = \Bigg\{2R_1 + iR_1\left[\cot(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \mp \csc(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\right] - iR_1\frac{1}{\left[\cot(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\mp \csc(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\right]}\Bigg\}^{-\frac{1}{m}}, \tag{75}$$

**Family 9**

$$u_9(x,t) = \Bigg\{\frac{1}{2(\omega^2 - a)} + \frac{\gamma}{2A(\omega^2 - a)} \times \left[\frac{-\sqrt{-\frac{A}{\gamma}}b_1\exp(\gamma\sqrt{-\frac{A}{\gamma}}\xi)+a_{-1}\exp(-\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{b_1\exp(\gamma\sqrt{-\frac{A}{\gamma}}\xi)+\frac{a_{-1}}{\sqrt{-\frac{A}{\gamma}}}\exp(-\gamma\sqrt{-\frac{A}{\gamma}}\xi)}\right]^2\Bigg\}^{-\frac{1}{m}}, \tag{76}$$

where $\xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x - \omega t)$.

If we set $b_1 = 1,\ a_{-1} = \pm\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (76), we obtain

$$u_{9(1)}(x,t) = \left\{\frac{1}{2(\omega^2 - a)}\operatorname{sech}^2(\gamma\sqrt{-\frac{A}{\gamma}}\xi)\right\}^{-\frac{1}{m}}, \tag{77}$$

and

$$u_{9(2)}(x,t) = \left\{ -\frac{1}{2(\omega^2 - a)} \operatorname{csch}^2(\gamma\sqrt{-\frac{A}{\gamma}}\xi) \right\}^{-\frac{1}{m}}. \tag{78}$$

Setting $b_1 = i,\ a_{-1} = \mp\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (76), we get

$$u_{9(3)}(x,t) = \left\{ \frac{1}{2(\omega^2 - a)} \sec^2(\gamma\sqrt{\frac{A}{\gamma}}\xi) \right\}^{-\frac{1}{m}}, \tag{79}$$

and

$$u_{9(4)}(x,t) = \left\{ \frac{1}{2(\omega^2 - a)} \csc^2(\gamma\sqrt{\frac{A}{\gamma}}\xi) \right\}^{-\frac{1}{m}}. \tag{80}$$

**Family 10**

$$\begin{aligned} u_{10}(x,t) \ = \ & \left\{ \frac{1}{2(\omega^2 - a)} + \frac{A}{2\gamma(\omega^2 - a)} \right. \\ & \left. \times \left[ \frac{b_1 \exp(\gamma\sqrt{-\frac{A}{\gamma}}\xi) + \frac{a_{-1}}{\sqrt{-\frac{A}{\gamma}}} \exp(-\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{-\sqrt{-\frac{A}{\gamma}} b_1 \exp(\gamma\sqrt{-\frac{A}{\gamma}}\xi) + a_{-1}\exp(-\gamma\sqrt{-\frac{A}{\gamma}}\xi)} \right]^2 \right\}^{-\frac{1}{m}}, \end{aligned} \tag{81}$$

where $\xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x - \omega t)$.

**Family 11**

$$\begin{aligned} u_{11}(x,t) \ = \ & \left\{ 2R_2 + \frac{R_2\gamma}{A} \left[ \frac{-\sqrt{-\frac{A}{\gamma}} b_1 \exp(\eta) + a_{-1}\exp(-\eta)}{b_1\exp(\eta) + \frac{a_{-1}}{\sqrt{-\frac{A}{\gamma}}}\exp(-\eta)} \right]^2 \right. \\ & \left. + \frac{R_2 A}{\gamma} \left[ \frac{b_1\exp(\eta) + \frac{a_{-1}}{\sqrt{-\frac{A}{\gamma}}}\exp(-\eta)}{-\sqrt{-\frac{A}{\gamma}} b_1\exp(\eta) + a_{-1}\exp(-\eta)} \right]^2 \right\}^{-\frac{1}{m}}, \end{aligned} \tag{82}$$

where $\eta = \gamma\sqrt{-\frac{A}{\gamma}}\xi, \qquad \xi = \frac{1}{4}\sqrt{\frac{1}{bA\gamma}}(x - \omega t), \qquad R_2 = \frac{1}{8(\omega^2 - a)}$.

If we set $b_1 = 1,\ a_{-1} = \pm\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (82), we obtain

$$u_{11(1)}(x,t) = \left\{ 2R_2 - R_2 \tanh^2(\gamma\sqrt{-\frac{A}{\gamma}}\xi) - R_2 \coth^2(\gamma\sqrt{-\frac{A}{\gamma}}\xi) \right\}^{-\frac{1}{m}}. \tag{83}$$

Setting $b_1 = i,\ a_{-1} = \mp\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (82), we get

$$u_{11(2)}(x,t) = \left\{2R_2 + R_2\tan^2(\gamma\sqrt{\frac{A}{\gamma}}\xi) + R_2\cot^2(\gamma\sqrt{\frac{A}{\gamma}}\xi)\right\}^{-\frac{1}{m}}. \tag{84}$$

**Family 12**

$$\begin{aligned} u_{12}(x,t) &= \left\{R + \frac{R\gamma}{A}\right. \\ &\times \left.\left[\frac{\frac{(\gamma a_0^2+Ab_0^2)}{4\gamma\sqrt{-\frac{A}{\gamma}}b_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)+a_0+\sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{\frac{(\gamma a_0^2+Ab_0^2)}{4Ab_{-1}}\exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)+b_0+b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}\right]^2\right\}^{-\frac{1}{m}}, \end{aligned} \tag{85}$$

where $\xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x - \omega t)$, $\qquad R = \frac{1}{2(\omega^2 - a)}$.

If we set $b_0 = 0,\ b_{-1} = 1,\ a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (85), we obtain

$$\begin{aligned} u_{12(1)}(x,t) &= \left\{R - R\left[\coth(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm \operatorname{csch}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\right]^2\right\}^{-\frac{1}{m}} \\ &= \left\{\frac{1}{(\omega^2 - a)\left[1 \mp \cosh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\right]}\right\}^{-\frac{1}{m}}. \end{aligned} \tag{86}$$

Setting $b_0 = 0,\ b_{-1} = i,\ a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (85), we get

$$u_{12(2)}(x,t) = \left\{R - R\left[\tanh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm i\operatorname{sech}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\right]^2\right\}^{-\frac{1}{m}}. \tag{87}$$

Setting $b_0 = 0,\ b_{-1} = 1,\ a_0 = \pm 2\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (85), we have

$$u_{12(3)}(x,t) = \left\{R + R\left[\tan(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \pm \sec(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\right]^2\right\}^{-\frac{1}{m}}. \tag{88}$$

Setting $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (85), we have

$$u_{12(4)}(x,t) = \left\{ R + R\left[\cot(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \mp \csc(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\right]^2 \right\}^{-\frac{1}{m}}. \tag{89}$$

**Family 13**

$$\begin{aligned} u_{13}(x,t) &= \left\{ R + \frac{RA}{\gamma} \right. \\ &\times \left. \left[ \frac{\frac{(\gamma a_0^2 + A b_0^2)}{4Ab_{-1}} \exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) + b_0 + b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)}{\frac{(\gamma a_0^2 + A b_0^2)}{4\gamma\sqrt{-\frac{A}{\gamma}}b_{-1}} \exp(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) + a_0 + \sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-2\gamma\sqrt{-\frac{A}{\gamma}}\xi)} \right]^2 \right\}^{-\frac{1}{m}}, \end{aligned} \tag{90}$$

where $\xi = \frac{1}{2}\sqrt{\frac{1}{bA\gamma}}(x - \omega t)$, $\qquad R = \frac{1}{2(\omega^2 - a)}$.

If we set $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (90), we obtain

$$u_{13(1)}(x,t) = \left\{ R - R\frac{1}{\left[\tanh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm i\,\mathrm{sech}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi)\right]^2} \right\}^{-\frac{1}{m}}. \tag{91}$$

Setting $b_0 = 0$, $b_{-1} = 1$, $a_0 = \pm 2\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (90), we get

$$u_{13(2)}(x,t) = \left\{ R + R\frac{1}{\left[\tan(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \pm \sec(2\gamma\sqrt{\frac{A}{\gamma}}\xi)\right]^2} \right\}^{-\frac{1}{m}}. \tag{92}$$

**Family 14**

$$\begin{aligned} u_{14}(x,t) &= \left\{ 2R_2 + \frac{R_2\gamma}{A}\left[ \frac{B\exp(\eta) + a_0 + \sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-\eta)}{Q\exp(\eta) + b_0 + b_{-1}\exp(-\eta)} \right]^2 \right. \\ &\left. + \frac{R_2 A}{\gamma}\left[ \frac{Q\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}{B\exp(\eta) + a_0 + \sqrt{-\frac{A}{\gamma}}b_{-1}\exp(-\eta)} \right]^2 \right\}^{-\frac{1}{m}}, \end{aligned} \tag{93}$$

where $B = \frac{(\gamma a_0^2 + Ab_0^2)}{4\gamma\sqrt{-\frac{A}{\gamma}}b_{-1}}$, $Q = \frac{(\gamma a_0^2 + Ab_0^2)}{4Ab_{-1}}$, $\eta = 2\gamma\sqrt{-\frac{A}{\gamma}}\xi$,
$\xi = \frac{1}{4}\sqrt{\frac{1}{bA\gamma}}(x - \omega t)$, $R_2 = \frac{1}{8(\omega^2 - a)}$.

If we set $b_0 = 0$, $b_{-1} = 1$, $a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (93), we obtain

$$\begin{aligned} u_{14(1)}(x,t) &= \Bigg\{ 2R_2 - R_2 \left[ \coth(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm \operatorname{csch}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \right]^2 \\ &\quad - R_2 \frac{1}{\left[ \coth(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm \operatorname{csch}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \right]^2} \Bigg\}^{-\frac{1}{m}} \end{aligned} \tag{94}$$

$$= \left\{ \frac{1}{2(\omega^2 - a)\left[ 1 - \cosh^2(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \right]} \right\}^{-\frac{1}{m}}. \tag{95}$$

Setting $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{-\frac{A}{\gamma}}$, and $\frac{A}{\gamma} < 0$ in equation (93), we get

$$\begin{aligned} u_{14(2)}(x,t) &= \Bigg\{ 2R_2 - R_2 \left[ \tanh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm i\operatorname{sech}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \right]^2 \\ &\quad - R_2 \frac{1}{\left[ \tanh(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \pm i\operatorname{sech}(2\gamma\sqrt{-\frac{A}{\gamma}}\xi) \right]^2} \Bigg\}^{-\frac{1}{m}}. \end{aligned} \tag{96}$$

Setting $b_0 = 0$, $b_{-1} = 1$, $a_0 = \pm 2\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (93), we have

$$\begin{aligned} u_{14(3)}(x,t) &= \Bigg\{ 2R_2 + R_2 \left[ \tan(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \pm \sec(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \right]^2 \\ &\quad + R_2 \frac{1}{\left[ \tan(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \pm \sec(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \right]^2} \Bigg\}^{-\frac{1}{m}}, \end{aligned} \tag{97}$$

Setting $b_0 = 0$, $b_{-1} = i$, $a_0 = \pm 2\sqrt{\frac{A}{\gamma}}$, and $\frac{A}{\gamma} > 0$ in equation (93), we have

$$\begin{aligned} u_{14(4)}(x,t) &= \Bigg\{ 2R_2 + R_2 \left[ \cot(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \mp \csc(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \right]^2 \\ &\quad + R_2 \frac{1}{\left[ \cot(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \mp \csc(2\gamma\sqrt{\frac{A}{\gamma}}\xi) \right]^2} \Bigg\}^{-\frac{1}{m}}. \end{aligned} \tag{98}$$

## 4. Conclusions

In this work, we studied a family of Boussinesq equation with nonlinear dispersion (1). The Riccati equation mapping method is used to carry out this work. Some new exact solutions including the solitary wave solutions and the periodic wave solutions are obtained. The majority of these results are very different to those in Ref. [11]. It is worth while to mention that this method is reliable and effective in solving nonlinear partial differential equations. The applied method will be used in further works to establish more entirely new solutions for other kinds of nonlinear partial differential equations.

## References

[1] A.M. Wazwaz, The tanh-coth method for new compactons and solitons solutions for the $K(n,n)$ and the $K(n+1,n+1)$ equations, *Appl. Math. Comput.*, **188** (2007), 1930-1940.

[2] H.Q. Zhang, Extended Jacobi elliptic function expansion method and its applications, *Commun Nonlinear Sci Numer Simulat*, **12** (2007), 627-735.

[3] M.A. Abdou, The extended F-expansion method and its application for a class of nonlinear evolution equations, *Chaos, Solitons and Fractals*, **31** (2007), 95-104.

[4] J.H. He, M.A. Abdou, New periodic solutions for nonlinear evolution equations using Exp-function method, *Chaos, Solitons and Fractals*, **34** (2007), 1421-1429.

[5] S. Zhang, Application of Exp-function method to a KdV equation with variable, *Physics Letters A*, **365** (2007), 448-453.

[6] S. Zhang, Exp-function method for solving Maccari's system, *Physics Letters A*, **371**, No-s: 1-2 (2007), 65-71.

[7] R. Kumar, R.S. Kaushal, A. Prasad, Some new solitary and travelling wave solutions of certain nonlinear diffusion-reaction equations using auxiliary equation method, *Physics Letters A*, **372** (2008), 3395-9.

[8] Y.L. Ma, B.Q. Li, A series of abundant exact travelling wave solutions for a modified generalized Vakhnenko equation using auxiliary equation method, *Appl. Math. Comput.*, **211** (2009), 102-107.

[9] W. Li, Y.M. Zhao, Y.M. Ding, Some new exact solutions of Exp-function type to Boussinesq equations with power law nonlinearity in two forms, *J. Math. Sci. Res.*, **14**, No. 2 (2010), 33-44.

[10] W. Li, Y.M. Zhao, Y.M. Ding, Application of exp-function method combined with F-expansion method for solving a generalized form of modified Degasperis-Procesi equation, *J. Math. Sci. Res.*, **14**, No. 8 (2010), 169-181.

[11] M.S. Bruzon, M.L. Gandarias, Symmetries for a family of Boussinesq equations with nonlinear dispersion, *Commun Nonlinear Sci Numer Simulat*, **14** (2009), 3250-3257.