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# ON THE HYPERCYCLICITY FOR A TUPLE OF OPERATORS

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**Abstract:** In this paper we characterize the equivalent conditions for a tuple of commutative bounded linear operators, satisfying the hypercyclicity criterion.

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**Key Words:** tuple, hypercyclic vector, hypercylicity criterion, hereditarily hypercyclicity

## 1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space $X$.

**Definition 1.1.** Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space $X$. We will let $\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}, ..., {T_n}^{k_n} : k_i \in \mathbb{Z}_+,\ \ i = 1, ..., n\}$ be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of $x$ under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector $x$ is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in $X$ and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, by $\mathcal{T}_d^{(k)}$ we will refer to the set of all $k$ copies of an element of $\mathcal{F}$, i.e. $\mathcal{T}_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}\}$. We say that $\mathcal{T}_d^{(k)}$ is hypercyclic provided there exist $x_1, ..., x_k \in X$ such that $\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\}$ is dense in the $k$ copies of $X$, $X \oplus ... \oplus X$.

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For simplicity we state and prove our results for a pair that is a tuple with $n = 2$, and the general case follows by a similar method. Note that if $T_1$, $T_2$ are commutative bounded linear operators on a Banach space $X$, and $\{m_j\}$, $\{n_j\}$ are two sequences of natural numbers, then we say $\{{T_1}^{m_j}{T_2}^{n_j} : j \geq 0\}$ is hypercyclic if there exists $x \in X$ such that $\{{T_1}^{m_j}{T_2}^{n_j}x : j \geq 0\}$ is dense in $X$.

**Definition 1.2.** We say that the pair $\mathcal{T} = (T_1, T_2)$ is topologically transitive if for every nonempty open subsets $U$ and $V$ of $X$ there exists $S \in \mathcal{F}$ such that $S(U) \cap V \neq \emptyset$.

**Definition 1.3.** We say that a pair $\mathcal{T} = (T_1, T_2)$ is hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences $(\{m_k\}, \{n_k\})$ of integers provided for all pair of subsequences $(\{m_{k_j}\}, \{n_{k_j}\})$ of $(\{m_k\}, \{n_k\})$, the sequence $\{{T_1}^{m_{k_j}}{T_2}^{n_{k_j}} : j \geq 1\}$ is hypercyclic. We say that a pair $\mathcal{T}$ is hereditarily hypercyclic, if it is hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences.

The formulation of the Hypercyclicty Criterion in the next section was given by N. S. Feldman ([4]). Here, we want to extend some properties of hypercyclic operators to a pair of commuting operators, and although the techniques work for any n-tuple of operators but for simplicity we prove our results only for the case $n = 2$. For some other topics we refer to [1–16].

## 2. Main Results

In this section we characterize the equivalent conditions for a pair of operators, satisfying the Hypercyclicity Criterion.

**Theorem 2.1.** (The Hypercyclicity Criterion for a Tuple) *Suppose $X$ is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ is a pair of continuous linear mappings on $X$. If there exist two dense subsets $Y$ and $Z$ in $X$, and a pair of strictly increasing sequences $\{m_j\}$ and $\{n_j\}$ such that: 1. $T_1^{m_j}T_2^{n_j} \to 0$ on $Y$ as $j \to \infty$,*

*2. There exists a sequence of function $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_j}T_2^{n_j}S_j z \to z$, then $\mathcal{T}$ is a hypercyclic tuple.*

**Lemma 2.2.** (see [18]) *Let $X$ be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ be the pair of operators $T_1$ and $T_2$. Then the followings are equivalent:*

*(i) $\mathcal{T}$ is hypercyclic.*

*(ii) for all nonempty open subsets $U$, $V$ in $X$, there exists a pair of sequences $(\{m_k\}, \{n_k\})$ of integers such that $T_1^{m_k} {T_2}^{n_k}(U) \cap V \neq \emptyset$ for all $k \geq 0$.*

*(iii) $\mathcal{T}$ is topologically transitive.*

**Theorem 2.3.** (see [18]) *Let $\mathcal{T} = (T_1, T_2)$ be a pair of operators acting on a separable infinite dimensional Banach space $X$. Then the followings are equivalent:*

*(i) $\mathcal{T}$ satisfies the Hypercyclicity Criterion.*

*(ii) $\mathcal{T}$ is hereditarily hypercyclic.*

*(iii) $\mathcal{T}_d^{(2)}$ is hypercyclic.*

**Lemma 2.4.** *Let $X$ be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ be the pair of operators $T_1$ and $T_2$. The followings are equivalent:*

*(i) for every pair of nonnegative sequence of integers $(\{m_k\}, \{n_k\})$ with $m_{k+1} - m_k \leq 2$ and $n_{k+1} - n_k \leq 2$ for every $k$, the sequence $\{{T_1}^{m_k} {T_2}^{n_k} : k \geq 0\}$ is hypercyclic.*

*(ii) for every nonempty open subsets $U$, $V$ of $X$, there exists a pair of integers $(m, n)$ such that $T_1^m T_2^n(U) \cap V \neq \emptyset$ and $T_1^{m+1} T_2^{n+1}(U) \cap V \neq \emptyset$.*

*Proof.* (i) $\to$ (ii): Suppose (ii) is not true and let $U$ and $V$ be such that $T_1^m T_2^n(U) \cap V$ and $T_1^{m+1} T_2^{n+1}(U) \cap V$ are never simultaneously nonempty, i.e., $T_1^m T_2^n(U) \cap V \neq \emptyset$ implies that $T_1^{m+1} T_2^{n+1}(U) \cap V = \emptyset$. Let $\{m_k\}$ and $\{n_k\}$ be respectively the sequence of integers $m$ and $n$ such that $T_1^m T_2^n(U) \cap V = \emptyset$. If the pair $(m, n)$ does not belong to the pair of sequences $(\{m_k\}, \{n_k\})$, then $T_1^m T_2^n(U) \cap V \neq \emptyset$ and so $T_1^{m+1} T_2^{n+1}(U) \cap V = \emptyset$. Hence $(m+1, n+1)$ belongs to $(\{m_k\}, \{n_k\})$ and this proves that $m_{k+1} - m_k \leq 2$ and $n_{k+1} - n_k \leq 2$ for every $k$. But $T_1^{m_k} T_2^{n_k}(U) \cap V = \emptyset$ for every $k$ and so $\{T_1^{m_k} T_2^{n_k}; k \geq 0\}$ can not be hypercyclic, that is a contradiction.

(ii) $\to$ (i): If (i) is false, let $\{m_k\}$ and $\{n_k\}$ be such that $m_{k+1} - m_k \leq 2$ and $n_{k+1} - n_k \leq 2$ for every $k$ and let $\{T_1^{m_k} T_2^{n_k}; k \geq 0\}$ is not hypercyclic. By Lemma 2.2, there exist $U$ and $V$ such that $T_1^{m_k} T_2^{n_k}(U) \cap V = \emptyset$ for all $k$. Now suppose that $T_1^m T_2^n(U) \cap V = \emptyset$ where the pair $m, n$ do not belong to $\{m_k\}, \{n_k\}$ respectively. So we have $(m+1, n+1) \in (\{m_k\}, \{n_k\})$ and $T_1^{m+1} T_2^{n+1}(U) \cap V = \emptyset$. Thus (ii) cannot be consistent. □

**Proposition 2.5.** *Let $\mathcal{T}$ be a pair of operators $T_1$ and $T_2$ on an infinite dimensional Banach space $X$. If $\mathcal{T}$ satisfies the Hypercyclicity Criterion, then $\mathcal{T}_d^{(4k)}$ is hypercyclic for all $k \geq 1$. The converse is also true.*

*Proof.* Suppose that $4k = MN$ where $M, N \geq 2$. Choose the sequences $\{m_k\}$ and $\{n_k\}$ of integers such that for every $k$, $m_{k+1} - m_k \leq M$ and $n_{k+1} - n_k \leq N$ with $M, N \geq 2$. Since $\mathcal{T}$ satisfies the Hypercyclicity Criterion, there exist sequences $\{p_k\}$ and $\{q_k\}$ such that $\mathcal{T}$ is hereditarily hypercyclic with respect to $(\{p_k\}, \{q_k\})$. Now consider the set $\{W_{ij} : \ i = 1, ..., M; \ j = 1, ..., N\}$ of nonempty open subsets of $X$. For $i = 1, ..., M$ and $j = 1, ..., N$, let $(U_{ij}, V_{ij})$ be pairs of sets belonging to the set $\{W_{ij} : \ i = 1, ..., M; \ j = 1, ..., N\}$. Then there exists a pair of subsequences $(\{p_{k_{t_1}}\}, \{q_{k_{t_1}}\})$ of $(\{p_k\}, \{q_k\})$ such that $T_1^{p_{k_{t_1}}} T_2^{q_{k_{t_1}}}(U_{11}) \cap V_{11} \neq \emptyset$ for every $t_1$. Now the sequence $\{T_1^{p_{k_{t_1}}} T_2^{q_{k_{t_1}}} : t_1 \geq 0\}$ is hypercyclic, since $\mathcal{T}$ is hereditarily hypercyclic with respect to $(\{p_k\}, \{q_k\})$. So, there exists a pair of subsequences $(\{p_{k_{t_2}}\}, \{q_{k_{t_2}}\})$ of $(\{p_{k_{t_1}}\}, \{q_{k_{t_1}}\})$ such that $T_1^{p_{k_{t_2}}} T_2^{q_{k_{t_2}}}(U_{12}) \cap V_{12} \neq \emptyset$ for every $t_2$. Continuing in this way we obtain a pair of subsequences $(\{p_{k_{t_N}}\}, \{q_{k_{t_N}}\})$ of $(\{p_{k_{t_{N-1}}}\}, \{q_{k_{t_{N-1}}}\})$ such that $T_1^{p_{k_{t_N}}} T_2^{q_{k_{t_N}}}(U_{1N}) \cap V_{1N} \neq \emptyset$ for every $t_N$. Now the sequence $\{T_1^{p_{k_{t_N}}} T_2^{q_{k_{t_N}}} : t_N \geq 0\}$ is hypercyclic, thus there exists a pair of subsequences $(\{p_{k_{t_{N+1}}}\}, \{q_{k_{t_{N+1}}}\})$ of $(\{p_{k_{t_N}}\}, \{q_{k_{t_N}}\})$ such that $T_1^{p_{k_{t_{N+1}}}} T_2^{q_{k_{t_{N+1}}}}(U_{21}) \cap V_{21} \neq \emptyset$ for every $t_{N+1}$. By continuing this manner after $MN$ steps we obtain a pair of sequences $(\{p_{k_{t_{MN}}}\}, \{q_{k_{t_{MN}}}\})$ such that for all $t_{MN}$, and $i = 1, ..., M$, $j = 1, ..., N$, we have $T_1^{p_{k_{t_{MN}}}} T_2^{q_{k_{t_{MN}}}}(U) \cap V \neq \emptyset$ for all $U, V \in \{W_{ij} : \ i = 1, ..., M; \ j = 1, ..., N\}$. This implies that indeed $\mathcal{T}_d^{(MN)}$ is hypercyclic. The converse follows from Theorem 2.3. □

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