International Journal of Pure and Applied Mathematics
Volume 76 No. 2 2012, 295-302
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

NON-UNIQUENESS OF CONFORMAL MAPPING AND POSSIBILITY OF A FIXED POINT MAPPING

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Abstract: Non-uniqueness of conformal mapping and possibility of a fixed point mapping is shown by concrete families of univalent analytic mapping functions with three real parameters, as long as univalent conformal mapping to a specified region from a semi-infinite region is given.

AMS Subject Classification: 30C20, 30C35
Key Words: conformal mapping, non-uniqueness, fixed point

1. Introduction

Conformal mapping under a boundary-fitted coordinate system is a very useful technique, especially to reduce to a standard configuration as a boundary value problem. Many aspects for it are given by P. Henrici[1], V.I. Ivanov and M.K. Trubetskov, see [2], P.K. Kythe [3], and R. Schinzinger, P.A.A. Laura [4]. In case of only specifying contour configuration, Schwarz−Christoffel transformation is a typical one from a semi-infinite half plane to a polygon, and many concrete mapping functions for various configurations are found in [5], although mapping is not unique as long as only contour configuration is specified a priori as pointed out by Y. Mochimaru, see [6].

Received: January 28, 2012

In the following, a family of conformal mapping with three real parameters is proposed to give a fixed point mapping and to show non-uniqueness of conformal mapping.

2. Analysis

2.1. General

Consider a conformal mapping from a circular region to a specified region, where any point on the circular contour corresponds to some point on the contour of the a-priori-specified region or on a newly produced apparent boundary (inside the region). The correspondence should be univalent except for points on the newly produced apparent boundary (zero-measure) if any. Proposed is the following: A family of univalent conformal mapping with three real parameters from a circular region in a complex plane $\xi, (|\xi| \leq 1)$ to a semi-infinite region in a complex plane $w, (\Im(w) \geq 0)$ is considered, where $|\xi| = 1$ is assumed to be mapped to $\Im(w) = 0, -\infty < \Re(w) < +\infty$. Next, let $f(w)$ be a complex analytic function of w defined in a semi-infinite plane ($\Im(w) \geq 0$) such that $z = f(w)$ prescribes a given target region in a complex plane univalently if $\Im(w) > 0$ and if $\Im(w) = 0, z = f(w)$ falls on the contour of the target region or on the newly produced apparent boundary in the target domain if any. Then successive mapping produces a prescribed mapping.

2.2. Specification of the Function $f(w)$

To show a possibility of non-uniqueness of the conformal mapping under only specification of the target contour and to show a possibility of existence of a fixed point conformal mapping, it is sufficient to give some concrete forms for $f(w)$. For a regular equilateral triangle (without an arbitrary constant multiplier and a shift factor, which may be complex $\neq 0$) a concrete form is given by

$$z = \int_{-\infty}^{w} \frac{dt}{(t+1)^{2/3}\, t^{2/3}\, (t-1)^{2/3}} \ , \ \Im(w) \geq 0 \ . \tag{1}$$

Hereafter powers stand for a principal value. If $|w| > 1$, equation (1) becomes

$$z = -\frac{1}{\Gamma(p_1)} \sum_{m=0}^{\infty} \frac{\Gamma(m+p_1)}{(2m+1)\Gamma(m+1)} w^{-2m-1} \ , \tag{2}$$

where $p_1 = 2/3$ and $\Gamma()$ is a Gamma function. If $|w| = 1$, equation (1) becomes

$$z = -\frac{\sqrt{\pi}\ \Gamma(1/3)}{2\ \Gamma(5/6)} + \int_0^{-i\ \mathrm{Ln}\ w} \frac{\mathrm{e}^{i(-\theta/3+\pi/6)}}{(2\sin\theta)^{2/3}}\ d\theta\ , \tag{3}$$

where Ln is a principal value of natural logarithm. If $|w| < 1$, equation (1) becomes

$$z = \frac{\sqrt{3\pi}\ \Gamma(1/3)}{2\ \Gamma(5/6)}\ i - \frac{1+\sqrt{3}\ i}{2\ \Gamma(p_1)}\ (w)^{1/3} \sum_{m=0}^{\infty} \frac{3}{6m+1}\ \frac{\Gamma(m+p_1)}{\Gamma(m+1)} w^{2m}\ . \tag{4}$$

For a rectangular equilateral triangle (without an arbitrary constant multiplier and a shift factor, which may be complex, $\neq 0$) a concrete form is

$$z = \int_{-\infty}^{w} \frac{dt}{(t+1)^{3/4} t^{1/2} (t-1)^{3/4}}, \Im(w) \geq 0\ . \tag{5}$$

If $|w| > 1$, equation (5) becomes

$$z = -\frac{1}{\Gamma(p_2)} \sum_{m=0}^{\infty} \frac{\Gamma(m+p_2)}{(2m+1)\Gamma(m+1)} w^{-2m-1}\ , \tag{6}$$

where $p_2 = 3/4$. If $|w| = 1$, equation (5) becomes

$$z = -\sqrt{2}\ K(1/\sqrt{2}\ ) + \int_0^{-i\ \mathrm{Ln}\ w} \frac{\mathrm{e}^{-i(\theta/4-\pi/8)}}{(2\sin\theta)^{3/4}} d\theta\ , \tag{7}$$

where $K()$ is a complete elliptic integral of the first kind, and especially

$$K(1/\sqrt{2}) = (\Gamma(1/4))^2/(4\sqrt{\pi}\ ).$$

If $|w| < 1$, equation (5) becomes

$$z = \sqrt{2}\ K(1/\sqrt{2}\ )\ i - \frac{1+i}{\Gamma(p_2)} \sqrt{2w} \sum_{m=0}^{\infty} \frac{\Gamma(m+p_2)}{(4m+1)\Gamma(m+1)} w^{2m}\ . \tag{8}$$

For a rectangle, the aspect ratio of which is $2K(k)/K(\sqrt{1-k^2})$, transformation is (without an arbitrary constant multiplier and a shift factor, which may be complex $\neq 0$)

$$z = -\int_0^{w} \frac{dt}{\sqrt{t+1}\sqrt{t-1}\sqrt{kt+1}\sqrt{kt-1}}\ , \tag{9}$$

where $\Im(w) \geq 0, 0 < k < 1$. If $|w| > 1/k$, equation (9) becomes

$$z = K(\sqrt{1-k^2}\ )\ i + \frac{1}{k}\sum_{m=0}^{\infty}\frac{w^{-2m-1}}{2m+1}\sum_{n=0}^{m}k^{-2n}\varphi(m-n)\varphi(n)\ , \qquad (10)$$

where $\varphi(N) \equiv (2N)!/(2^N N!)^2, (N:\ \text{integer} \geq 0)$. If $|w| = 1/k$, equation (9) becomes

$$z = K(k) + K(\sqrt{1-k^2})i + \int_0^{-\ i\ \mathrm{Ln}\ (kw)} \frac{\mathrm{e}^{i(-2\theta-3\pi)/4}}{\sqrt{2\sin\theta}\sqrt{1-k^2\mathrm{e}^{-2i\theta}}}\ d\theta\ . \qquad (11)$$

If $1 < |w| < 1/k$, equation (9) becomes

$$z = C + i\ g(w;k)\ , \qquad (12)$$

$$\begin{aligned} g(w;k) \quad \equiv \quad & \left\{\sum_{m=0}^{\infty}k^{2m}\varphi^2(m)\right\}\mathrm{Ln}\ w \\ & + \sum_{m=1}^{\infty}\frac{(kw)^{2m}}{2m}\sum_{n=m}^{\infty}k^{2(n-m)}\varphi(n-m)\varphi(n) \\ & - \sum_{m=1}^{\infty}\frac{w^{-2m}}{2m}\sum_{n=0}^{\infty}k^{2n}\varphi(n+m)\varphi(n)\ , \end{aligned} \qquad (13)$$

$$C \equiv K(k) + \frac{2i}{1+k}\ F\left(\sin^{-1}\sqrt{\frac{1+k}{1-k}\ \frac{\omega-1}{\omega+1}}, \frac{1-k}{1+k}\right) - i\ g(\omega;k)\ , \qquad (14)$$

where $\omega = 1/\sqrt{k}$, and $F(,)$ is an incomplete elliptic integral of the first kind. If $|w| = 1$, equation (9) becomes

$$z = K(k) - \int_0^{-i\ \mathrm{Ln}\ w} \frac{\mathrm{e}^{i(2\theta-\pi)/4}}{\sqrt{2\sin\theta}\sqrt{1-k^2\mathrm{e}^{2i\theta}}}d\theta\ . \qquad (15)$$

If $|w| < 1$, equation (9) becomes

$$z = \sum_{m=0}^{\infty}\frac{(kw)^{2m+1}}{2m+1}\sum_{n=0}^{m}k^{-2n-1}\varphi(n)\varphi(m-n)\ . \qquad (16)$$

2.3. Mapping from a Circular to a Semi-Infinite Region through a Univalent Analytic Function with Three Real Parameters

Case (A): the complex circular region is assumed to be expressed by $\xi = e^{\alpha+i\beta}, -\infty < \alpha \le 0, -\pi < \beta \le \pi$. The proposed univalent function is given by

$$\mathrm{am}(u/i, \sqrt{1-k^2}\ ) = \tan^{-1}\left(-\tanh\frac{\alpha+i\beta}{2}\right), \tag{17}$$

$$w = a\ \mathrm{sn}^2\left[\{u+K(k)\}\frac{K(k_0)}{2K(k)}, k_0\right] + b, \tag{18}$$

where am(,) is a Jacobian elliptic amplitude function, $\tan^{-1}()$ stands for a principal value and $-K(k) \le \Re(u) \le K(k)$, $0 \le \Im(u) \le K(\sqrt{1-k^2}\ )$, $0 < k < 1$, $k_0 \equiv 2\sqrt{k}/(1+k)$, $a > 0$, $b:$ real. Clearly $\alpha = 0$ corresponds to $|\Re(u)| = K(k)$ or $\Im(u) = 0$ or $\Im(u) = K(\sqrt{1-k^2})$, and consequently $\Im(w) = 0$. For Jacobian elliptic am function, the following applies:

$$\mathrm{am}(2K(k)v/\pi, k) = v - i\ \mathrm{Ln}\prod_{m=1}^{\infty}\left[\frac{1-q^{(2m-1)}\exp\{(-1)^m 2iv\}}{1-q^{(2m-1)}\exp\{(-1)^{m-1}2iv\}}\right], \tag{19}$$

where

$$|\Re(v)| \le \pi/2,\ |\Im(v)| < (K(\sqrt{1-k^2}\ )/K(k))(\pi/2),$$
$$q \equiv \exp\left\{-\pi K(\sqrt{1-k^2}\ )/K(k)\right\}.$$

Case (B): The following is proposed:

$$\mathrm{am}(u,k) \equiv -\ i\ \mathrm{Ln}\sqrt{-\ i\ \tanh\frac{\alpha+i\beta}{2}}, \tag{20}$$

where $-\infty < \alpha \le 0, -\pi < \beta \le \pi$, or equivalently

$$\mathrm{am}(u,k) \equiv \sin^{-1}\left(\frac{i-\tanh\dfrac{\alpha+i\beta}{2}}{2\sqrt{-i\tanh\dfrac{\alpha+i\beta}{2}}}\right), \tag{21}$$

$$w = a\ \mathrm{sn}\left[-i\{u-K(k)\}, \sqrt{1-k^2}\ \right] + b, \tag{22}$$

where $a > 0, b:$ real, $0 < k < 1, 0 \le \Re(u) \le K(k), |\Im(u)| \le K(\sqrt{1-k^2}\ )$. Clearly $\alpha = 0$ corresponds to $\Re(u) = 0$ or $\Re(u) = K(k)$ or $|\Im(u)| = K(\sqrt{1-k^2})$, and consequently $\Im(w) = 0$.

2.4. Possibility of a Fixed Point Conformal Mapping

For given ξ, variation dw is given by

$$dw = \frac{\partial w}{\partial a}\ da + db + \frac{\partial w}{\partial k}\ dk\ . \tag{23}$$

For case (A)

$$\frac{\partial w}{\partial a} = \mathrm{sn}^2(r, k_0)\ ,$$

$$\frac{\partial w}{\partial k} = 2\ a\ \mathrm{sn}(r, k_0)\ \frac{\partial}{\partial k}\mathrm{sn}(r, k_0)\ ,$$

$$r \equiv \{u + K(k)\}\ \frac{K(k_0)}{2K(k)}\ ,\ k_0 \equiv \frac{2\sqrt{k}}{1+k}\ ,$$

$$u = i \int_0^{\tan^{-1}(-\tanh\frac{\alpha+i\beta}{2})} \frac{d\theta}{\sqrt{1-(1-k^2)\sin^2\theta}}\ .$$

In general, if h and ψ are functions of k,

$$\begin{aligned}\frac{\partial}{\partial k}\mathrm{sn}(h,\psi) &= \mathrm{cn}(h,\psi)\mathrm{dn}(h,\psi) \\ &\quad \times \left[\frac{\partial h}{\partial k} - \psi\frac{\partial \psi}{\partial k}\int_0^{\mathrm{am}(h,\psi)} \frac{\sin^2\theta}{\sqrt{1-\psi^2\sin^2\theta}^{\,3}}\ d\theta\right]\ ,\end{aligned}$$

$$\int_0^{\mathrm{am}(h,\psi)} \frac{\sin^2\theta}{\sqrt{1-\psi^2\sin^2\theta}^{\,3}} d\theta$$

$$= \frac{1}{\psi^2}\left[\frac{E\{\mathrm{am}(h,\psi),\psi\}}{1-\psi^2} - \frac{\psi^2}{1-\psi^2}\frac{\mathrm{sn}(h,\psi)\mathrm{cn}(h,\psi)}{\mathrm{dn}(h,\psi)} - h\right]\ ,$$

where cn(,),dn(,) are Jacobian elliptic functions, and for $K(k)$

$$\frac{d\ K(k)}{dk} = \frac{1}{k}\left\{\frac{E(k)}{1-k^2} - K(k)\right\}\ .$$

For case (B)

$$\frac{\partial w}{\partial a} = \mathrm{sn}(\phi, \sqrt{1-k^2})\ ,$$

$$\phi \equiv i\,\{K(k) - u\}\ ,$$

$$u = F\left(- i \text{ Ln } \sqrt{- i \tanh \frac{\alpha + i\beta}{2}}, k \right) \ ,$$

$$\text{am}(\phi, \sqrt{1-k^2}) = i \text{ Ln} \left\{ \frac{1 + \text{sn}(\phi/i, k)}{\text{cn}(\phi/i, k)} \right\} \ ,$$

$$\int_0^{\text{am}(\phi,\sqrt{1-k^2})} \frac{\sin^2\theta}{\sqrt{1-(1-k^2)\sin^2\theta}^3} d\theta = i \left[\frac{1}{k^2} \{K(k)\right.$$

$$\left. -F(\varphi_0, k)\} - \frac{1}{k^2(1-k^2)} \{E(k) - E(\varphi_0, k)\} \right] \ ,$$

$$\varphi_0 \equiv \sin^{-1} \left[\frac{1}{\sqrt{(1-k^2)\cosh^2 \left\{ \text{am}(\phi, \sqrt{1-k^2}\,)/i \right\} + k^2}} \right] \ ,$$

where $E()$: a complete elliptic integral of the second kind, $E(,)$: an incomplete elliptic integral of the second kind. Thus, $dw = 0$ has a solution if and only if

$$\frac{da}{\Im\left(\dfrac{\partial w}{\partial k}\right)} = \frac{db}{\Im\left(\dfrac{\partial w}{\partial a}\overline{\dfrac{\partial w}{\partial k}}\right)} = \frac{dk}{\Im\left(\overline{\dfrac{\partial w}{\partial a}}\right)} \ , \tag{24}$$

where $\overline{}$ indicates a complex conjugate. In case any denominator is zero, its numerator is interpreted as zero. Concrete numerical examples for $dw = 0$ are from case (A)

$$\frac{da}{-2.43} = \frac{db}{2.73} = \frac{dk}{-0.23}$$

for $a = 1, b = 0, k = 0.6$ at $\xi = 0(\alpha = -\infty$, independent of β), and from case (B)

$$\frac{da}{3.13} = \frac{dk}{-1.25} \ , \quad db = 0$$

for $a = 1, b = 0, k = 0.6$, at $\xi = 0, (\alpha = -\infty$, independent of $\beta)$, which gives a fixed point mapping.

3. Conclusions

Among conformal mappings, mapping from a circular region to a specified region is not unique as shown by the family of mapping equations (18) and (22). All the more, the family of conformal mapping given by equations (18) and (22) can be a fixed point mapping at least for the center ($\xi = 0$) of the original circular domain $\xi(|\xi| \leq 1)$.

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