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# THE BLACK-SCHOLES FORMULA AND THE GREEK PARAMETERS FOR A NONLINEAR BLACK-SCHOLES EQUATION

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**Abstract:** We study the Greek (risk) parameters of a nonlinear Black-Scholes partial differential equation whose nonlinearity is as a result of transaction costs. These parameters are derived from the Black-Scholes formula of the nonlinear Black-Scholes equation $u_t + \frac{1}{2}\sigma^2 s^2 u_{ss}(1 + 2\rho s u_{ss}) = 0$ by differentiating the formula with respect to either a variable or a parameter in the equation. The Black-Scholes formula and all the Greek parameters are of the form $\frac{1}{\rho} f(s,t)$ and therefore they blow at $\rho = 0$.

**AMS Subject Classification:** 35K10, 35K55
**Key Words:** nonlinear black-scholes equation, black-scholes formula, illiquid markets, Greek parameters, transaction cost model

## 1. Introduction

Two primary assumptions are used in formulating classical arbitrage pricing theory: frictionless and competitive markets. In a frictionless market, there are no transaction costs and restrictions on trade while in a competitive market, a trader can buy or sell any quantity of a security without changing its price.

The notion of liquidity risk is introduced on relaxing the two assumptions above. This means that the Greek (risk) parameters derived from the Black-Scholes formulae under the classical theory in [1] become unrealistic due to the

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introduction of the liquidity risk. Hence, the Black-Scholes formula derived from the nonlinear Black-Scholes equation resulting from illiquid markets is appropriate in explaining this liquidity risk.

Greek parameters from the Black-Scholes formula of the nonlinear Black-Scholes partial differential equation are currently unknown.

The purpose of this paper is to obtain the Greek parameters from the Black-Scholes formula of the nonlinear Black-Scholes equation arising from transaction costs. This is done by differentiating the formula with respect to either the parameters or variables in the equation.

This paper is outlined as follows. Section 2 describes the modified option pricing theory. The Black-Scholes formula and the Greek parameters are presented in Section 3. Section 4 concludes the paper.

## 2. Modified Option Valuation Model

Nonlinearities in diffusion models can arise from source terms, insect dispersal, heat conduction and illiquid market effects.

In this work, we will consider the continuous-time (quadratic) transaction-cost model for modelling illiquid markets. Two assets are used in the model: a bond (or a risk-free money market account with spot rate of interest $r \geq 0$) whose value at time $t$ is $B_t \equiv 1$, and a stock (i.e. a risky and illiquid asset). The bond's market is assumed to be liquid (or perfectly elastic) [2].

Cetin *et al* [2] has put forward the predominant model in the transaction-cost model where a fundamental stock price process $s_t^0$ follows the dynamics

$$ds_t^0 = \mu s_t^0 dt + \sigma s_t^0 dW_t, \qquad t \in [0, T],$$

where $\mu$ is drift, $\sigma$ is volatility, and $W_t$ is the Wiener process. When trading $\alpha$ shares, the transaction price to be paid by the investor at time $t$ is

$$\overline{s}_t(\alpha) = e^{\rho\alpha} s_t^0, \quad \alpha \in \mathbb{R},$$

where $\rho$ is a liquidity parameter with $0 \leq \rho < 1$. A bid-ask-spread with size depending on $\alpha$ is essentially modeled by the transaction price above. For a Markovian trading strategy $\Phi_t = \phi(t, s_t^0)$ for a smooth function $\phi = u_s$, we have $\phi_s = u_{ss}$, where $u_s = \frac{\partial u}{\partial s}$, $\phi_s = \frac{\partial \phi}{\partial s}$, and $u_{ss} = \frac{\partial^2 u}{\partial s^2}$.

If the stock and bond positions are $\Phi_t$ and $\beta_t$ respectively where $\Phi_t$ is a semimartingale, then the paper value $V_t^M = \Phi_t s_t^0 + \beta_t$. The change in the quadratic variation

$$\langle \Phi \rangle t = \int_0^t \left(\phi_s(\tau, s_\tau^0)\sigma s_\tau^0\right)^2 d\tau$$

is $d\langle\Phi\rangle t = \left(u_{ss}(t, s_t^0)\sigma s_t^0\right)^2 dt$. Applying Itô formula to $u(t, s_t^0)$ gives

$$du(t, s_t^0) = u_s(t, s_t^0)ds_t^0 + \left(u_t(t, s_t^0) + \tfrac{1}{2}\sigma^2(s_t^0)^2 u_{ss}(t, s_t^0)\right) dt, \tag{1}$$

where $u_t = \frac{\partial u}{\partial t}$. In the limit, the wealth dynamics of a self-financing strategy is

$$dV_t^M = \Phi_t ds_t^0 - \rho s_t^0 d\langle\Phi\rangle t. \tag{2}$$

Since $V_t^M = u(t, s_t^0)$, substitute $d\langle\Phi\rangle t$ into (2) and equate the deterministic components of the resulting equation and equation (1) to get

$$u_t + \tfrac{1}{2}\sigma^2 s^2 u_{ss}(1 + 2\rho s u_{ss}) = 0, \qquad u(s_T^0, T) = h(s_T^0), \tag{3}$$

where $h(s_T^0)$ is a terminal claim whose hedge cost $u(s_t^0, t)$ is the solution to (3). The magnitude of the market impact is determined by $\rho s$. Large $\rho$ implies a big market-impact of hedging. If $\rho \to 0$ or no hedging demand, the asset's price follows the standard Black-Scholes model in [1] with constant volatility $\sigma$.

## 3. The Black-Scholes Formula and the Greek Parameters

### 3.1. The Black-Scholes Formula

It has been shown in Theorem 4.1 of [3] that the solution to equation (3) is given by

$$u(s,t) = \frac{\left(-\sqrt{s}e^{\frac{ct+s_0}{2}} + s(1-\ln s)\left(\frac{1}{4} - \frac{c}{\sigma^2}\right) + st\left(\frac{\sigma^2}{16} - \frac{c^2}{\sigma^2}\right) - \frac{\sigma^2}{16c}e^{\mathrm{ct+s_0}}\right)}{\rho}, \tag{4}$$

where $c$ is the speed of the wave, and $s_0$ is an integration constant. This solution is called the *Black-Scholes formula* and can be used for valuing a call option $u(s,t)$ at $t > 0$.

### 3.2. The Greek Parameters

We obtain the *delta* of the call option $u(s,t)$ by differentiating the Black-Scholes formula (4) with respect to the spatial variable $s$. Hence,

$$u_s = \frac{1}{\rho}\left(-\frac{1}{2\sqrt{s}}e^{\frac{ct+s_0}{2}} - \ln s\left(\frac{1}{4} - \frac{c}{\sigma^2}\right) + t\left(\frac{\sigma^2}{16} - \frac{c^2}{\sigma^2}\right)\right) \quad \text{for} \quad \rho, s, \sigma > 0.$$

When $u_s$ is differentiated with respect to $s$ we obtain *gamma* as

$$u_{ss} = \frac{1}{\rho s}\left(\frac{1}{4\sqrt{s}}e^{\frac{ct+s_0}{2}} + \frac{c}{\sigma^2} - \frac{1}{4}\right) \quad \text{for} \quad \rho, s, \sigma > 0.$$

The parameter *theta* is given by

$$u_t = \frac{1}{\rho}\left(-\frac{c}{2}\sqrt{s}e^{\frac{ct+s_0}{2}} + s\left(\frac{\sigma^2}{16} - \frac{c^2}{\sigma^2}\right) - \frac{\sigma^2}{16}e^{ct+s_0}\right) \quad \text{for} \quad \rho, \sigma > 0$$

when the Black-Scholes formula (4) is differentiated with respect to time $t$. If the price of the asset does not move, the option price will change by theta with time $t$.

Differentiating $u_{ss}$ with respect to $s$ gives option *speed* as

$$u_{sss} = \frac{\partial^3 u}{\partial s^3} = \frac{1}{\rho s^2}\left(-\frac{3}{8\sqrt{s}}e^{\frac{ct+s_0}{2}} - \frac{c}{\sigma^2} + \frac{1}{4}\right) \quad \text{for} \quad \rho, s, \sigma > 0.$$

Gamma is used by traders to estimate how much they will rehedge by if the stock price moves. An option delta may change by more or less the amount the traders have approximated the value of the stock price to change. If it is by a large amount that the stock price moves, or the option nears the strike and expiration, the delta becomes unreliable and hence the use of the speed.

When the Black-Scholes formula (4) is differentiated with respect to $\sigma$ we get the *vega* of a call option $u(s,t)$ as

$$\frac{\partial u}{\partial \sigma} = \frac{1}{\rho}\left(\frac{2c}{\sigma^3}s(1-\text{ln}s) + st\left(\frac{\sigma}{8} + \frac{2c^2}{\sigma^3}\right) - \frac{\sigma}{8c}e^{ct+s_0}\right) \quad \text{for} \quad \rho, s, \sigma, c > 0.$$

## 4. Conclusion

We have studied the Greek (risk) parameters such as delta, gamma, theta, speed, and vega. These parameters are for a call option in illiquid markets whose illiquidity is arising from transaction costs. The parameters have been derived from the Black-Scholes formula given in equation (4) using differentiation. The Black-Scholes formula and all the Greek parameters are of the form $\frac{1}{\rho}f(s,t)$ where $f(s,t)$ is a smooth function of $s$ and $t$. The formula (4) and the Greek parameters blow at $\rho = 0$.

In conclusion, further research needs to be done to evaluate the impact of Greek parameters.

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