$\mathcal{AP}$
ijpam.eu

IMPROVED RESULTS IN SCHEME THEORY

Martin Dowd

1613 Wintergreen Pl.
Costa Mesa, CA 92626, USA

Abstract: In previous papers by the author, schemes were used to specify some new axioms for set theory, to give a lower bound on the Mahlo rank of a weakly compact cardinal, and to give a chain in the Galvin-Hajnal order with properties of interest. Here, various improvements are made.

1. Introduction

Schemes were defined by the author in [6], where they were called systems of operations. A related notion is the "dressed ordinals" of [14].

In [6], it was shown using schemes that the set of greatly Mahlo cardinals below a weakly compact cardinal is a Π_1^1–enforceable set. This was also shown in [2]. In [7], it was shown that a more restricted set of cardinals, called $\mathcal{H}^\theta$-Mahlo cardinals, is Π_1^1-enforceable. It was also shown that the set of greatly Mahlo cardinals is, mod the thin ideal, contained in M_σ for any maximal set M_σ of Mahlo rank $\sigma < \kappa^+$ (see below for terminology). In [8], schemes were used to postulate some new axioms for set theory. In [9], schemes were used to prove a lower bound on the Mahlo rank of an $\mathcal{H}^n$-Mahlo cardinal. In [10], schemes were used to construct a chain of functions in the Galvin-Hajnal order which has properties of interest.

In this paper, various improvements are made to the methods and results of the above mentioned papers. Notation as in the preceding papers will be used, as follows:

- Ord for the class of ordinals, Lim for the class of limit ordinals, Lim for the operator taking a class of ordinals to the class of its limit points (so that Lim = Lim(Ord)), Inac for the strongly inaccessible cardinals, Pow for the power set of a set, Cf for the cofinality of a set of ordinals, Dom for the domain of a function, Id for the identity function on a given domain.
- For $\kappa \in$ Inac, In_κ for Inac $\cap\, \kappa$, and $\mathrm{In}_\kappa^\emptyset$ for $\mathrm{In}_\kappa \cup \{\emptyset\}$.
- $\triangle$ for diagonal intersection, $\triangledown$ for diagonal union.

The reader is assumed to be familiar with the basic properties of club subsets of a regular uncountable cardinal, and thin and stationary sets.

Throughout the paper, κ will denote an element of Inac.

2. Schemes

In this section, the basic facts concerning schemes are reviewed, with some streamlining from earlier versions. A scheme, of rank σ, for κ, is a pair $\Sigma = \langle\sigma, \phi\rangle$ where $\sigma < \kappa^+$ and ϕ is a function whose domain is the set of limit ordinals $\alpha \le \sigma$. For $\alpha \in \mathrm{Dom}(\phi)$, $\phi(\alpha)$ is an increasing function with domain an ordinal $\eta \le \kappa$, and whose range is an unbounded subset of α. If $\mathrm{Cf}(\alpha) < \kappa$ then $\eta < \kappa$, and if $\mathrm{Cf}(\alpha) = \kappa$ then $\eta = \kappa$. Sc_κ denotes the set of schemes for κ.

For a scheme $\Sigma = \langle\sigma, \phi\rangle$ in Sc_κ, and $\alpha \le \sigma$, let $\Sigma_{\le\alpha} = \langle\alpha, \phi \upharpoonright (\alpha + 1)\rangle$.

For $F\colon \mathrm{Pow}(\kappa) \mapsto \mathrm{Pow}(\kappa)$ and $\Sigma \in \mathrm{Sc}_\kappa$, the function $F^\Sigma\colon \mathrm{Pow}(\kappa) \mapsto \mathrm{Pow}(\kappa)$ may be defined by the following recursion. In cases 2 and 3 α_ξ is written for $\phi(\alpha)(\xi)$.

Case 0 ($\sigma = 0$): $F^\Sigma = \mathrm{Id}$.

Case 1 ($\sigma = \tau + 1$): $F^\Sigma = F \circ F^{\Sigma_{\le\tau}}$.

Case 2 ($\sigma \in \mathrm{Lim}$, $\mathrm{Cf}(\sigma) < \kappa$): $F^\Sigma(X) = \cap_{\xi<\eta} F^{\Sigma_{\le\sigma_\xi}}(X)$.

Case 3 ($\sigma \in \mathrm{Lim}$, $\mathrm{Cf}(\sigma) = \kappa$): $F^\Sigma(X) = \triangle_{\xi<\kappa} F^{\Sigma_{\le\sigma_\xi}}(X)$.

Define the subset $T_\Sigma \subseteq \kappa$ recursively as follows.

0: $\emptyset$.

1: $T_{\Sigma_{\le\tau}}$.

2: $(\eta + 1) \cup \cup_{\xi<\eta} T_{\Sigma_{\le\sigma_\xi}}$.

3: $\triangledown_{\xi<\kappa} T_{\Sigma_{\le\sigma_\xi}}$.

It is readily verified that T_Σ is thin.

Say that a scheme Σ' is a prefix of Σ, written $\Sigma' \sqsubseteq \Sigma$, if $\Sigma' = \Sigma_{\le\alpha}$ for some $\alpha \le \sigma$. Given a limit ordinal η and a chain S_α for $\alpha < \eta$ under the order $\sqsubseteq$,

the join $\sqcup S_\alpha$ is that "partial" scheme, where $\sigma = \cup\sigma_\alpha$ and $\phi = \cup\phi_\alpha$. $\sqcup S_\alpha$ may be completed to a scheme by specifying $\phi(\sigma)$.

A scheme $\Sigma \downarrow \lambda$ in Sc_λ will be defined by recursion on Σ, for those $\lambda \in \mathrm{In}_\kappa$ such that $\lambda \notin T_\Sigma$. Write $\Sigma \downarrow \lambda$ as $\langle \sigma', \phi' \rangle$.

0: The scheme with $\sigma' = 0$.

1: $\Sigma_{\leq\tau} \downarrow \lambda$ with τ' replaced by $\tau' + 1$.

2: $\sqcup_{\xi<\eta}\Sigma_{\leq\sigma_\xi} \downarrow \lambda$, with $\phi'(\sigma')(\xi)$ set to $(\phi(\sigma)(\xi))'$.

3: $\sqcup_{\xi<\lambda}\Sigma_{\leq\sigma_\xi} \downarrow \lambda$, with $\phi'(\sigma')(\xi)$ set to $(\phi(\sigma)(\xi))'$.

Let $\mathcal{L}_1$ be those $F\colon \mathrm{Pow}(\kappa) \mapsto \mathrm{Pow}(\kappa)$. such that for any $\beta < \kappa$, $F(X)\cap\beta = F(Y)\cap\beta$ whenever $X\cap\beta = Y\cap\beta$. Given such, a function $(F\restriction\beta)\colon\mathrm{Pow}(\beta) \mapsto \mathrm{Pow}(\beta)$ may be defined for any $\beta < \kappa$, by letting $(F\restriction\beta)(X\cap\beta) = F(X)\cap\beta$. It is easily seen that $\mathcal{L}_1$ is closed under composition, pointwise intersection, and pointwise diagonal intersection; whence by induction on Σ, if $F \in \mathcal{L}_1$ and $\Sigma \in \mathrm{Sc}_\kappa$ then $F^\Sigma \in \mathcal{L}_1$.

Lemma 1. *Suppose $F \in \mathcal{L}_1$, and $\Sigma \in \mathrm{Sc}_\kappa$. For $\lambda \in \mathrm{In}_\kappa$, if $\lambda \notin T_\Sigma$ then $F^\Sigma \restriction \lambda = F^{\Sigma\downarrow\lambda}$.*

Proof. By induction on Σ; see lemma 3 of [9] for some details. $\qquad\square$

3. Maximal Sets and Canonical Functions

This section also reviews and extends various facts. Suppose $<$ is a well-founded relation on a set S. A rank function ρ may be defined on the elements of S, by the recursion $\rho(x) = \mu\rho(\forall w < x(\rho(w) < \rho))$. Letting $x^<$ denote $\{w : w < x\}$, $\rho(x) = \rho[x^<]$. Also, $\rho(x) = \sup\{\rho(w) + 1 : w < x\}$. The rank $\rho(<)$ of $<$ may be defined as the smallest ordinal such that $\rho(w) < \rho$ for all $w \in S$, and equals $\rho[S]$, and also $\sup\{\rho(w)+1 : w \in S\}$. The following lemma is considered folklore, and is stated for emphasis.

Lemma 2. $\rho(<) < |S|^+$.

Proof. Since $\rho(x) \neq \rho(y) \Rightarrow x \neq y$, $|\rho[S]| \leq |S|$, and the claim follows. $\qquad\square$

For $X, Y \subseteq \kappa$ say that $X \subseteq_t Y$ if $X - Y$ is thin. For $X \subseteq \mathrm{In}_\kappa^\emptyset$ let $\mathrm{H}(X) = \{\lambda \in X\colon X \cap \lambda \text{ is a stationary subset of } \lambda\}$. For $X, Y \subseteq \mathrm{In}_\kappa^\emptyset$ say that $X <_R Y$ if $Y \subseteq_t \mathrm{H}(X)$. The reader is assumed to be familiar with the basic properties of H and $<_R$. In particular, $<_R$ is well-founded and transitive. Let ρ_R denote the rank function, and write $\rho_R(\kappa)$ for $\rho_R(<_R)$. $\rho_R(\kappa)$ is a measure of how "Mahlo"

κ is: $\rho_R(\kappa) \geq 1$ iff κ is Mahlo, $\rho_R(\kappa) \geq \kappa^+$ iff κ is greatly Mahlo, etc. By lemma 2, $\rho_R(\kappa) < (2^\kappa)^+$, and if GCH then $\rho_R(\kappa) < \kappa^{++}$.

It is useful to consider the order $<_R$, "relativized" to a subset $W \subseteq \mathrm{In}_\kappa^\emptyset$; this is simply $<_R \restriction W$. The rank function will be denoted ρ_R^W. In most uses W is a stationary subset, although some facts don't require this assumption.

In the order $<_R^W$, a stationary set $M \subseteq \mathrm{In}_\kappa^\emptyset$ is said to be maximal of rank ρ if $\rho_R^M(M) = \rho$, and $X \subseteq_t M$ whenever $X \subseteq W$ is a stationary subset with $\rho_R^W(X) \geq \rho$.

Lemma 3. *If $H^\Sigma(W)$ is stationary then it is maximal of rank σ in $<_R \restriction W$. If W is maximal of rank ρ in $<_R$ then $H^\Sigma(W)$ is maximal of rank $\rho + \sigma$ in $<_R$.*

Proof. Theorem 1 of [9] holds for the order $<_R \restriction W$, with $\mathrm{In}_\kappa^\emptyset$ replaced by W in the statement of the theorem; the modification to the proof are straightforward. The lemma follows easily. $\square$

Note that if $\rho_R^W(\kappa) < \kappa^+$ then $\rho_R^W(\kappa)$ equals the smallest σ such that $H^\Sigma(W)$ is thin. In [7] a stationary subset $S \subseteq \mathrm{In}_\kappa^\emptyset$ with $\rho_R(S) = 0$ and $\rho_R(H(S)) = 2$ is constructed, assuming $V = L$ and κ not weakly compact.

Given a sequence f_ξ of functions on Ord to Ord, as usual define the sup "pointwise", i.e., $(\sup_\xi f_\xi)(\gamma) = \sup_\xi(f_\xi(\gamma))$. Given a sequence $\langle f_\xi : \xi < \kappa \rangle$ let $\mathrm{dsup}_{\xi < \kappa} f_\xi$ be the function f where $f(\lambda) = \sup_{\xi < \lambda} f_\xi(\lambda)$.

Suppose f and g are functions from κ to κ. Say that $f <_* g$ if $\{\lambda : f(\lambda \geq g(\lambda)\}$ is thin. By ω-completeness of the thin ideal, $<_*$ is well-founded; it is also transitive. Let ρ_* denote the rank function.

The relations $\leq_*$ and $\equiv_*$ are defined similarly to $<_*$; $\leq_*$ is a well-founded quasi-order, and $\equiv_*$ is the canonical congruence relation. $<_*$ is not the strict part of $\leq_*$. Readily verified basic facts include the following.

- For $\eta < \kappa$, $\sup_{\xi < \eta} f_\xi$ is a least upper bound under $\leq_*$ of $\{f_\xi\}$.
- $\mathrm{dsup}_{\xi < \kappa} f_\xi$ is a least upper bound under $\leq_*$ of a sequence $\langle f_\xi \rangle$.
- If $f \leq_* g <_* h$ or $f <_* g \leq_* h$ then $f <_* h$.

A function f is said to be canonical of rank ρ if $\rho_{<_*}(f) = \rho$, and if $\rho_*(g) \geq \rho$ then $f \leq_* g$.

For a scheme $\Sigma \in \mathrm{Sc}_\kappa$, define a function $f_\Sigma : \mathrm{In}_\kappa \mapsto \kappa$ by recursion on Σ as follows.

0: The identically 0 function.
1: $f_{\Sigma \leq \tau} + 1$.
2: $\sup_{\xi < \eta} f_{\Sigma \leq \sigma_\xi}$.
3: $\mathrm{dsup}_{\xi < \kappa} f_{\Sigma \leq \sigma_\xi}$.

Lemma 4. f_Σ *is canonical of rank σ.*

Proof. This follows by induction on Σ, using well-known facts which can be found in [12] for example. $\square$

Lemma 5. *Suppose $\Sigma \in Sc_\kappa$, $\lambda \in In_\kappa$, and $\lambda \notin T_\Sigma$.*
a. $f_\Sigma \upharpoonright \lambda = f_{\Sigma\downarrow\lambda}$.
b. $f_\Sigma(\lambda)$ *equals the rank of $\Sigma \downarrow \lambda$.*

Proof. Both claims follow by induction on Σ; see lemma 5 of [10] for some details. $\square$

Lemma 6. *Suppose $X \subseteq In_\kappa^\emptyset$, $\Sigma \in Sc_\kappa$, $\lambda \in X \cap In_\kappa$, and $\lambda \notin T_\Sigma$. Then $\lambda \in H^\Sigma(X)$ iff $\rho_R^{X\cap\lambda}(\lambda) \geq f_\Sigma(\lambda)$.*

Proof. The proof is by induction on Σ, with cases as follows. In case 0, both statements of the equivalence are true under the assumption $\lambda \in X$. In case 1, $\lambda \in HH^{\Sigma \leq \tau}(X)$ iff $\lambda \in H^{\Sigma \leq \tau}(X)$ and $H^{\Sigma \leq \tau}(X) \cap \lambda$ is stationary iff (by lemma 1) $\lambda \in H^{\Sigma \leq \tau}(X)$ and $H^{\Sigma \leq \tau \downarrow \lambda}(X \cap \lambda)$ is stationary. Using lemma 5.b and the induction hypothesis, the first conjunct holds iff $\rho_R^{X\cap\lambda}(\lambda) \geq \tau'$ where τ' is the rank of $\Sigma_{\leq\tau} \downarrow \lambda$. It follows that the conjunction holds iff $\rho_R^{X\cap\lambda}(\lambda) \geq \sigma'$ where $\sigma' = \tau' + 1$ is the rank of $\Sigma \downarrow \lambda$; iff (using 5.b again) $\rho_R^{X\cap\lambda}(\lambda) \geq f_\Sigma(\lambda)$. In case 2, $\lambda \in \cap_{\xi<\eta}H^{\Sigma \leq \sigma_\xi}(In_\kappa^\emptyset)$ iff $\rho_R(\lambda) \geq \sup_{\xi<\eta}\{f_{\Sigma_{\leq\sigma_\xi}}(\lambda)\}$. Case 3 is similar to case 2. $\square$

The order $<_* \upharpoonright X^\kappa$ may be considered. In this case, X will be assumed to be stationary. In particular, κ must be Mahlo. If f is a canonical function of rank σ on κ then $f \upharpoonright X$ is a canonical function of rank σ in this order. In particular, this holds for f_Σ.

Lemma 7. *For any Σ, $\rho_R^X(\kappa) > \sigma$ iff $H^\Sigma(X)$ is stationary iff $\{\lambda \in X \cap In_\kappa : \rho_R^X(\lambda) \geq f_\Sigma(\lambda)\}$ is stationary.*

Proof. The first equivalence is immediate, and the second follows by lemma 6. $\square$

4. Lower Bound on the Mahlo Rank of a Weakly Compact Cardinal

In this section the methods of [9] will be used, in further iterations, to improve the lower bound on the Mahlo rank of a weakly compact cardinal given there. Although the bound is undoubtedly still weak, the methods might be of interest as examples in attempting to develop a general theory.

Let $\odot$ denote the ordinal exponentiation function. Define $\uparrow$ by the recursion $\alpha \uparrow 0 = 1$, $\alpha \uparrow (\beta + 1) = \alpha \odot (\alpha \uparrow \beta)$, and $\alpha \uparrow \beta = \sup_{\beta' < \beta} \alpha \uparrow \beta'$ for limit β. In [9] it was shown that if κ is a weakly compact cardinal then $\rho(\kappa) \geq \kappa^+ \uparrow \omega$.

For a cardinal κ let $\phi_0^\kappa(\alpha)$ denote the function κ^α. The function $\phi_0^\omega(\alpha)$ is the initial function of the "Veblen hierarchy" (q.v. see [13]). The Veblen function for any "base" cardinal κ may be defined the same way, namely, $\phi_\alpha^\kappa(\xi)$ is the ξth element in the enumeration of the values which are fixed points of each ϕ_β^κ for $\beta < \alpha$.

Let E_α be defined by the following recursion.

1. $E_0 = \kappa^+ \uparrow \omega$.
2. $E_\alpha = E_\beta \uparrow \omega$ if $\alpha = \beta + 1$.
3. $E_\alpha = \cup_{\beta < \alpha} E_\beta$ for $\alpha \in \mathrm{Lim}$.

When the base is ω, it is well-known that $E_\alpha = \phi_1(\alpha)$ (see [15]). For convenience, a proof will be given for arbitrary base.

Lemma 8. For any α, $\kappa^{+E_\alpha} = E_\alpha$.

Proof. This is clear for $\alpha = 0$. Suppose $\alpha = \beta + 1$ where $\beta \geq 1$. Then $P_\alpha = P_\beta^{P_\alpha} \geq \kappa^{+P_\alpha} \geq P_\alpha$. If α is a limit ordinal then $\kappa^{+P_\alpha} = \sup\{\kappa^{+P_\beta}\} = \sup\{P_\beta\} = P_\alpha$. $\square$

Theorem 9. $E_\alpha = \phi_1^\kappa(\alpha)$.

Proof. The proof is by induction on α. For the basis $\alpha = 0$, the least fixed point of ϕ_0 is easily seen to be $\kappa^+ \uparrow \omega$. Suppose $\alpha = \beta + 1$. It is clear that $E_\beta + 1 \leq E_\alpha$, whence (with $\odot$ being associated to the right) $\kappa^+ \odot \cdots \odot \kappa^+ \odot (E_\beta + 1) \leq E_\beta \odot \cdots \odot E_\beta \odot E_\beta$, whence $E_\alpha \geq \phi_1^\kappa(\alpha)$. We claim that for $n \geq 2$, $E_\beta \odot \cdots \odot E_\beta$ (n occurrences of E_β) $\leq \kappa^+ \odot \cdots \odot \kappa^+ \odot \kappa^+ \odot (E_\beta + 1)$, and $E_\alpha = \phi_1^\kappa(\alpha)$ follows. The claim is proved by induction on n. For the basis $n = 2$, using lemma 8, $E_\beta \odot E_\beta = (\kappa^+ \odot E_\beta) \odot E_\beta = \kappa^+ \odot E_\beta \odot 2 \leq \kappa^+ \odot E_\beta \odot \kappa^+ = \kappa^+ \odot (\kappa^+ \odot E_\beta) \odot \kappa^+ = \kappa^+ \odot \kappa^+ \odot (E_\beta \cdot \kappa^+) = \kappa^+ \odot \kappa^+ \odot ((\kappa^+ \odot E_\beta) \cdot \kappa^+) = \kappa^+ \odot \kappa^+ \odot \kappa^+ \odot (E_\beta + 1)$. In the case $n + 1$, $E_\beta \odot E_\beta \odot \cdots \odot E_\beta = (\kappa^+ \odot E_\beta) \odot E_\beta \odot \cdots \odot E_\beta = \kappa^+ \odot (E_\beta \cdot E_\beta \odot \cdots \odot E_\beta) = \kappa^+ \odot E_\beta \odot \cdots \odot E_\beta \leq \kappa^+ \odot \kappa^+ \odot \cdots \odot \kappa^+ \odot \kappa^+ \odot (E_\beta + 1)$. The case when α is a limit ordinal is immediate. $\square$

The reader is referred to [9] for definitions and properties of the following:
$$\mathcal{L}_n, \ F_n \mapsto F_n^*, \ I_n, \ \cap, \ \triangle, \ \mathcal{H}_n, \ \mathcal{R}_n, \ \rho_n.$$
Other notation following [9] will be used.

For $1 \leq j \leq k < \omega$ and Σ a scheme for κ of rank σ, let $I_{j;\Sigma}$ be defined by the following recursion:

1. $I_{j;0} = I_j$, or H if $j = 1$

2. $I_{jk;\Sigma} = I_{k;\Sigma}(I_{k-1;\Sigma}) \cdots (I_{j;\Sigma})$

3. $I_{j\omega;\Sigma} = \cap_{j \leq k < \omega} I_{jk;\Sigma}$

4. $I_{j;\Sigma} = I_{j\omega;\Sigma_{\leq\tau}}$ if $\sigma = \tau + 1$

5. $I_{j;\Sigma} = \cap_{\xi < \eta} I_{j;\Sigma_{\leq\alpha_\xi}}$ if $\mathrm{Cf}(\sigma) < \kappa$

6. $I_{j;\Sigma} = \triangle_{\xi < \kappa} I_{j;\Sigma_{\leq\alpha_\xi}}$ if $\mathrm{Cf}(\sigma) = \kappa$

It is convenient to define P_α to be 1 if $\alpha = 0$, $E_{\alpha-1}$ if $0 < \alpha < \omega$, and E_α if $\alpha \geq \omega$.

Lemma 10. $I_{j;\Sigma} \in \mathcal{R}_j$, and $\rho_j(I_{j;\Sigma}) \geq P_\sigma$.

Proof. The proof is by induction on Σ. The basis $\sigma = 0$ is immediate. For the case $\sigma = \tau + 1$, for $l \geq 0$ let $Q_{l;\tau} = P_\tau \cdot \kappa^+ \odot (P_\tau \cdot \kappa^+ \odot (\cdots P_\tau \cdot \kappa^+ \odot P_\tau))$, where there are $l + 1$ P_τ's. By results of [9], $\rho_j(I_{jk;\Sigma}) = Q_{k-j;\tau}$. Using lemma 8, it follows that $Q_{l;\tau}$ equals P_τ if $l = 0$, P_τ^2 if $l = 1$, and $P_\tau \uparrow l$ if $l \geq 2$. It then follows that $\rho_j(I_j; \sigma) = \rho_j(I_{j\omega}; \tau) \geq \sup_k \rho_j(I_{jk}; \tau) \geq \sup_k Q_{k-j;\tau} = \sup_k P_\tau \uparrow (k - j) = P_\tau \uparrow \omega = P_\sigma$. The cases where σ is a limit ordinal follow by the definition of P_τ and results of [9]. □

A sequence of classes X_ξ for $\xi \in \mathrm{Ord}$ can be coded as a class; in a simple method, the code X equals $\{\langle \xi, x \rangle : x \in X_\xi\}$. This remains true for classes X_ξ in (i.e., subsets of) V_κ, where $\xi < \kappa$. If $\lambda \in \mathrm{In}_\kappa$ (in fact more generally), $X \cap V_\lambda = \{\langle \xi, x \rangle : \xi < \lambda, x \in X_\xi \cap V_\lambda\}$.

Suppose κ is Π_1^1-indescribable. Say that $X \subseteq \kappa$ is Π_1^1-enforceable if there is a Π_1^1 formula $\Phi(P)$ with a single class free variable P, and a class $\mathring{P}$, such that $\models_{V_\kappa} \Phi(\mathring{P})$ and, and for $\lambda \in \mathrm{In}_\kappa$, if $\models_{V_\lambda} \Phi(\mathring{P} \cap V_\lambda)$ then $\lambda \in X$. This is a variation of the definition suitable for the present purposes, and well-known to be equivalent to other variations (see [11]).

In demonstrating that various predicates are Π_1^1, Σ_1^1 functions and Δ_1^1 predicates may be used. Some predicates are in fact Δ_0^1 (only first order quantifiers). For example, suppose $\Phi(P_1, \ldots, P_k)$ is a formula with several class free variables. Let $\mathring{P}$ be the code for $\langle \mathring{P}_1, \ldots, \mathring{P}_k \rangle$ as described above. Let $\Phi'(P)$ be the formula obtained by replacing each subformula $w \in P_i$ by $\langle i, w \rangle \in P$. The latter subformula is first order. In particular, multiple first and second order free variables may be used to specify formulas.

For the next lemma, some facts noted in [7] will be reviewed. There is a Δ_0^1 predicate stating that the class X represents a well-order on a subset of κ. This states that X is a class of ordered pairs, which as a binary relation is transitive and reflexive, total, and has no descending chains of length ω. The formula defines the desired class, in any V_κ where $\kappa \in \mathrm{Inac}$.

There is a Δ_0^1 predicate stating that the class X represents a scheme for κ. Namely, it represents a pair $\langle \sigma, \phi \rangle$ where σ is represented as above, and ϕ is a function whose domain is the limit points $\alpha < \sigma$, where $\phi(\alpha)$ is a function with domain either an ordinal, or all ordinals, etc.

An element $F_n \in \mathcal{L}_n$ for $n > 0$ can be coded as a class in V_κ. Indeed, it may be seen that $F_n \upharpoonright \lambda \in V_{\lambda+1+3n}$; the code F_n^c may be taken as $\{\langle \lambda, F_n \upharpoonright \lambda \rangle : \lambda \in \mathrm{In}_\kappa\}$. The predicate $\mathrm{App}_n(G_{n-1}^c, F_n^c, F_{n-1}^c)$ where $G_{n-1} = F_n(F_{n-1})$ is first order. Also, $(F_n \upharpoonright \lambda)^c = F_n^c \cap V_\lambda$.

Let $\mathcal{E}_0$ be the collection of Π_1^1-enforceable subsets of κ, and for an integer n, inductively let $\mathcal{E}_{n+1}$ be the elements $F \in \mathcal{L}^{n+1}$ such that $F[\mathcal{E}_n] \subseteq \mathcal{E}_n$. It is not difficult to show that for $n > 0$, $F_n \in \mathcal{E}_n$ iff for all $\langle F_j : j < n \rangle$ with $f_j \in \mathcal{E}_j$ for all $j < n$, $F_n(F_{n-1}) \cdots (F_1)(X) \in \mathcal{E}_0$ (use induction on n).

The following are well-known or follow by straightforward arguments.
- $\mathcal{E}_0$ is closed under the operations $\cap_{\xi<\eta} X_\xi$ for $\eta < \kappa$, and $\triangle_{\xi<\kappa} X_\xi$.
- For $n > 0$, $\mathcal{E}_n$ is closed under the operations $\cap_{\xi<\eta} F_{n\xi}$ for $\eta < \kappa$, and $\triangle_{\xi<\kappa} F_{n\xi}$.
- For $n > 0$, $\mathcal{E}_n$ is closed under $\circ$.
- For $n > 0$, for any scheme Σ for κ, if $F \in \mathcal{E}_n$ then $F^\Sigma \in \mathcal{E}_n$.

Lemma 11. *Suppose κ is weakly compact.*
1. $H \in \mathcal{E}_1$.
2. *For $n \geq 1$, if $F_n \in \mathcal{E}_n$ then $F^* \in \mathcal{E}_n$.*
3. *For $n \geq 1$, $I_{n+1} \in \mathcal{E}_{n+1}$.*

Proof. Part 1 is well-known. For part 2, suppose $F_j \in \mathcal{E}_j$ for $j < n$. Then $\models_{V_\kappa} \forall \Sigma (F_n^c)^\Sigma (F_{n-1}^c) \cdots (F_1^c)(X) \neq \emptyset$. Then $\{\lambda : \models_{V_\lambda} \forall \Sigma (F_n^c \cap V_\lambda)^\Sigma (F_{n-1}^c \cap V_\lambda) \cdots (F_1^c \cap V_\lambda)(X \cap \lambda) \neq \emptyset\} \in \mathcal{E}_0$. So $F_n^*(F_{n-1}) \cdots (F_1)(X) \in \mathcal{E}_0$. It follows by remarks above that $F^* \in \mathcal{E}_n$. Part 3 follows from part 2. $\square$

Lemma 12. *If κ is weakly compact then $I_{1;\Sigma} \in \mathcal{E}_1$.*

Proof. We show by induction that in the clauses in the definition of $I_{j;\Sigma}$, the left side is in $\mathcal{E}_j$. Clause 1 follows immediately by lemma 11. Clause 2 follows by a straightforward induction. Clause 3, 5, and 6 follows by remarks preceding lemma 11. Clause 4 is immediate. $\square$

Let $P_{\kappa^+} = \sup\{P_\sigma : \sigma < \kappa^+\}$.

Theorem 13. *If κ is weakly compact then $\rho(\kappa) \geq P_{\kappa^+}$.*

Proof. This follows using lemmas 10 and 12. $\square$

5. Chains in the Galvin-Hajnal Order

For κ Mahlo, let $\mathcal{F}$ denote $\{f : \text{In}_\kappa \mapsto \kappa : f(\lambda) < \lambda^{++}\}$. The Galvin-Hajnal order on $\mathcal{F}$ will be denoted $<_*$ as usual. A chain in $\mathcal{F}$ is a sequence $\langle f_\xi : \xi \le \sigma \rangle$ such that if $\xi' < \xi$ then $f_{\xi'} <_* f_\xi$; σ is called the rank of the chain.

A chain will be said to be regular if the following hold inductively.

0 ($\sigma = 0$): $f_0 \equiv_* 0$, where 0 denotes the identically 0 function.

1 ($\sigma = \tau + 1$): $f_\sigma \equiv_* f_\tau + 1$.

2 ($\sigma \in \text{Lim}$, $\text{Cf}(\sigma) \le \kappa$): $f_\sigma \equiv_* \sup_{\xi < \sigma} f_\xi$.

A chain will be said to be representing if it is regular, and for every $\xi \le \sigma$, and every normal ultrafilter U on κ, f_ξ represents ξ in the ultrapower V^κ/U.

A chain will be said to be strongly representing if it is representing, and except for a thin set of λ, if U' is a normal ultrafilter on λ, $f_\xi \upharpoonright \lambda$ represents $f_\xi(\lambda)$ in V^λ/U'.

Chains (resp. regular chains, representing chains, strongly representing chains) will also be said to be of type 1 (resp. 2, 3, 4). Given a chain $\langle f_\xi : \xi \le \sigma \rangle$, for $\xi \le \sigma$ let S_ξ denote $\{\lambda \in \text{In}_\kappa : o(\lambda) \ge f_\xi(\lambda)\}$. Trivially, $\xi' < \xi$ implies $S_\xi \subseteq_t S_{\xi'}$.

For κ is a measurable cardinal, let $o(\kappa)$ denote the Mitchell order. The following is an improved version of results from [10], with some inaccuracies in the proofs corrected.

Theorem 14. *Suppose κ is a measurable cardinal, and $\langle f_\xi : \xi \le \sigma \rangle$ is a strongly representing chain with $\sigma \le o(\kappa)$. Then S_ξ is stationary, and $\xi' < \xi$ implies $S_\xi >_R S_{\xi'}$.*

Proof. Let U_1 be a normal ultrafilter on κ with $O(U_1) = \sigma$. By lemma 19.34 of [Jech], o represents $o(U_1)$ in V^κ/U_1. In particular $\{\lambda \in \text{In}_\kappa : o(\lambda) = f_\sigma(\lambda)\}$ is stationary, and the first claim follows. For the second claim, it suffices to show that for $\xi < \sigma$, $S_{\xi+1} \subseteq_t \text{H}(S_\xi)$. Suppose $\lambda \in S_{\xi+1}$. Then $o(\lambda) \ge f_{\xi+1}(\lambda)$ so except for a thin set of λ there is a normal ultrafilter U' on λ with $o(U') = f_\xi(\lambda)$. By hypothesis, except for a thin set of λ, $f_\xi \upharpoonright \lambda$ represents $f_\xi(\lambda)$ in V^λ/U'. It follows that $S_\xi \cap \lambda \in U'$. $\square$

Write the Cantor normal form to the base κ^+ for the ordinal σ as $\kappa^{+\epsilon_k} \cdot \delta_k + \cdots + \kappa^{+\epsilon_0} \cdot \delta_0$. For $\sigma < \kappa^+ \uparrow \omega$, cases for a recursion on the Cantor normal form may be given as follows. In each case, the preceding cases are assumed to be false.

Case 1: $k > 0$

Case 2: $\epsilon_0 = 0$

Case 3: $\delta_0 > 1$
Case 4: $\epsilon_0 = 1$
Case 5: $\epsilon_0 < \kappa^{+\epsilon_0}$

For $\sigma < \kappa^+ \uparrow \omega$ a function $f_\sigma \in \mathcal{F}$ may be defined using the above recursion.

1. $f_{\kappa^{+\epsilon_k}\cdot\delta_k} + \cdots + f_{\kappa^{+\epsilon_0}\cdot\delta_0}$
2. f_Σ where Σ is a chosen scheme of rank σ
3. $f_{\kappa^{+\epsilon}} \cdot f_\delta$
4. $f_\sigma(\lambda) = \lambda^+$
5. $f_\sigma(\lambda) = \lambda^{+f_\epsilon(\lambda)}$

Lemma 15.   *For $\sigma < \kappa^+ \uparrow \omega$ the following hold.*

0. $f_0 \equiv_* 0$.
1. $f_{\sigma+1} \equiv_* f_\sigma + 1$.
2. If $\sigma \in Lim$ and $Cf(\sigma) \leq \kappa$ then $f_\sigma \equiv_* \sup_{\sigma' < \sigma} f_{\sigma'}$.
3. If $Cf(\sigma) = \kappa^+$ and $\sigma' < \sigma$ then $f_{\sigma'} \leq_* f_\sigma$.

It follows that $\langle f_\sigma \rangle$ is a type 2 chain.

Proof. The proof is by induction on σ. Case 0 is immediate. Remaining cases will be denoted i.j, where i is the case of the lemma and j is the case of the Cantor normal form recursion. Case j=1 falls into two subcases, 1.a, $\epsilon_0 = 0$, and 1.b, $\epsilon_0 > 0$, In case 1.1.a, $f_{\sigma+1} \equiv_* f_{\kappa^{+\epsilon_k}\cdot\delta_k} + \cdots + f_{\delta_0+1} \equiv_* f_\sigma + 1$. In case 1.1.b, $f_{\sigma+1} \equiv_* f_{\kappa^{+\epsilon_k}\cdot\delta_k} + \cdots + f_{\kappa^{+\epsilon_0}\cdot\delta_0} + f_1 \equiv_* f_\sigma + 1$. Case 1.2 is similar to case 1.1.a, and cases 1.3, 1.4, and 1.5 are similar to case 1.1.b. Case 2.2 follows by properties of schemes. Cases 2.1.1, 2.1.b, and 2.3 then follow by ordinal arithmetic. Case 2.4 is impossible. Case 2.5 follows by induction and ordinal arithmetic. Case 3.4 follows because $f_\sigma(\lambda) < \lambda^+$ for $\sigma < \kappa^+$. Case 3.5 follows by induction and ordinal arithmetic. $\square$

Lemma 16.   *If U is a normal ultrafilter on κ then f_σ represents σ in V^κ/U. It follows that $\langle f_\sigma \rangle$ is a type 3 chain.*

Proof. First, if F is a definable function on ordinals and $f_1, \ldots, f_n$ represent $\alpha_1, \ldots, \alpha_n$ then the function h where $h(\xi) = F(f_1(\xi), \ldots, f_n(\xi))$ for $\xi < \kappa$ represents $F(\alpha_1, \ldots, \alpha_n)$; this follows by Los' theorem. Second, since U is normal, the identity function represents κ. Third, by the first fact, f_{κ^+} (more properly any extension of it to κ) represents κ^+. Fourth, if $\sigma < \kappa^+$ then f_σ represents σ (induction on Σ; see lemma 6 of [10]). The claim follows by induction on the five cases of the recursive definition of f_σ, using the first fact for $+$, $\cdot$, and ordinal exponentiation. $\square$

For $\sigma < \kappa^+ \uparrow \omega$, a set T_σ will be defined. An ordinal $\sigma \downarrow \lambda < \lambda^+ \uparrow \omega$ will be defined by recursion on σ, for those $\lambda \in \text{In}_\kappa$ such that $\lambda \notin T_\sigma$. The definition is by recursion on the Cantor normal form, with cases as follows.

In case 1, $T_\sigma = \cup_i T_{\kappa^{+\epsilon_i} \cdot \delta_i}$, and $\sigma \downarrow \lambda = \sum_{i=k}^0 (\kappa^{+\epsilon_i} \cdot \delta_i) \downarrow \lambda$.

In case 2, $T_\sigma = T_\Sigma$, and $\sigma \downarrow \lambda$ equals the rank of $\Sigma \downarrow \lambda$.

In case 3, $T_\sigma = T_{\kappa^{+\epsilon_0}} \cup T_{\delta_0}$, and $\sigma \downarrow \lambda = (\kappa^{+\epsilon_0} \downarrow \lambda) \cdot (\delta_0 \downarrow \lambda)$.

In case 4, $T_\sigma = \emptyset$, and $\sigma \downarrow \lambda = \lambda^+$.

In case 5, $T_\sigma = T_{\epsilon_0}$, and $\sigma \downarrow \lambda = \lambda^{+\epsilon_0 \downarrow \lambda}$.

Lemma 17. *For $\lambda \notin T_\sigma$, $f_\sigma \upharpoonright \lambda = f_{\sigma \downarrow \lambda}$ and $f_\sigma(\lambda) = \sigma \downarrow \lambda$.*

Proof. By a straightforward induction on σ, with cases as above. □

The following is an improvement to the bound of theorem 11 of [10].

Theorem 18. $\langle f_\sigma \rangle$ *is a type 4 chain.*

Proof. This follows by lemmas 15, 16, and 17 □

6. Continuity of H^*

In this section some further remarks to those of [9] will be made on the question of the continuity of the Gaifman operation. The following theorem is a special case of lemma 4 of of [9]. For convenience a proof is given.

Theorem 19. *Suppose Σ is a scheme for κ, $X \subseteq \text{In}_\kappa^\emptyset$, and $\lambda \in \text{In}_\kappa$. If $\lambda \notin T_\Sigma$ and $\lambda \in H^*(X)$ then $\lambda \in H^\Sigma(X)$.*

Proof. The proof is by induction on Σ. Case 0 follows by definition of H^* and $H^0 = \text{Id}$. For case 1, by the definition of H^* and lemma 1 of [DowdIR], $H^\tau(X) \cap \lambda$ is stationary. By definition of T_σ, $\lambda \notin T_\tau$. It follows inductively that $\lambda \in H^\tau(X)$. Hence, $\lambda \in H(H^\tau(X)) = H^{\tau+1}(X)$. For case 2, for any $\xi < \eta$, $\lambda \notin T_{\sigma_\xi}$, so $\lambda \in H^{\sigma_\xi}(X)$; thus, $\lambda \in \cap_{\xi < \eta} H^{\sigma_\xi}(X)$. For case 3, for any $\xi < \lambda$, $\lambda \notin T_{\sigma_\xi}$, so $\lambda \in H^{\sigma_\xi}(X)$; thus, $\lambda \in \triangle_{\xi < \kappa} H^{\sigma_\xi}(X)$. □

The "continuity" of H^* at a Mahlo cardinal κ is the assertion that, for any $X \subseteq \text{In}_\kappa^\emptyset$, if $S \subseteq_t H^\Sigma(X)$ for all $\Sigma \in \text{Sc}_\kappa$, then $S \subseteq_t H^*(X)$. The notation "Gcont" will be used to denote this statement. The following theorem was stated in [9]; a detailed proof will be given here, along with an additional fact.

Theorem 20. *For $X \subseteq \text{In}_\kappa^\emptyset$ the following are equivalent.*
a. *If $S \subseteq_t H^\Sigma(X)$ for all $\Sigma \in \text{Sc}_\kappa$, then $S \subseteq_t H^*(X)$.*

b. *For any stationary set S such that $S \cap H^*(X) = \emptyset$ there is a stationary subset $S' \subseteq S$ and a $\Sigma \in Sc_\kappa$, such that $S' \cap H^\Sigma(X) = \emptyset$.*

c. *For any stationary set S such that $S \cap H^*(X) = \emptyset$ there is a stationary subset $S' \subseteq S$ and a $\Sigma \in Sc_\kappa$, such that $\rho_R^{X \cap \lambda}(\lambda) < f_\Sigma(\lambda)$ for all $\lambda \in S'$.*

If $H^*(X)$ is stationary these hold iff

d. $\rho_R^X(H^*(X)) = \kappa^+$.

Proof. a⇔b follows by logic, simple set theory, and the fact that $S \not\subseteq_t T$ iff $S' \cap T \neq \emptyset$ for some stationary set $S' \subseteq S$. b⇔c follows using lemma 6. $\rho_R^X(H^*(X)) \geq \kappa^+$ because for all Σ, $H^*(X)) \subseteq_t H^\Sigma$ and $\rho_R^X(H^*(X)) = \sigma$. Suppose $\rho_R^X(H^*(X)) > \kappa^+$. Let $S \subseteq X$ be a stationary subset such that $S <_R H^*(X)$ and $\rho_R^X(S) = \kappa^+$. For any $\Sigma \in Sc_\kappa$ there is a stationary subset $T \subseteq X$ such that $S \subseteq_t T$ and $\rho_R^X(T) = \sigma$. By maximality $S \subseteq_t T \subseteq_t H^\Sigma(X)$. But clearly $S \not\subseteq_t H^*(X)$. This proves a⇔d.   □

Let $f_+ : \kappa \mapsto \kappa$ be the function where $f_+(\alpha) = |\alpha|^+$. For a regular uncountable cardinal κ, The following two principles may be defined.

D1. If $f <_* f_+$ then for some $\Sigma \in Sc_\kappa$, $\{\alpha : f(\alpha) < f_\Sigma(\lambda)\}$ is club.

D2. If $f <_* f_+$ then for some $\Sigma \in Sc_\kappa$, $\{\alpha : f(\alpha) < f_\Sigma(\lambda)\}$ is stationary.

The names are those used in [3]. D1 is also called the bounding principle. D2 is well-known to be equivalent to the "weak Chang conjecture" at κ (see [5]).

It was claimed in [9] that D2 implies condition (c) of theorem 20. We have not been able to prove this, but only the following.

Theorem 21. *If D1 holds at κ then Gcont holds at κ.*

Proof. We may suppose X is stationary, since otherwise Gcont holds trivially. Then $X - H(X)$ is stationary (see theorem 2 of of [9]). Let $f(\alpha)$ equal $\rho_R^{X \cap \alpha}(\alpha)$ if $\alpha \in X - H(X)$, else 0. Since $f < f_+$, for some Σ, $D = \{\alpha : f(\alpha) < f_\Sigma(\lambda)\}$ is club, and so $(X - H(X)) \cap D$ is stationary. Thus, condition (c) of theorem 20 follows from D1.   □

The fact that D1 is required for the preceding theorem suggests that Gcont may be false in L. Indeed, it might be possible using the methods of theorems VI.6.1' and VII.1.2' of [4], to construct in L a stationary subset $S \subseteq X - H^*(X)$ such that $S \subseteq H^\Sigma(X)$ for all Σ in a chain of schemes.

For another fact of interest, it is shown in [1] that if if $V = L$ and κ is not ineffable (indeed if $\diamondsuit_\kappa''$ holds), then there is a function $f_\diamondsuit : \kappa \mapsto \kappa$ such that $f_\diamondsuit(\alpha) < f_+(\alpha)$ for all $\alpha < \kappa^+$, and $f_\Sigma <_* f_\diamondsuit$ for all Σ. In particular, D2 is false

at κ. Letting $S_\diamond = \{\lambda \in \text{In}_\kappa : \rho_R(\lambda) \geq f_\diamond(\lambda)\}$, it follows using lemma 7 that $S_\diamond \subseteq_{\text{t}} \text{H}^\Sigma(\text{In}_\kappa^\emptyset)$ for all Σ. But it is not clear whether $S_\diamond \not\subseteq_{\text{t}} \text{H}^*(\text{In}_\kappa^\emptyset)$ follows from the construction of $f_\diamond$.

Although peripheral to the subject under consideration, the following theorem provides an example of another use of lemma 7.

Theorem 22. *If κ is the smallest greatly Mahlo cardinal then D1 is false at κ.*

Proof. Since κ is the smallest greatly Mahlo cardinal, $\rho_R(\lambda) <_* \lambda^+$ for $\lambda \in \text{In}_\kappa$. On the other hand, since κ is greatly Mahlo, for any Σ, $\rho_R(\kappa) > \sigma$, so by lemma 7 $\{\lambda : \rho_R(\lambda) \geq f_\Sigma(\lambda)\}$ is stationary. Thus, D1 does not hold. $\quad\square$

7. Axiom G

In this section some improvements to Section 8 of [8] will be made. For a class Z let LimZ denote the map $X \mapsto \text{Lim}(X) \cap Z$. Recall that for $F : \text{Pow}(\text{Inac}) \mapsto \text{Pow}(\text{Inac})$, F^* is the function such that, for, $X \subseteq \text{Inac}$, $F^*(X) = \{\kappa \in \text{Inac} \cap X : F^\Sigma(X \cap \kappa)$ is stationary for all $\Sigma \in \text{Sc}_\kappa$. For functions F, G from classes to classes, say that $F \subseteq G$ if $F(X) \subseteq G(X)$ for all X. Familiarity with class schemes is assumed. The following are readily verified, where Σ is any class scheme .

- If $Y \subseteq Z$ then LimY $\subseteq$ LimZ.
- If $F \subseteq G$ then $F^\Sigma \subseteq G^\Sigma$.

Recall the operator H of [8], where $\kappa \in \text{H}(X)$ iff $\kappa \in \text{Inac} \cap X$ and $X \cap \kappa$ is stationary. It is readily seen that for $X \subseteq \text{Inac}$, $\text{H}(X) = \text{LimI}^*(X)$. It was erroneously stated in Section 8 of [8] that LimI* equals H_0. The fact of interest is that Mahl $= \text{H}_0(\text{Inac}) = \text{H}(\text{Inac})$, so that axioms M1-M4 and G agree.

Following is a generalization of lemma 1 of [8] (which is the case $Z = \text{Ord}$). An error in the proof of part 2 is repaired.

Lemma 23. *Suppose $Y \subseteq Z$.*

0. *Y is Y-closed.*
1. *If X is Y-closed then LimZ(X) is Y-closed.*
2. *If $\eta \in \text{Ord}$ and $\langle X_\xi : \xi < \eta \rangle$ is a sequence (coded as a class) of Y-closed classes then $\cap_{\xi<\eta} X_\xi$ is Y-closed.*
3. *If $\langle X_\xi : \xi \in \text{Ord} \rangle$ is a sequence (coded as a class) of Y-closed classes then $\triangle_{\xi \in \text{Ord}} X_\xi$ is Y-closed.*

Proof. In case 0, $\text{Lim}(Y) \cap Y \subseteq Y$. In case 1, by hypothesis $\text{Lim}(X) \cap Y \subseteq X$. So $\text{Lim}(\text{Lim}(X) \cap Z) \cap Y \subseteq \text{Lim}(X) \cap Y \subseteq \text{Lim}(X) \cap Z$. For case 2, $\text{Lim}(\cap_\xi X_\xi) \cap Y \subseteq \text{Lim}(X_\xi) \cap Y \subseteq X_\xi$ for any ξ. For case 3, suppose $\alpha \in \text{Lim}(\triangle_\xi X_\xi) \cap Y$. Let α_η be a sequence in $\triangle_\xi X_{\xi \cap Y}$ converging to α. If $\xi < \alpha$ then some suffix of the sequence converges in X_ξ to α, so $\alpha \in X_\xi$. But this shows that $\alpha \in \triangle_\xi X_\xi$. $\square$

Following is a strengthened version of lemma 4 of [8].

Lemma 24. *Suppose $Y \subseteq Z$, and Y is stationary.*
0. Y is Y-club.
1. If X is Y-club then $LimZ(X)$ is Y-club.
2. If $\eta \in Ord$ and $\langle X_\xi : \xi < \eta \rangle$ is a sequence (coded as a class) of Y-club classes then $\cap_{\xi < \eta} X_\xi$ is Y-club.
3. If $\langle X_\xi : \xi \in Ord \rangle$ is a sequence (coded as a class) of Y-club classes then $\triangle_{\xi \in Ord} X_\xi$ is Y-club.

Proof. By lemma 23, it suffices to show that the resulting class is unbounded. In case 0, Y is unbounded since it is stationary. In case 1, since Y is stationary $\text{Lim}(X) \cap Y$ is unbounded, whence $\text{Lim}(X) \cap Z$ is unbounded. For case 2, $W = \cap_{\xi < \eta} \text{Lim}(X_\xi)$ is club, so $W \cap Y$ is unbounded; and $W \cap Y \subseteq \cap_{\xi < \eta} X_\xi$. For case 3, $W = \triangle_\xi \text{Lim}(X_\xi)$ is club, so $W \cap Y$ is unbounded; and $W \cap Y \subseteq \triangle_\xi X_\xi$. $\square$

Following is a strengthened version of lemma 6 of [8]. An error in the proof is repaired.

Lemma 25. *Suppose $Y \subseteq Z \subseteq Inac$. The following are equivalent.*
a. Y is stationary.
b. For any scheme Σ, $LimZ^\Sigma(Y)$ is Y-club.
c. For any scheme Σ, $LimZ^\Sigma(Y) \neq \emptyset$.
d. For any scheme Σ, $LimI^\Sigma(Y) \neq \emptyset$.

Proof. b follows from a by lemma 24. c follows from b trivially. d follows from c by remarks above. a follows from d by lemma 5 of [8]. $\square$

Corollary 26. *Suppose $X \subseteq Z \subseteq Inac$, and X is stationary. Then for any scheme Σ, $LimZ^\Sigma(X)$ is stationary.*

Proof. This follows by the lemma, and lemma 3 of [8]. $\square$

Corollary 27. *If $X \subseteq Z \subseteq Inac$ then $LimZ^*(X) = H(X)$.*

Proof. $\kappa \in \mathrm{H}(X)$ iff $\kappa \in X$ and $X \cap \kappa$ is stationary, iff $\kappa \in X$ and $\mathrm{LimZ}^{\Sigma}(X \cap \kappa)$ is stationary for all $\Sigma \in \mathrm{Sc}^{\kappa}$, iff $\kappa \in \mathrm{LimZ}^*(X)$. $\qquad\square$

Recall that M_{Σ} is defined by recursion on Σ as follows.

0. Inac.
1. $\mathrm{LimM}^*_{\Sigma_{\leq\tau}}(M_{\Sigma_{\leq\tau}})$.
2. $\cap_{\xi<\eta} M_{\Sigma_{\leq\sigma_\xi}}$.
3. $\triangle_{\xi<\eta} M_{\Sigma_{\leq\sigma_\xi}}$.

Axiom A_{Σ} is given by the following clauses.

0. M_{Σ} is M_{Σ}-club.
1. If X is M_{Σ}-club then $\mathrm{LimM}_{\Sigma}(X)$ is M_{Σ}-club.
2. If $\eta \in \mathrm{Ord}$ and $\langle X_\xi : \xi < \eta \rangle$ is a sequence (coded as a class) of M_{Σ}-club classes then $\cap_{\xi<\eta} X_\xi$ is M_{Σ}-club.
3. If $\langle X_\xi : \xi \in \mathrm{Ord} \rangle$ is a sequence (coded as a class) of M_{Σ}-club classes then $\triangle_\xi X_\xi$ is M_{Σ}-club.

It is readily verified that $M_{\Sigma} \subseteq \mathrm{Inac} \cup \{0\}$.

Theorem 28. *Suppose Σ is a scheme.*
a. $M_{\Sigma} = H^{\Sigma}(Inac)$.
b. A_{Σ} *holds iff M_{Σ} is stationary.*

Proof. Part a follows by induction on Σ, using corollary 27 at stages in case 1. For part b, suppose A_{Σ} holds; then by lemma 25, with $Y = Z = M_{\Sigma}$, M_{Σ} is stationary. Suppose M_{Σ} is stationary; then by lemma 24, with $Y = Z = M_{\Sigma}$, axiom A_{Σ} holds. $\qquad\square$

It follows that an inaccessible cardinal κ is greatly Mahlo iff V_{κ} satisfies axiom G, a question left open in [8].

A justification of axiom G might involve an "informal" induction on Σ. However, a difficulty arises in justifying the claim that M_{Σ} is unbounded. The idea is, that V is M_{Σ}; but formulating this seems to involve additional complications. Axiom G appears likely to be true; but some reformulation might be required before a sufficiently "rigorous" justification can be given. The axiom scheme G_s, where Σ is any scheme definable from a parameter, might be justifiable by the methods already considered. Further consideration will be left to further research.

8. Another Rank

In [7] an approach was considered to the construction of stationary sets. Some additional remarks will be made here. Suppose $\kappa \in \text{Inac}$. A superscheme, of rank σ, for κ, is a pair $\Sigma = \langle \sigma, \phi \rangle$ where $\sigma < \kappa^{++}$ and ϕ is a function whose domain is the set of limit ordinals $\alpha \leq \sigma$, of cofinality at most κ. For $\alpha \in \text{Dom}(\phi)$, $\phi(\alpha)$ is as in the definition of a scheme. Ssc_κ denotes the set of superschemes for κ.

For convenience conventions as in [7] will be used. A superscheme is used to iterate H on a subset $X \subseteq \text{In}_\kappa$. For $\alpha \leq \sigma$ a recursive definition will be given of the result X_α of iterating to stage α. The subset $X_{\alpha,\lambda} \subseteq \lambda$ for each inaccessible cardinal $\lambda < \kappa$ will be defined also.

$$
\begin{array}{lll}
 & X_\alpha & X_{\alpha,\lambda} \\
\alpha = 0 & X & X \cap \lambda \\
\alpha = \beta + 1 & \text{H}(X_\beta) & \text{H}(X_{\beta,\lambda}) \\
\text{Cf}(\alpha) < \kappa & \cap_{\xi < \eta} X_{\alpha_\xi} & \cap_{\xi < \eta} X_{\alpha_\xi, \lambda} \text{ if } \eta < \lambda \\
 & & X \cap \lambda \text{ if } \lambda \leq \eta < \kappa \\
\text{Cf}(\alpha) = \kappa & \triangle_{\xi < \kappa} X_{\alpha_\xi} & \triangle_{\xi < \lambda} X_{\alpha_\xi, \lambda}
\end{array}
$$

If $\text{Cf}(\alpha) = \kappa^+$, $\lambda \in X_\alpha$ iff $\lambda \in X \cap \kappa$ and $X_{\beta,\lambda}$ is stationary for all $\beta < \alpha$; and $X_{\alpha,\lambda} = X_\alpha \cap \lambda$.

For convenience some theorems from [7] will be reproduced, with some simplifications and corrections. As an initial observation, $X_\alpha \subseteq_t X$ and $X_{\alpha,\lambda} \subseteq_t X \cap \lambda$ for all $\lambda \in \text{In}_\kappa$. The proof is a simple induction.

Given $\Sigma \in \text{Ssc}_\kappa$, for $\alpha < \sigma$ define the subset $T_\alpha \subseteq \kappa$ recursively as follows.
   0: $\emptyset$.
   1: T_β.
   2: $(\eta + 1) \cup \cup_{\xi < \eta} T_{\alpha_\xi}$.
   3: $\nabla_{\xi < \kappa} T_{\alpha_\xi}$.
   4: $\emptyset$.
It is readily verified that T_α is thin.

   Theorem 29.   *Suppose $\Sigma \in \text{Ssc}_\kappa$ and $\lambda \in \text{In}_\kappa$. $X_\alpha \cap \lambda \subseteq X_{\alpha,\lambda}$, and if $\lambda \notin T_\alpha$ equality holds.*

   Proof. By induction on α.
   If $\alpha = 0$, $X_0 \cap \lambda = X_{0,\lambda}$ by definition.
   If $\alpha = \beta + 1$, $X_{\beta+1} \cap \lambda = \text{H}(X_\beta) \cap \lambda = \text{H}(X_\beta \cap \lambda) \subseteq \text{H}(X_{\beta,\lambda}) = X_{\beta+1,\lambda}$.
   Suppose $\text{Cf}(\alpha) < \kappa$. If $\eta < \lambda$ then $X_\alpha \cap \lambda = (\cap_{\xi < \eta} X_{\alpha_\xi}) \cap \lambda = \cap_{\xi < \eta}(X_{\alpha_\xi} \cap \lambda) \subseteq \cap_{\xi < \eta} X_{\alpha_\xi, \lambda} = X_{\alpha,\lambda}$. If $\eta \geq \lambda$ the next to last expression may be replaced

by $X_0 \cap \lambda$.

If $\text{Cf}(\alpha) = \kappa$ then $X_\alpha \cap \lambda = (\triangle_{\xi < \kappa} X_{\alpha_\xi}) \cap \lambda = \triangle_{\xi < \lambda} (X_{\alpha_\xi} \cap \lambda) \subseteq \triangle_{\xi < \lambda} X_{\alpha_\xi, \lambda} = X_{\alpha, \lambda}$.

If $\text{Cf}(\alpha) = \kappa^+$ then $X_{\alpha, \lambda} = X_\alpha \cap \lambda$ by definition.

It is readily verified in all cases that equality holds if $\lambda \notin T_\alpha$. □

Theorem 30. *Suppose $\Sigma \in Ssc_\kappa$ and $\beta \leq \alpha < \sigma$. Then $X_\alpha \subseteq_t X_\beta$.*

Proof. Since the claim for $\beta = \alpha$ is immediate, $\beta < \alpha$ may be assumed. The proof is by induction on α.

If $\alpha = 0$, $\beta = \alpha$.

If $\alpha = \gamma + 1$, if $\beta \leq \gamma$ then $X_\alpha \subseteq X_\gamma \subseteq_t X_\beta$.

Suppose $\text{Cf}(\alpha) < \kappa$. If $\beta < \alpha$ then $\beta < \alpha_\xi$ for some ξ and $X_\alpha \subseteq X_{\alpha_\xi} \subseteq_t X_\beta$.

Suppose $\text{Cf}(\alpha) = \kappa$. If $\beta < \alpha$ then $\beta < \alpha_\xi$ for some ξ and $X_\alpha \subseteq_t X_{\alpha_\xi} \subseteq_t X_\beta$.

Suppose $\text{Cf}(\alpha) = \kappa^+$. The claim is proved by induction on β.

If $\beta = 0$ then $X_\alpha \subseteq_t X$.

For $\beta = \gamma + 1$, suppose $\lambda \in X_\alpha$. Then except for a thin set of λ, $\lambda \in X_\beta$. Also, $X_{\beta, \lambda}$ is stationary, so except for a thin set of λ $X_\beta \cap \lambda$ is stationary. Thus, except for a thin set of λ, if $\lambda \in X_\alpha$ then $\lambda \in HX_\beta = X_{\beta+1}$.

If $\text{Cf}(\alpha) < \kappa$ then inductively on β, $X_\alpha \subseteq_t X_{\beta_\xi}$ for $\xi < \eta$, whence $X_\alpha \subseteq_t X_\beta$.

The case $\text{Cf}(\beta) = \kappa$ is similar to the case $\text{Cf}(\alpha) < \kappa$.

Suppose $\text{Cf}(\beta) = \kappa^+$. If $\lambda \in X_\alpha$ then $X_{\gamma, \lambda}$ is stationary for $\gamma < \alpha$, and a fortiori for $\gamma < \beta$, so $\lambda \in X_\beta$; that is, $X_\alpha \subseteq X_\beta$. □

As a corollary, if $\beta < \alpha$ then $\beta + 1 \leq \alpha$, so $X_\alpha \subseteq_t X_{\beta+1} = \text{H}(X_\beta)$, which shows that $X_\alpha >_R X_\beta$. Thus, as long as X_α remains stationary, a superscheme yields an ascending chain. The "superscheme rank" $\rho_{\text{Ssc}}(\kappa)$ of an inaccessible cardinal κ may be defined as the smallest ordinal such that for every $\rho' < \rho$ there is a superscheme of rank $\rho' < \rho$ such that $X_{\rho'}$ is stationary.

In [7] it was claimed that, mod $\equiv_t$, X_α is independent of Σ. However the proof is not correct, and this question is open.

References

[1] J. Baumgartner, Ineffability properties of cardinals II, In: *Butts and Hintikka*, Logic, Foundations of Mathematics and Computability Theory, Reidel, Dordrecht, Holland (1977), 87-106.

[2] J. Baumgartner, A. Taylor, S. Wagon, On splitting stationary subsets of large cardinals, *J. Symb. Logic*, **42** (1977), 203-214.

[3] O. Deiser, D. Donder, Canonical functions, non-regular ultrafilters and Ulam's problem on ω_1, *Journal of Symbolic Logic*, **68** (2003), 713-739.

[4] K. Devlin, *Constructibility*, Springer-Verlag (1984).

[5] H. Donder, J. Levinski, Some principles related to Chang's conjecture, *Annals of Pure and Applied Logic*, **45** (1989), 39-101.

[6] M. Dowd, Remarks on Levy's reflection axiom, *Mathematical Logic Quarterly*, **39** (1993), 79-95.

[7] M. Dowd, Iterating Mahlo's operation, *Int. J. Pure Appl. Math.*, **9**, No. 4 (2003), 469-512.

[8] M. Dowd, Some new axioms for set theory, *Int. J. Pure Appl. Math.*, **66**, No. 2 (2011), 121-136.

[9] M. Dowd, A lower bound on the Mahlo rank of a weakly compact cardinal, *Int. J. Pure Appl. Math.*, 68, No. 4 (2011), 415-438.

[10] M. Dowd, Normal ultrafilters and Mahlo rank, *Int. J. Pure Appl. Math.*, **68**, No. 4 (2011), 487-491.

[11] F. Drake, *Set Theory. An Introduction to Large Cardinals*, North Holland (1974).

[12] T. Jech, *Set Theory*, Springer (2003).

[13] K. Schutte, *Proof Theory*, Springer (1977).

[14] H. Simmons, Iteration templates as generalized ordinal notations, *Manuscript* (2009).

[15] http://en.wikipedia.org/wiki/Epsilon_0