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THE BLACK-SCHOLES OPERATOR AS THE GENERATOR OF A C_0-SEMIGROUP AND APPLICATIONS

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Abstract: Considering the Black-Scholes equation as a Cauchy problem for different options: european, asian and vanilla, in this work we see that the second order differential operator

$$\hat{A}u = ax^2D^2u + bxDu - bu, \quad a, b \in \mathbb{R}$$

on the Schwartz space $S_0(0,\infty)$, generates a C_0-semigroup. We also find the corresponding solution as some class of evolution equation.

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1. Introduction

The Black-Scholes model is a mathematical description of financial markets and derivative investment instruments. Let $(\Omega, \mathcal{F}, P, \mathcal{F}_{t\geq 0})$ be a filtered probability space and let $W(t)$ be a brownian motion in $\mathbb{R}$. We will consider the stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \tag{1.1}$$

the dW term here stands-in for any and all sources of incertainty in the price

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history of the stock. The price processes given by the geometric brownian motion $S(t)$, $S(0)=x_0$, with μ and σ constants.

1. If $\sigma=0$, then the behaviour of the asset price is totally deterministic and we have the ordinary differential equation

$$\frac{dS}{S}=\mu dt$$

this can be solved to give $S=x_0e^{\mu t}$, where x_0 is the asset price at time $t=0$.

2. The equation (1.1) can be considered to be scheme for constructing time series that may realised by share prices.

The Black-Scholes equation was deduced in [4], [10] and is the linear partial differential equation

$$\frac{\partial C}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0, \qquad (1.2)$$
$$0\leq t<T, S\geq 0,$$

where $C(S,t)$ is the value of european type option on the asset price S at time t, μ is the instantaneous mean return, σ is the stock volatility [11].

With the assumptions of the Black-Scholes model, the second order partial differential equation (1.2) holds whenever C is twice differentiable with respect to S and once with respect to t.

In this work we consider the Black-Scholes equation as an operator acting on a Banach space. Using the theory of strongly continuous semigroup (in compact form, C_0-semigroup; see [7] and [8]) we characterize the semigroup whose generator is the operator mentioned above and thus, applying the semigroup to the domain of such operator, we obtain the price of different options of investment.

Using different techniques, the Cauchy problem related to the equation (1.2) has been considered by several authors. In reference [2], the problem for the case of european options was solved; using special transformations, the authors found the semigroup generated by the operator related to the equation (1.2). For asian options, and using special transformation methods the problem has been solved in [1]. In reference [3], making use of the Fourier transformations, the authors probe that the second order part of the Black-Scholes operator is a sectorial one making in the case of american options. In reference [5] the Black-Scholes operator, for the case of barrier options is seen as a Hamiltonian an its relation with supersymmetric quantum mechanics studied. The rest of the work is as follows: in Section 2 two results characterizing the semigroup whose generator is the Black-Scholes operator are given, and in Section 3, several applications, to different options are studied.

2. Some Results about C_0-Semigroups Related with the Black-Scholes Operator

In this section we consider the Black-Scholes equation (1.2) as the Cauchy problem

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0, \quad (x,t) \in (0,\infty) \times (0,T),$$

$$u(x,T) = h(x), \quad x \in (0,\infty),$$

where σ and T are the constants resulting of the model, h is a suitable measurable function ; T is the time of maturity of the option. In order to solve this problem we have to obtain the price of the option. Thus, in this section we see that a C_0-semigroup $\{U_t\}_{t\geq 0}$ exists, acting on a Banach space, such that the dynamics of the market should be given by $(U_{T-t}h)(x) = u(x,t)$.

Let $a = \frac{1}{2}\sigma^2 > 0$ and $b = r \in \mathbb{R}$, consider the operator

$$\hat{A}u = ax^2D^2u + bxDu - bu$$

which will be called the Black-Scholes operator densely defined on the Banach space $L^p(0,\infty)$. Let $\zeta = \frac{1}{2a}(b-a)$ and define the operator

$$\begin{aligned} D(A) &:= \mathcal{S}_0, \\ Af &:= x^2D^2f + (2\zeta+1)xDf - (2\zeta+1)f, \end{aligned}$$

where $\mathcal{S}_0$ is the Schwartz space on $(0,\infty)$ (with the norm of supremum)

$$\mathcal{S}_0 = \mathcal{S}(0,\infty) = \{f \in C^\infty(0,\infty) : \ \forall\, \alpha, \beta \in \mathbb{N}, \ \sup_x |x^\alpha D^\beta f(x)| < \infty\}.$$

We can see that $aA = \hat{A}$ on $D(A)$ and also prove that A can be written as

$$A = (B+\zeta)^2 - (1+\zeta)^2$$

where B is the generator of a C_0-group. We see that A and therefore $\hat{A}$ generates a semigroup $\{U_t\}_{t\geq 0}$ on $\mathcal{S}_0$.

We now go to establish two fundamental theorems, both of them useful in order to find the C_0-semigroup generated by the Black-Scholes operator $\hat{A}$ and with it the dynamics of the evolution (option prices).

Theorem 1. *Let $(V_tf)(x) := f(xe^t)$ for all $f \in \mathcal{S}_0$, $x \geq 0$ and $t \in \mathbb{R}$. Then $||V_t|| = 1$ for all t and $\{V_t\}_{t\in\mathbb{R}}$ is a C_0-group on $\mathcal{S}_0$ with generator*

$$Bf := xDf,\ D(B) := \mathcal{S}(0,\infty) = \mathcal{S}_0.$$

Proof. It is easy to see that V_t form a group as well as $||V_t|| = 1$ for all $t \in \mathbb{R}$. Let $f \in C_0([0,\infty))$,[1] then, when $t \to 0$:

$$||V_tf - If||_\infty = \sup_{x\geq 0} |f(xe^t) - f(x)| = |f(x_0e^t) - f(x_0)| \to 0.$$

Since that $C_0([0,\infty))$ is dense on $\mathcal{S}_0$, then V_t form a C_0-group on $\mathcal{S}_0$. Denote by F its generator. Let us now define the C_0-semigroups $V_t^+ := V_t$ and $V_t^- := V_{-t}$ for all $t \geq 0$ with infinitesimal generator F and $-F$ respectively.

Take us $f \in D(F)$ and $g = (I - F)f \in \mathcal{S}_0$. Because $\{V_t^+\}_{t\geq 0}$ is a C_0-semigroup semigroup of contractions, for all $x > 0$, using properties of the resolvent, we have

$$f(x) = \int_0^\infty e^{-t} g(xe^t)dt = x\int_x^\infty \frac{g(u)}{u^2}du.$$

Taking the last result and using the fundamental theorem of calculus, we have, after some changes of variable

$$\lim_{h\to 0}\frac{1}{h}(f(x+h) - f(x)) = \lim_{h\to 0}\Big(\frac{x+h}{h}\int_{x+h}^\infty \frac{g(u)}{u^2}du - \frac{x}{h}\int_x^\infty \frac{g(u)}{u^2}du\Big)$$

$$= \lim_{h\to 0}\Big(\frac{f(x+h)}{x+h} - \frac{x}{h}\int_0^h \frac{g(u+x)}{(u+x)^2}du\Big) = \frac{f(x)}{x} - \frac{g(x)}{x}.$$

therefore $f \in C^1(0,\infty)$; in addition, the function $x \longmapsto xf'(x) = f(x) - g(x) \in \mathcal{S}_0$, which implies that $f \in D(B)$ and also that $(Bf)(x) = xf'(x) = f(x) - g(x) = (Ff)(x)$, so that we have proved $F \subset B$, analogously we can proof that $-F \subset B$.

Inversely, let $f \in D(B)$ and now define $g := Bf$. For all $t, x \geq 0$ we have

$$\int_0^t (V_s^+f)(x)ds = \int_0^t g(xe^s)ds = \int_0^t xe^s f'(xe^s)ds$$

$$= \int_x^{xe^t} f'(r)dr = f(xe^t) - f(x).$$

Therefore, we conclude that $f \in D(F)$ and $Ff = g = Bf$, hence the inclusion $B \subset F$ has been proved. In the same way, we can see that $B \subset -F$ and as a result $B = F$. □ □

[1]Here $\mathsf{C}_0([0,\infty))$ are the functions of compact support on $[0,\infty)$

Using the above theorem we have that $(B+\zeta, D(B))$ generates the C_0-group $\{e^{\zeta t}V_t\}_{t\in\mathbb{R}}$. Then the operator $((B+\zeta)^2, D(B^2))$ generates a C_0-semigroup $\{U_{1,t}\}_{t\geq 0}$ on $\mathcal{S}_0$ which acts in accordance with

$$(U_{1,t}f) = \int_{-\infty}^{\infty} \phi(s)e^{\zeta s\sqrt{2t}}(V_{s\sqrt{2t}}f)ds$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is the Gaussian function $\phi(s) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}s^2}$; as a consequence of the general theory of operators we have

$$D(B^2) = \{f \in D(B) : Bf \in D(B)\}$$

for which one have that $D(B) = D(aA)$ and thus $D(B) = D(\hat{A})$. Furthermore, considering that $B = xD$ we have

$$A = x^2D^2 + (2\zeta+1)xD - (2\zeta+1) = (B+\zeta)^2 - (1-\zeta)^2.$$

We can see that A generates an analytic C_0-semigroup [6] of bounded operators $\{U_{2,t}\}_{t\geq 0}$ with angle $\delta = \frac{\pi}{2}$ on $\mathcal{S}_0$ which can be represented by $U_{2,t}f := e^{-(1+\zeta)^2}U_{1,t}f$, and hence for $t \geq 0$

$$U_{2,t}f = e^{-t(1+2\zeta)} \int_{-\infty}^{\infty} \phi(s)(V_{s\sqrt{2t}+2\zeta t}f)ds.$$

Also, since that $\hat{A} = aA$, then $\hat{A}$ generates an analytic C_0-semigroup of operators $\{U_{2,at}\}_{t\geq 0}$, which, recalling that $\zeta = \frac{1}{2a}(b-a)$ and for each $f \in \mathcal{S}_0$, $t \geq 0$ then we have

$$(U_tf)(x) = e^{-bt} \int_{-\infty}^{\infty} \phi(s)(V_{s\sqrt{2at}+(b-a)t}f)(x)ds$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2}(V_{s\sqrt{2at}+(b-a)t}f)(x)ds.$$

Thus, the following theorem has been proved.

Theorem 2. *The operator $(\hat{A}, D(A))$ generates a C_0-semigroup of operators $\{U_t\}_{t\geq 0}$ analytic with $\delta = \frac{\pi}{2}$ on $\mathcal{S}_0$, given for every $f \in \mathcal{S}_0$, $t \geq 0$ and $x \geq 0$ by*

$$(U_tf)(x) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2}(V_{s\sqrt{2at}+(b-a)t}f)(x)ds.$$

Let us consider, as a particular case, the following C_0-semigroup, sometimes called the heat semigroup [7]. Considering the space $X = L^p(\mathbb{R})$, $1 \leq p < \infty$, for $t > 0$ and defining $\{U_t\}_{t>0}$ for $f \in L^p(\mathbb{R})$ through

$$(U_t f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4t}} f(y)dy,$$

we can see that $U_t f = G_t * f$, ($*$ being the convolution) where $G_t(x) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}$. Since that $G_t \in L^1(\mathbb{R})$, then $U_t f \in L^p(\mathbb{R})$ when $f \in L^p(\mathbb{R})$.

In addition

$$\|U_t f\|_p = \|G_t * f\|_p \leq \|G_t\|_1 \|f\|_p.$$

Given that

$$\|G_t\|_1 = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{4t}} dx = 1,$$

it follows that $\|U_t\| \leq 1$, i.e. , $\{U_t\}_t$ is bounded on $L^p(\mathbb{R})$ for every $t > 0$. Now, if we choice $U_0 = I$; then if $t = 0$ or $s = 0$ we have that $U_{t+s} = U_t U_s$. Now, in the general case, we consider the Schwartz space

$$\mathcal{S}(\mathbb{R}) = \{f \in C^{\infty}(\mathbb{R}) : \ \forall\, \alpha, \beta \in \mathbb{N}, \quad \sup_x |x^{\alpha} D^{\beta} f(x)| < \infty\}$$

which is a density subspace of $L^p(\mathbb{R})$. For $f \in \mathcal{S}(\mathbb{R})$, because of the associativity of the convolution for $s, t > 0$ one can find $(U_s U_t)f = G_s * (G_t * f) = (G_s * G_t) * f$; after some calculations we have

$$(G_s * G_t)(x) = \frac{1}{4\pi\sqrt{st}} \int_{-\infty}^{\infty} exp(\frac{-(x-y)^2}{4s}) exp(\frac{-y^2}{4t}) dy$$

$$= \frac{1}{4\pi\sqrt{st}} \int_{-\infty}^{\infty} exp(\frac{-t(x-y)^2 - sy^2}{4st}) dy$$

$$= \frac{1}{4\pi\sqrt{st}} \int_{-\infty}^{\infty} exp(\frac{-tx^2 + 2txy - (s+t)y^2}{4st}) dy$$

$$= \frac{1}{4\pi\sqrt{st}} \int_{-\infty}^{\infty} exp(\frac{-(s+t)}{4st} Y^2) exp(-\frac{(s+t)}{4st} \frac{st}{(s+t)^2} x^2) dY$$

$$= \frac{1}{2\pi\sqrt{s+t}} exp\{\frac{-x^2}{4(s+t)}\} \int_{-\infty}^{\infty} e^{-z^2} dz, \quad (z = \sqrt{\frac{s+t}{4st}} Y)$$

$$= \frac{\sqrt{\pi}}{2\pi\sqrt{s+t}} exp\{\frac{-x^2}{4(s+t)}\} = \frac{1}{\sqrt{4\pi(s+t)}} exp\{\frac{-x^2}{4(s+t)}\} = G_{s+t}(x)$$

and thus $(U_sU_t)f = G_{s+t} * f = U_{s+t}f$ for $f \in \mathcal{S}(\mathbb{R})$; as $\mathcal{S}(\mathbb{R})$ is a dense subspace of $L^p(\mathbb{R})$, the result can be extended to the entire space $L^p(\mathbb{R})$.

Using some results of the test functions [9] we have $G_t * f \longrightarrow f$ as $t \to 0^+$. Therefore $\{U_t\}_{t\geq 0}$ is a C_0-semigroup on $L^p(\mathbb{R})$.

3. Applications to the Black-Scholes Equation

In this section we use the results of the last one in order to find the price of one option of inversion modeled by the Black-Scholes equation.

3.1. European Option

In the previous section we saw that the Black-Scholes equation becomes the Cauchy problem

$$a\hat{A}f(x,t) + f_t(x,t) = 0, \quad f(x,T) = h(x).$$

For some $h \in \mathcal{S}_0$, using the theorem 2 we see that $\hat{A}$ generates an C_0-semigruoup $\{U_t\}_{t\geq 0}$, and therefore, at the time $t \in [0,T]$ the option of prices (for european type) is given by

$$\begin{aligned}(U_th)(x) &= e^{-b(T-t)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} f(e^{s\sqrt{2a(T-t)}-(b-a)(T-t)}x)ds \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} f(e^{s\sigma\sqrt{(T-t)}-(r-\frac{1}{2}\sigma^2)(T-t)}x)ds.\end{aligned}$$

3.2. Asian Option

An asian option (or average value option) is a special type of option contract. For asian options the payoff is determined by the average underlying price over some pre-set period of time. This is different to the case of the usual european option as well as the american option, where the payoff of the option contract depends on the price of the underlying instrument at maturity; asian options are thus one of the basic forms of exotic options.

The Black-Scholes equation for the arithmetic asian option is given by

$$\begin{aligned}&u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0, \quad (x,t) \in (0,\infty) \times (0,T) \\ &u(x,T,A_T) = f(x, \frac{A_T}{T}), \qquad\qquad x \in (0,\infty)\end{aligned}$$

where x is a stock price, r is interest rate, σ is the asset volatility, T the expiration date, $A = \int_0^T x(t)dt$ is a running sum of the asset price and $f(x, \frac{A_T}{T})$ is the payoff function depends on the type of the arithmetic asian option.

In reference [12] the transformations

$$u(x,t) = x^n f(\mathbf{x},t)e^{b\frac{x}{A}},\ \mathbf{x} = \frac{x}{A^2},\ n = -\frac{r}{\sigma^2},\ b = \frac{2}{\sigma^2}$$

were proposed. With these, the last equation becomes a parabolic partial differential equation and moreover

$$f(\mathbf{x},t) = e^{-\frac{1}{2}(r-\frac{r^2}{\sigma^2})t} g(\mathbf{x},t)$$

where $g(\mathbf{x},t)$ satisfies

$$g_t + \frac{1}{2}\sigma^2\mathbf{x}^2 g_{\mathbf{xx}} = 0.$$

If we follow [12] again, with the changes $\tau = T - t$ and $z = \ln \mathbf{x}$ the Black-Scholes equation for asian options turns into the heat equation:

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial \eta^2}, \quad g(\eta,0) = f(\eta) = f(\eta,T) \tag{3.1}$$

where $\eta = \frac{\sigma}{\sqrt{2}}\xi$ being $\xi = z - \frac{1}{2}\sigma^2\tau$.

Considering the heat semigruoup we see that the solution of equation (3.1) is given by

$$g(\eta,\tau) = (G_\tau * f)(\eta) = (U_\tau f)(\eta),$$

Doing the changes of variables backward the solution of the Black-Scholes equation for the asian option is

$$u(x,t) = x^n f(\mathbf{x},t)e^{b\frac{x}{A}} = x^n e^{-\frac{1}{2}(r-\frac{r^2}{\sigma^2})t} g(\mathbf{x},t)e^{b\frac{x}{A}},$$

i.e. the solution is given by

$$(U_t f)(x) = (U_{T-t}f)(\frac{\sigma}{\sqrt{2}}[\ln(\frac{x}{A^2}) - \frac{1}{2}\sigma^2(T-t)]);$$

here U_t it is the heat semigroup on $(0,\infty)$, that is, $U_t f = G_t * f$, where $G_t(x) = \frac{1}{\sqrt{4\pi t}} exp(\frac{-x^2}{4t})$.

3.3. Vanilla Option

Sometimes an option will be called a vanilla option (no-exotic option), which simply means that it has no special features. It has normal terms, including strike price and expiration date. For the moment we restrict our attention to the vanilla european call $c(x,t)$, with exercise price E and expiry date T.

The Black-Scholes equation and boundary conditions for a european call with value $c(x,t)$ are, for $(x,t) \in (0,\infty) \times (0,T)$

$$c_t(x,t) + \frac{1}{2}\sigma^2 x^2 c_{xx}(x,t) + rxc_x(x,t) - rc(x,t) = 0,$$

$$c(0,t) = 0,\; c(x,t) \sim x - Ee^{-r(T-t)},\; as\ x \to \infty$$

$$c(x,T) = max(x-E,0).$$

If we do the changes

$$x = Ee^z, \quad t = T - \frac{2\tau}{\sigma^2}, \quad c(x,t) = Ev(z,\tau),$$

the Black-Scholes equation transforms, in this case, into the partial differential equation

$$v_\tau = v_{zz} + (k-1)v_z - kv, \quad k = \frac{2r}{\sigma^2};$$

with the boundary conditions:

$$v(z,0) = max(e^z - 1, 0),$$

$$\text{and } v(z,\tau) \sim e^z, \;\; v(z,\tau) = 0, \text{ as } x \to \pm\infty \text{ respectively.}$$

We see that the solution is given by the C_0-semigroup ("heat like")

$$u(z,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(y\sqrt{2\tau} + z) e^{-\frac{1}{2}y^2} dy$$

with $y = \frac{x-z}{\sqrt{2\tau}}$, $u_0(z) = max(e^{\frac{1}{2}(k+1)z} - e^{\frac{1}{2}(k-1)z}, 0)$.

In the case of a vanilla european put $p(x,t)$ option, the Black-Scholes equation is the same with the unique change in the inicial condition; instead of $c(x,T) = max(x-E,0)$ now the condition is $p(x,T) = max(E-x,0)$.

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