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ON SOME A-STATISTICAL APPROXIMATION PROCESSES

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Abstract: In the present paper we are concerned with A-statistical convergence of two sequences of linear positive operators. The first is of discrete type and the second is of integral type.

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1. Introduction

In [7] Lupaş proposed studying the following sequence of linear and positive operators

$$(\Lambda_n f)(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), \quad x \geq 0,\ f : \mathbb{R}_+ \to \mathbb{R}, \tag{1}$$

where $\mathbb{R}_+ = [0, \infty)$, $(\alpha)_0 = 1$ and $(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$, $k \geq 1$. Concerning the raised problem, in [1] some quantitative estimates for the rate of convergence were given. Also, Agratini [2] presented an integral extension in Kantorovich sense of these operators defined as follows

$$(K_n f)(x) = n 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} \int_{k/n}^{(k+1)/n} f(t)dt, \quad x \geq 0 \tag{2}$$

and f belongs to the class of local integrable function defined on $\mathbb{R}_+$.

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During the last decade, the so-called A-statistical convergence has been shown to be useful in summing non-convergent sequences of positive linear operators. The roots of this approach go back to Cesàro's matrix summability method.

The aim of this note is to study the A-statistical convergence of the above two sequences, $(\Lambda_n)_{n\geq 1}$ and $(K_n)_{n\geq 1}$, respectively.

2. Preliminaries

First of all we recall basic information about the convergence in statistical sense.

A sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ is said to be *statistically convergent* to a real number L if, for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : \ |x_n - L| \geq \varepsilon) = 0,$$

where

$$\delta(S) := \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{n} \chi_S(j)$$

is the density of $S \subseteq \mathbb{N}$. Here χ_S stands for the characteristic function of the set S. We denote this limit by $st - \lim_n x_n = L$.

The idea of statistical convergence was introduced independently by H. Fast [4] and H. Steinhauss [8].

Further on, let $A = (a_{k,n})_{k,n\in\mathbb{N}}$ be a non-negative regular summability matrix. Recall that the regularity conditions on a matrix A are known as Silverman-Toeplitz conditions in the functional analysis.

For a given sequence of real numbers, $x := (x_n)_{n\in\mathbb{N}}$, the sequence $Ax := ((Ax)_k)$ defined by the formula

$$(Ax)_k := \sum_{n=1}^{\infty} a_{k,n} x_n$$

is called the A-transform of x whenever the series converges for each $k \in \mathbb{N}$. A real sequence $(x_n)_{n\in\mathbb{N}}$ is said to be *A-statistically convergent* to the real number L if, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \sum_{n:|x_\mathsf{n} - L|\geq\varepsilon} a_{k,n} = 0.$$

We denote this limit by $st_A - \lim_n x_n = L$, see [5].

In the special case $A = C_1$, Cesàro matrix of order one, the A-statistical convergence reduces to the statistical convergence.

The study of the statistical convergence for sequences of linear positive operators was attempted in the year 2002 by A.D. Gadjiev and C. Orhan [6].

We are going to present the notations and the spaces of functions which will be used throughout the paper. In what follows the symbol I denotes $\mathbb{R}$ or $\mathbb{R}_+$. A function $\rho : I \to \mathbb{R}$ is called a *weight function* if it is continuous on I, $\rho(x) \geq 1$ for $x \in I$ and $\lim\limits_{|x|\to\infty} \rho(x) = \infty$. Set

$$B_\rho(I) = \{f : I \to \mathbb{R} : \ |f(x)| \leq M_f \rho(x), \ (\forall) \ x \in I\},$$

$$C_\rho(I) = \{f \in B_\rho(I) : \ f \text{ is continuous on } I\}.$$

Endowed with the norm $\|\cdot\|_\rho$, where

$$\|f\|_\rho := \sup_{x\in I} \frac{|f(x)|}{\rho(x)},$$

$B_\rho(I)$ and $C_\rho(I)$ are Banach spaces.

By using A-statistical convergence, O. Duman and C. Orhan proved the following Bohman-Korovkin type theorem [3; Theorem 3].

Theorem 1. *Let $A = (a_{k,n})_{k,n\in\mathbb{N}}$ be a non-negative regular summability matrix and let $(L_n)_{n\in\mathbb{N}}$ be a sequence of linear positive operators, $L_n : C_{\rho_1}(\mathbb{R}) \to C_{\rho_2}(\mathbb{R})$, where ρ_1 and ρ_2 are weight functions which satisfy the relation*

$$\lim_{|x|\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = 0. \tag{3}$$

Then

$$st_A - \lim_n \|L_n f - f\|_{\rho_2} = 0 \text{ for any } f \in C_{\rho_1}(\mathbb{R}) \tag{4}$$

if and only if

$$st_A - \lim_n \|L_n F_\nu - F_\nu\|_{\rho_1} = 0, \quad \nu = 0, 1, 2, \tag{5}$$

where $F_\nu(x) = x^\nu \rho_1(x)(1 + x^2)^{-1}$, $\nu = 0, 1, 2$.

Following the line of the proof given by the authors, we easily deduce that in the above theorem $\mathbb{R}$ can be replaced by $\mathbb{R}_+$. Consequently, using our notation, the theorem is working for the set I.

3. Results

For our purpose we choose $I = \mathbb{R}_+$.

For any integer $s \geq 0$ we denote by e_s the monomial of s degree, $e_s(t) = t^s$, $t \in \mathbb{R}_+$.

As regards Λ_n, $n \in \mathbb{N}$, operators, the following identities take place

$$\Lambda_n e_0 = e_0, \quad \Lambda_n e_1 = e_1, \quad \Lambda_n e_2 = e_2 + \frac{2}{n} e_1. \tag{6}$$

The first two relations are taken from [7] and the last relation was proved in [1; Lemma 1]. Also, for K_n, $n \in \mathbb{N}$, operators one has

$$K_n e_0 = e_0, \quad K_n e_1 = e_1 + \frac{1}{2n} e_0, \quad K_n e_2 = e_2 + \frac{3}{n} e_1 + \frac{1}{3n^2} e_0, \tag{7}$$

see [2; Lemma 1].

To establish our results we use the following weight functions

$$\rho_1(x) = 1 + x^2, \quad \rho_2(x) = 1 + x^{2\lambda}, \quad \lambda > 1, \quad x \in \mathbb{R}_+, \tag{8}$$

consequently the identity (3) is satisfied.

Theorem 2. *The operators Λ_n, $n \in \mathbb{N}$, defined by (1) satisfy the following identity*

$$st_A - \lim_n \|\Lambda_n f - f\|_{\rho_2} = 0 \text{ for any } f \in C_{\rho_1}(\mathbb{R}_+), \tag{9}$$

where ρ_1 and ρ_2 are defined by (8).

Proof. We use Theorem 1. On the basis of (8), we get $F_\nu = e_\nu$, $\nu = 0, 1, 2$. According to (6), one has

$$\|\Lambda_n e_k - e_k\|_{\rho_1} = 0 \text{ for } k = 0 \text{ and } k = 1,$$

consequently (5) takes place for $\nu = 0$ and $\nu = 1$. Further on, we have

$$\|\Lambda_n e_2 - e_2\|_{\rho_1} = \frac{2}{n} \sup_{x \geq 0} \frac{x}{1 + x^2} \leq \frac{1}{n}.$$

For a given $\varepsilon > 0$ we define the sets

$$K_1 := \{n \mid \|\Lambda_n e_2 - e_2\|_{\rho_1} \geq \varepsilon\}, \quad K_2 = \left\{n \mid \frac{1}{n} \geq \varepsilon\right\}.$$

Clearly, $K_1 \subset K_2$. Then, for each $j \in \mathbb{N}$, we get

$$\sum_{n \in K_1} a_{j,n} \le \sum_{n \in K_2} a_{j,n}. \tag{10}$$

Since $st_A - \lim_n \frac{1}{n} = 0$, taking in (10) the limit as j tends to infinity, we conclude

$$\lim_{j \to \infty} \sum_{n \in K_1} a_{j,n} = 0$$

which gives relation (5) for $\nu = 2$. By applying Theorem 1, our result (9) follows. □

Theorem 3. *The operators K_n, $n \in \mathbb{N}$, defined by (2) satisfy the following identity*

$$st_A - \lim_n \|K_n f - f\|_{\rho_2} = 0 \text{ for any } f \in C_{\rho_1}(\mathbb{R}_+), \tag{11}$$

where ρ_1 and ρ_2 are defined by (8).

Proof. We check the three conditions from (5). Relations (7) imply

$$\|K_n e_0 - e_0\|_{\rho_1} = 0 \quad \text{and} \quad \|K_n e_1 - e_1\|_{\rho_1} = \frac{1}{2n},$$

consequently (5) holds true for $\nu = 0$ and $\nu = 1$. On the other hand, we can write

$$\|K_n e_2 - e_2\|_{\rho_1} = \sup_{x \ge 0} \left(\frac{3x}{n(1+x^2)} + \frac{1}{3n^2(1+x^2)} \right) \le \frac{3}{2n} + \frac{1}{3n^2} \le \frac{2}{n}.$$

Using the same technique as in the proof of Theorem 2, we get

$$st_A - \lim_n \|K_n e_2 - e_2\|_{\rho_1} = 0.$$

Theorem 1 leads us to the desired result (11). □

4. Concluding Remark

Statistical convergence is a concept more general than the uniform convergence. The advantage of replacing the uniform convergence by statistical convergence consists in fact that the second is able to improve the technique of signal approximation in different functions spaces. We have shown that the uniform approximation processes $(\Lambda_n)_{n \ge 1}$, $(K_n)_{n \ge 1}$ as presented in papers [7], [1], [2], enjoy a more general property, namely the A-statistical approximation process. In fact, for these two sequences the identity (4) was proved.

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