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NUMERICAL AND QUALITATIVE STABILITY ANALYSIS OF RING AND LINEAR NEURAL NETWORKS WITH A LARGE NUMBER OF NEURONS

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Abstract: We describe the stability domain in the parameter space of a ring neural network with an unlimited number of neurons by means of the stability cones. We found that the stability domain expands, when the ring of neurons is broken. Networks are described by matrix delay differential higher order equations.

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1. Introduction

Our aim is to describe the stability domain of a large neural net, artificial and biological, with the ring architecture in its parameter space. Many results have been reported in the literature on the dynamics of ring neural networks (see [1]–[11]). The stability of four-neuron ring has been investigated in [1]–[9]. The ring consisting of arbitrary number of neurons was considered in [11], but they only deal with symmetric interaction strength of neuron with its neighbors.

We consider the asymmetric and antisymmetric interactions of the neuron with the right and left neighbors. We study the changes in the conditions of stability caused by the break in the ring. One more feature of our work is the stability analysis of ring and linear networks with an unlimited number of neurons.

Our main research tool is a stability cone [12]. Similar methods are used to study the stability of discrete-time models of the ring system of neurons in [13] (see also [14]).

The organization of this paper is as follows. Section 2 introduces a model of the ring of neurons and contains our main results on the ring neural network stability. We specify a region in the parameter space where the asymptotic stability of our model holds for any number of neurons in the ring. If the parameters are outside this area, then the system is unstable provided a number of neurons in the ring is sufficiently large. In Section 3, we show that breaking the link in the ring of neurons contributes to the stability of the system. Section 4 provides the description of the numerical methods that lead to the results of Sections 2 and 3. Numerical simulation not only illustrates the theory, but also plays an important role in searching the regions of stability. In Section 5, we prove all the theorems. Section 6 contains a discussion of some directions for future research.

2. Problem Statement and Main Results about the Stability of a Neural Ring

We study the local stability of artificial and biologic neural networks with delayed interactions. Let us start with a linear system (see [1] and [11]) for studying a ring of $n > 2$ neurons (Fig. 1) with two delays:

$$\dot{x}_j(t) + x_j(t) + a\,x_{j-1}(t-\tau_1) + b\,x_{j+1}(t-\tau) = 0 \qquad (j \bmod n). \tag{1}$$

Here the variable x_j $(1 \leqslant j \leqslant n)$ represents the signal of the j-th neuron, the coefficients a and b stand for the connection strength between the neuron and

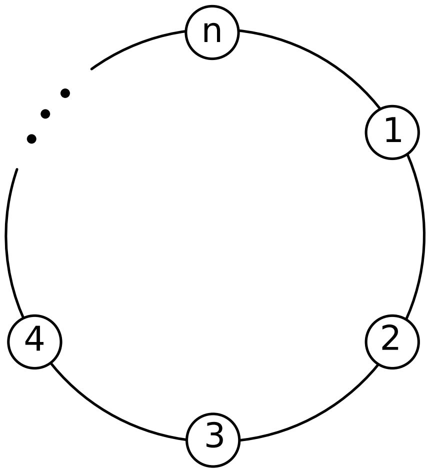

Figure 1: The neural ring.

the left and right neighbors respectively, τ_1 and τ are delays ($0 \leqslant \tau_1, \tau < \infty$). As usual, we call a linear differential equation (asymptotically) stable if his null solution is (asymptotically) stable.

Analytical study of the stability domain of (1) in a parameter space seems to be quite nontrivial, as the scalar equation $\dot{x}(t) + a\,x(t-\tau_1) + b\,x(t-\tau) = 0$ has a very complicated stability domain (see [15] and [16]). Therefore the basic model (1) needs to be simplified. We will consider two modifications of (1), in the first of which $\tau_1 = 0$, in the second $\tau_1 = \tau$:

$$\dot{x}_j(t) + x_j(t) + a\,x_{j-1}(t) + b\,x_{j+1}(t-\tau) = 0 \qquad (j \bmod n), \tag{2}$$

$$\dot{x}_j(t) + x_j(t) + a\,x_{j-1}(t-\tau) + b\,x_{j+1}(t-\tau) = 0 \qquad (j \bmod n). \tag{3}$$

Equation (2) can be used as a model for networks where the delay in neuron interaction with the right neighbor is considerably smaller than with the left one. Equation (3) reflects the situation when the delays in interaction with both neighbors are close to each other. In both equations (2), (3) the connection strengths of neuron with right and left neighbors are not necessarily the same. The latter property distinguishes our model from [1], [8] and [11]. Another difference is that the delay in self connection is not considered in (2), (3).

Systems (2), (3) have the form

$$\dot{x}(t) + Ax(t) + Bx(t-\tau) = 0, \tag{4}$$

$n \times n$ matrices A, B being simultaneously triangularizable. Let us introduce a special notation for $n \times n$ matrix of line shift operator:

$$P = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{5}$$

Systems (2), (3) can be rewritten for $x(t) = (x_1(t), \dots, x_n(t))^T$ as

$$\dot{x}(t) + (I + aP)\, x(t) + bP^{n-1}\, x(t - \tau) = 0, \tag{6}$$

$$\dot{x}(t) + I\, x(t) + (aP + bP^{n-1})\, x(t - \tau) = 0, \tag{7}$$

respectively, where I is a unit $n \times n$ matrix. Mori et al. [17] proved that (4) is asymptotically delay-independently stable provided that

$$\min_{1 \leqslant j \leqslant n} \{\alpha_{jj} - \sum_{k=1, k \neq j}^{n} |\alpha_{jk}|\} > \max_{1 \leqslant j \leqslant n} \sum_{k=1}^{n} |\beta_{jk}|, \tag{8}$$

α_{jk}, β_{jk} being entries of A, B respectively. We can rephrase the results as follows.

Theorem 1. *[17]. If $|a| + |b| < 1$, then both (5), (6) and (5), (7) are asymptotically stable for all $n > 2$ and $\tau \geqslant 0$.*

Our first result based on the stability cone is the following.

Theorem 2. *If $|a + b| > 1$, then both (5), (6) and (5), (7) are unstable for all $\tau \geqslant 0$ provided n is sufficiently large.*

Theorem 2 will be proved in Section 5. Theorems 1 and 2 do not give information about the behavior of (5), (6) and (5), (7) when both inequalities $|a + b| < 1$ and $|a| + |b| > 1$ take place. For this case we made the geometrical and numerical experiments described in Section 4. Here we present the results of these experiments.

The stability domains in a-b plane are shown in Figures 2 and 3 for certain values of τ. The domain D_τ extends the domain $|a| + |b| < 1$ to the North-West and South-East up to the boundaries, depending on τ, shown in Figures 2 and 3. If (a, b) lies inside D_τ in Figure 2, then system (5), (6) is asymptotically stable for all $n > 2$. If (a, b) lies outside D_τ in Figure 2, then (5), (6) is unstable provided n is large enough.

Analogous assertions are valid for Figure 3 and system (5), (7).

The domain D_τ is central-symmetric: if $(a,b) \in D_\tau$, then $(-a,-b) \in D_\tau$. The stability domain for (5), (6) is wider than that of (5), (7) with the same delay τ.

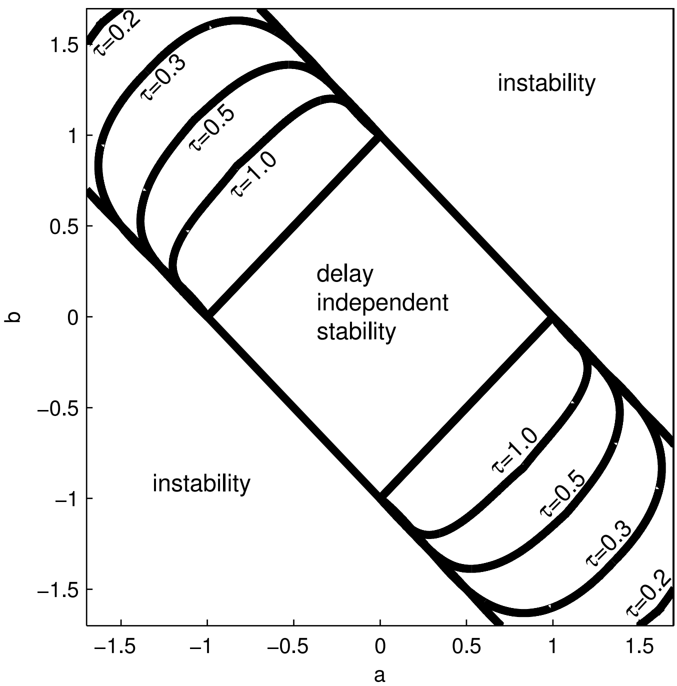

Figure 2: Stability domains of (5), (6) for n large enough.

The line $a = -b$ is important for both systems in question, as the stability points are concentrated in a neighborhood of this line. It is then natural to consider the following two systems:

$$\dot{x}_j(t) + x_j(t) + a\,(x_{j-1}(t) - x_{j+1}(t-\tau)) = 0 \qquad (j \bmod n), \tag{9}$$

$$\dot{x}_j(t) + x_j(t) + a\,(x_{j-1}(t-\tau) - x_{j+1}(t-\tau)) = 0 \qquad (j \bmod n). \tag{10}$$

Definition 3. A real-valued function $a_1(\tau)$ is said to be a stability boundary of (9) for n large enough, if (9) is asymptotically stable for all $n > 2$ provided that $|a| < a_1(\tau)$, and simultaneously (9) is unstable for n large enough,

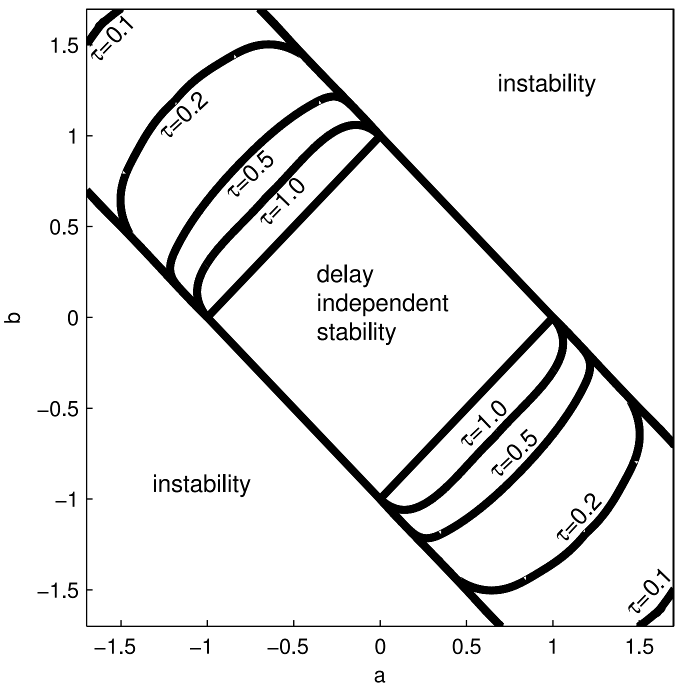

Figure 3: Stability domains of (5), (7) for n large enough.

if $|a| > a_1(\tau)$. Let us define by analogy $a_2(\tau)$ as a stability boundary of (10) for n large enough in the same way as $a_1(\tau)$ for (9).

The results of numerical and geometric experiments for (9), (10) are given in Table 1. Obviously, $\lim_{\tau\to\infty} a_1(\tau) = \lim_{\tau\to\infty} a_2(\tau) = 1/2$. Next theorem deals with the behavior of (9), (10) with $\tau \to 0$ which is now not quite trivial.

Theorem 4.

$$\lim_{\tau\to 0} a_1(\tau)\sqrt{2\tau} = \lim_{\tau\to 0} a_2(\tau)2\sqrt{\tau} = 1. \quad (11)$$

Theorem 4 will be proved in Section 5. Table 1 confirms the estimates given in Theorem 4. For example, Table 1 yields $a_1(\tau)\sqrt{2\tau}$ value 1.0017 and $a_2(\tau)2\sqrt{\tau}$ value 1.0033 with $\tau = 0.01$.

τ	0.01	0.1	0.2	0.5	1	2	3
$a_1(\tau)$	7.0830	2.2742	1.6337	1.0829	0.8229	0.6597	0.5988
$a_2(\tau)$	5.0166	1.6337	1.1921	0.8226	0.6595	0.5678	0.5380

Table 1: Stability boundaries of (9), (10) for n large enough.

3. Break in the Ring

If the connection in a ring is broken between, say, first and last neurons we get a linear neural network (Figure 4) [18], [19]. One may expect that a linear neural network behave the same way as a ring net when the number of neurons in both nets is unbounded, since a line can be considered as limit case of a circle with infinite radius. In this Section we will show that these expectations do not take place.

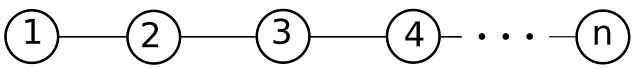

Figure 4: The linear neural network.

Equation (5), (6) with deleted connections between x_1 and x_n transforms to some equation of the form (4), where the matrices A, B are not simultaneously triangularizable in general. Our stability cone method does not work for the stability analysis of such systems. That is why we restrict ourselves to the analysis of (5), (7) with a broken connection. So, consider the following analog of (5), (7):

$$\begin{aligned} \dot{x}_1(t) + x_1(t) + b\,x_2(t-\tau) &= 0, \\ \dot{x}_j(t) + x_j(t) + a\,x_{j-1}(t-\tau) + b\,x_{j+1}(t-\tau) &= 0, \quad 1 < j < n, \\ \dot{x}_n(t) + x_n(t) + a\,x_{n-1}(t-\tau) &= 0, \qquad n > 2. \end{aligned}$$

Rewrite the system above as

$$\dot{x}(t) + I\,x(t) + Q\,x(t-\tau) = 0, \tag{12}$$

where Q is $n \times n$ matrix of the form

$$Q = \begin{pmatrix} 0 & b & 0 & \dots & 0 & 0 \\ a & 0 & b & \dots & 0 & 0 \\ 0 & a & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & b \\ 0 & 0 & 0 & \dots & a & 0 \end{pmatrix}. \tag{13}$$

In order to formulate the next theorem, we need to define a real-valued function $F(\tau)$ of the delay $\tau \in (0, \infty)$:

$$F(\tau) = \frac{1}{4\sin^2 \omega(\tau)}, \tag{14}$$

where $\omega(\tau)$ is the least positive root of the equation

$$\tau = \omega \tan \omega. \tag{15}$$

Theorem 5. *1. If* $|ab| < \frac{1}{4}$, *then* (12), (13) *is asymptotically stable for all* $n > 2$ *and* $\tau \geqslant 0$.
2. If $ab > \frac{1}{4}$, *then* (12), (13) *is unstable for all* $\tau \geqslant 0$ *provided* n *is sufficiently large.*
3. If $ab < 0$ *and* $|ab| < F(\tau)$, *then* (12), (13) *is asymptotically stable for all* $n > 2$.
4. If $ab < 0$ *and* $|ab| > F(\tau)$, *then* (12), (13) *is unstable provided* n *is sufficiently large.*

Theorem 5 will be proved in Section 5.

The delay-independent stability domain drown by first part of Theorem 5 is wider than the domain $|a| + |b| < 1$ given by sufficient stability condition (8). Figure 5 shows the results of Theorem 5.

Comparing Figures 3 and 5 we see that the break of ring expands the stability region of neural network. One more observation: if either $a = 0$ or $b = 0$, then the stability of (12), (13) is provided by Theorem 5. This means that if the ring is broken and if the interaction of any neuron with the right (left) neighbor is blocked, then the neural network is delay-independently stable. This effect is absent in uncorrupted ring network (see Theorem 2).

Consider the case $a = -b$ in (12), (13). Put $H(\tau) = \sqrt{F(\tau)}$ (see (14), (15)). It is easily seen that $\lim_{\tau\to\infty} H(\tau) = 1/2$ and $\lim_{\tau\to 0} H(\tau)2\sqrt{\tau} = 1$.

$H(\tau)$ plays the same role in (12), (13) with $a = -b$, as $a_2(\tau)$ in (10) (see Definition 1). Comparison between $H(\tau)$ and $a_2(\tau)$ (see (11)) reveals only a

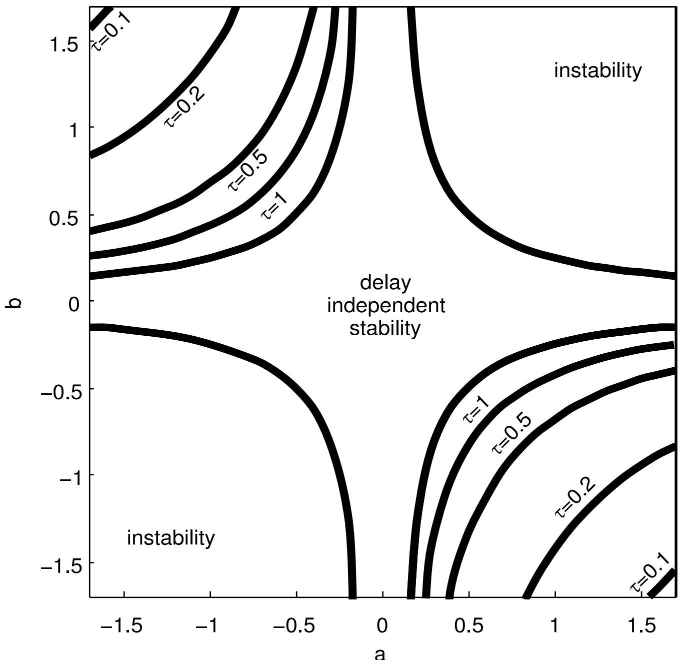

Figure 5: Stability domains of (12), (13) for n large enough.

little difference in the stability conditions between closed and broken neural network, if the connections of every neuron with neighbors are antisymmetric, i.e. if $a = -b$ in (12), (13).

4. Numerical Methods

Let us describe some ideas on how to use the stability cone [12] for stability diagnosis of (4).

Definition 6. [12]. The stability cone for (4) is a set of points $M = (u_1, u_2, u_3) \in \mathbb{R}^3$ such that

$$u_1 + iu_2 = (i\omega - h)\exp(-i\omega), \qquad u_3 = h, \tag{16}$$

where h, ω satisfy the inequalities

$$h \geqslant -\omega/\tan\omega, \quad -\pi < \omega < \pi. \tag{17}$$

We assume A, B in (4) to be simultaneously triangularizable $n \times n$ matrices. After making a simultaneous transformation of A, B, we get the triangle forms A_T, B_T with the diagonal entries λ_j, μ_j $(1 \leqslant j \leqslant n)$ of the matrices A_T, B_T respectively.

Construct a set of n points $M_j = (u_{1j}, u_{2j}, u_{3j}) \in \mathbb{R}^3$ such that

$$u_{1j} + iu_{2j} = \tau\mu_j \exp(i\tau \operatorname{Im}\lambda_j), \quad u_{3j} = \tau \operatorname{Re}\lambda_j, \quad 1 \leqslant j \leqslant n. \tag{18}$$

According to [12], system (4) is asymptotically stable if and only if all the points M_j $(1 \leqslant j \leqslant n)$ lie inside the stability cone (16), (17). If there exists a point M_j $(1 \leqslant j \leqslant n)$ lying outside the stability cone, then equation (4) is unstable.

Program 'Stability Cone' [20] is based on algorithm described above. Input data for the program are eigenvalues of A, B in an order determined by their simultaneous linear transformation to the triangle form. The output of the program is a union of intervals

$$\theta = \cup_{k=1}^{N}(\tau_k,\ \tau_{k+1}), \tag{19}$$

such that if $\tau \in \theta$, then (4) is asymptotically stable, and if $\tau \notin [\tau_k,\ \tau_{k+1}]$ for every k $(1 \leqslant k \leqslant N)$, then (4) is unstable.

Stability diagnosis of (2), (3) with an unbounded order n requires a modification of the algorithm described above.

The eigenvalues of P (see (5)) are $\lambda_j = \exp(i\frac{2\pi j}{n})$, $1 \leqslant j \leqslant n$. Therefore after transforming of four circullant matrices $(I + a\,P)$, $b\,P^{n-1}$, I, $a\,P + b\,P^{n-1}$ to diagonal form one receives the following diagonal entries respectively:

$$\lambda_j^{'} = 1 + a\ \exp(i\frac{2\pi j}{n}), \quad \mu_j^{'} = b\ \exp(-i\frac{2\pi j}{n}), \quad 1 \leqslant j \leqslant n, \tag{20}$$

$$\lambda_j^{''} = 1, \quad \mu_j^{''} = a\ \exp(i\frac{2\pi j}{n}) + b\ \exp(-i\frac{2\pi j}{n}), \quad 1 \leqslant j \leqslant n. \tag{21}$$

To analyze the stability of (5), (6) we construct a set of points $M_j^{'} = (u_{1j}^{'}, u_{2j}^{'}, u_{3j}^{'}) \in \mathbb{R}^3$, $1 \leqslant j \leqslant n$, defined by (see (18), (20))

$$u_{1j}^{'} + iu_{2j}^{'} = \tau b\ \exp(i(-\frac{2\pi j}{n} + a\tau \sin\frac{2\pi j}{n})), \quad u_{3j}^{'} = \tau(1 + a\cos\frac{2\pi j}{n}). \tag{22}$$

Similarly, for (5), (7) we define the points $M_j^{''} = (u_{1j}^{''}, u_{2j}^{''}, u_{3j}^{''}) \in \mathbb{R}^3$, by (see (18), (21))

$$u_{1j}^{''} + iu_{2j}^{''} = \tau(a\ \exp(i\frac{2\pi j}{n}) + b\exp(-i\frac{2\pi j}{n})), \quad u_{3j}^{''} = \tau, \quad 1 \leqslant j \leqslant n. \tag{23}$$

To analize the stability of (5), (6) with indefinitely large values of n it makes sense to construct a continuous closed curve $M'(t) = (u_1'(t), u_2'(t), u_3'(t))$, letting $t = \frac{2\pi j}{n}$ in (22):

$$u_1'(t) + iu_2'(t) = \tau b \exp(i(-t + a\tau \sin t)),$$
$$u_3'(t) = \tau(1 + a\cos t), \quad 0 \leqslant t \leqslant 2\pi. \tag{24}$$

Let us construct the curve $M''(t)$ for (5), (7) in the same manner as $M'(t)$ for (5), (6):

$$u_1''(t) + iu_2''(t) = \tau(a \exp(it) + b \exp(-it)),$$
$$u_3''(t) = \tau, \quad 0 \leqslant t \leqslant 2\pi. \tag{25}$$

The set of points M_j' constructed by (22) becomes dense in (24) as $n \to \infty$. Therefore the question about the stability of (5), (6) is reduced to a geometrical problem. Namely, if all the points of (24) lie inside the stability cone (16), (17), then (5), (6) is asymptotically stable for all $n > 2$. If there exists a point of the curve (24) lying outside the cone, then system (5), (6) is unstable for n large enough (see Figure 6).

In turn, this geometrical problem is solved by numerical methods. By (24) we calculate $\arg(u_1'(t) + iu_2'(t))$ for (5), (6) and find one or two points in the cone surface at a height of $h = u_3'(t)$ with the same argument. A comparison of values $|u_1 + iu_2|$ for the points on the curve and on the cone surface gives answer if system (5), (6) is stable. System (5), (7) is studied similarly.

Program 'Stability of Ring Neural Networks' [20] realizes this algorithm. Using this program we constructed the boundaries of the stability domains D_τ in Figures 2 and 3 and stability boundaries $a_1(\tau)$, $a_2(\tau)$ in Table 1.

5. Proofs of the Theorems

5.1. Proof of Theorem 2

Proof. Our proof starts with the observation that the inequality

$$u_1 + u_3 \geqslant 0 \tag{26}$$

holds on the surface of the stability cone and inside the cone. Combining (24) with $a + b > 1$ and $\tau > 0$ yields $u_1'(\pi) + u_3'(\pi) = -\tau(b + a - 1) < 0$. The last inequality shows that point $M'(\pi)$ lies outside the stability cone (see (26)),

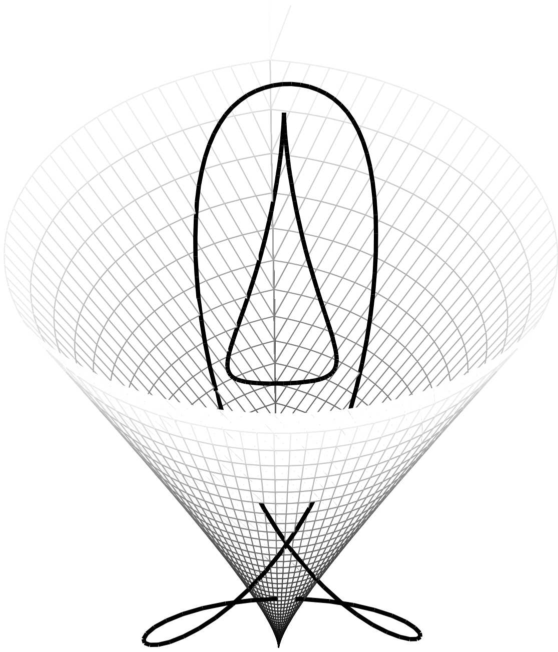

Figure 6: The stability cone and two curves (24). The first curve with $\tau = 1.5$, $a = -1.4$, $b = 0.7$ lies partially outside the cone, therefore (5), (6) unstable for n large enough. The second curve with $\tau = 2$, $a = 0.5$, $b = -0.2$ lies totally inside the cone, therefore (5), (6) is asymptotically stable for all $n > 2$.

hence under the condition $a + b > 1$ system (5), (6) is unstable, if n is large enough. Transformation $t \to t + \pi, a \to -a, b \to -b$ maps the curve (24) to itself, therefore the stability domain of (5), (6) is central-symmetric. Hence from $a + b < -1$ the instability of (5), (6) follows provided n is large enough. Likewise, letting $t = \pi/2$ in (25) we get instability of (5), (7), if n is large enough and $|a + b| > 1$.

Consider the case $\tau = 0$. The eigenvalues ν_j of $I + aP + bP^{n-1}$ are given by (see (20), (21))

$$\nu_j = \lambda_j^{'} + \mu_j^{'} = \lambda_j^{''} + \mu_j^{''} = 1 + a \exp(i\frac{2\pi j}{n}) + b \exp(-i\frac{2\pi j}{n}). \qquad (27)$$

If n is large enough, then the value of $2\pi j/n$ can be made arbitrarily close to π or equal to π by appropriate choice of j. Hence from (27) it follows that $\mathrm{Re}\,\nu_{\mathrm{j}} < 0$ at some j provided $a + b < -1$. Hence, systems (5), (6) and (5), (7) are unstable provided $a + b < -1$ and n large enough. Similarly (5), (6) and

(5), (7) can be proved to be unstable when $a+b>1$. Theorem 2 is proved. □

5.2. Proof of Theorem 4

Proof. 1. To determine the asymptotics of $a_1(\tau)$ we first compute the points of tangency of the stability cone with curve (24) when $b=-a_1(\tau)$, $t=\pi/2$. On curve (24)

$$\arg(u_1'(\frac{\pi}{2})+iu_2'(\frac{\pi}{2}))-\frac{\pi}{2}=\tau a_1(\tau). \tag{28}$$

If $\tau\to 0$ and $h=u_3(\pi/2)=\tau$, then (16), (17) yields

$$\arg(u_1+iu_2)-\frac{\pi}{2}=-\omega+\arg(i\omega-\tau)-\frac{\pi}{2}\sim\frac{\tau}{\omega}-\omega. \tag{29}$$

Here $\alpha\sim\beta$ means $(\alpha/\beta)\to 1$. Tangency of the curve and the cone implies the equality of arguments (28), (29) as well as the equality $|u_1+iu_2|=|u_1'(\pi/2)+iu_2'(\pi/2)|$. Therefore

$$\tau a_1(\tau)\sim\frac{\tau}{\omega}-\omega,\quad \tau a_1(\tau)=\sqrt{\omega^2+\tau^2}\sim\omega. \tag{30}$$

Applying (30) we obtain $\omega\sim\sqrt{\tau/2}$, which gives the desired estimate $a_1(\tau)\sim 1/\sqrt{2\tau}$.

2. In order to get asymptotics of $a_2(\tau)$, put $a=a_2(\tau)$, $b=-a_2(\tau)$ in (25). We obtain the segment

$$u_1''(t)=0,\quad u_2''(t)=2\tau a_2(\tau)\sin t,\quad u_3''(t)=0,\quad 0\leqslant t\leqslant 2\pi. \tag{31}$$

Consider $u_3=\tau$, $u_1=0$ on the surface of cone (16), (17). One gets $\omega^2\sim\tau$, therefore

$$u_2=\omega\cos\omega+\tau\sin\omega\sim\sqrt{\tau}. \tag{32}$$

By (32) and (31) with $t=\pi/2$ we have $a_2(\tau)\sim 1/(2\sqrt{\tau})$. Theorem 4 is proved. □

5.3. Proof of Theorem 5

Proof. Let us begin the proof by studying the eigenvalues of Q.

Lemma 1. *The eigenvalues of $n\times n$ matrix Q are*

$$\mu_{jn}=2\sqrt{ab}\cos\frac{\pi j}{n+1},\quad 1\leqslant j\leqslant n. \tag{33}$$

Proof. Characteristic polynomial of Q (see (13)) is $Q_n(\mu) = |\mu I - Q|$. It satisfies the relations ($n = 1, 2, \ldots$)

$$Q_1(\mu) = \mu, \quad Q_2(\mu) = \mu^2 - ab, \quad Q_{n+2}(\mu) = \mu Q_{n+1}(\mu) - ab\, Q_n(\mu). \tag{34}$$

If $ab = 0$, then the conclusion of lemma is evident. Assume $ab \neq 0$. Let us change the variable to construct a new sequence of polynomials:

$$\mu = 2y\sqrt{ab}, \quad Q_n(2y\sqrt{ab}) = (\sqrt{ab})^n U_n(y). \tag{35}$$

Combining (34) with (35) gives ($n = 1, 2, \ldots$)

$$U_1(y) = 2y, \quad U_2(y) = 4y^2 - 1, \quad U_{n+2}(y) = 2yU_{n+1}(y) - U_n(y). \tag{36}$$

We conclude from (36) that U_n are the the Chebyshev polynomials of the second kind. The zeros of U_n are known:

$$y_{jn} = \cos\frac{\pi j}{n+1}, \quad 1 \leqslant j \leqslant n. \tag{37}$$

Comparison of (37) with (35) completes the proof of Lemma 1. □

We now proceed to the proof of Theorem 5. Eigenvalues of $n \times n$ matrix I are $\lambda_{jn} = 1$. If $|ab| < 1/4$, then from (33) we have

$$\frac{|\mu_{jn}|}{\operatorname{Re}\lambda_{jn}} = 2\sqrt{ab}\,|\cos\frac{\pi j}{n+1}| < 1, \tag{38}$$

which implies the delay-independent asymptotic stability (see [12]). Part 1 of Theorem 5 is proved.

In order to prove parts 2–4 of Theorem 5 we construct the points $M_{jn} = (u_{1jn}, u_{2jn}, u_{3jn}) \in \mathbb{R}^3$ ($1 \leqslant j \leqslant n$), such that (see (33))

$$u_{1jn} + iu_{2jn} = \tau\mu_{jn}\exp(i\tau\operatorname{Im}\lambda_{jn}) = 2\tau\sqrt{ab}\cos\frac{\pi j}{n+1},$$
$$u_{3jn} = \tau\operatorname{Re}\lambda_{jn} = \tau. \tag{39}$$

Suppose that $ab > 1/4$. By (39) there exists n_0, such that for every $n > n_0$

$$u_{1nn} = 2\tau\sqrt{ab}\cos\frac{\pi n}{n+1} < -\tau. \tag{40}$$

At the same time $u_{2nn} = 0$, $u_{3nn} = \tau$.

But the inequality

$$u_1 \geqslant -u_3 = -\tau \tag{41}$$

holds on the surface of the stability cone and inside the cone at a height of $u_3 = \tau$ with $u_2 = 0$.

Comparison of (41) with (40) shows that the point M_{nn} lies outside the stability cone if $n > n_0$. Hence system (12), (13) is unstable provided n is sufficiently large. Part 2 of Theorem 5 is proved.

From (39) by Lemma 1 with $ab < 0$ it follows that

$$u_{1jn} = 0, \quad u_{2jn} = 2\tau\sqrt{|ab|}\cos\frac{\pi j}{n+1}, \quad u_{3jn} = \tau. \tag{42}$$

At the same time the equality

$$|u_2| = \frac{\omega}{\cos\omega}, \tag{43}$$

holds on the surface of the stability cone at a height of $u_3 = \tau$ with $u_1 = 0$. Since $u_1 = 0$, in (43) ω is given by (15). From (42) and (14) we deduce that

$$u_{2jn} = \tau\cos\frac{\pi j}{n+1}\sqrt{\frac{|ab|}{F(\tau)}\frac{1}{\sin\omega}}, \tag{44}$$

If $|ab| < F(\tau)$, then from (44), (43) and (15) we obtain $|u_{2jn}| < |u_2|$ for all j, n, hence all the points M_{jn} lie inside the stability cone, therefore system (12), (13) is asymptotically stable for all $n > 2$. Part 3 of Theorem 5 is proved.

If $|ab| > F(\tau)$, then there exists n_0 such that $\cos\frac{\pi}{n+1} > \sqrt{\frac{F(\tau)}{|ab|}}$ for all $n > n_0$. By (42), it follows that

$$u_{21n} > 2\tau\sqrt{F(\tau)}. \tag{45}$$

From (45), (14) and (15) we have $u_{21n} > \frac{\omega}{\cos\omega}$, which gives $u_{21n} > u_2$, by (43). Hence M_{1n} lies outside the stability cone, therefore (12), (13) is unstable for all $n > n_0$. Theorem 5 is proved. □

6. Conclusion

To the best of authors' knowledge, neither ring nor linear neural networks with an indefinitely large number of neurons have been considered in the literature up to this time. The stability problem of other neural network configurations forming standard graphs (complete, bipartite, star graphs) is worth studying also. Our future work will focus on the stability analysis of discrete-time ring neural networks with a large number of neurons.

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