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INDEPENDENT AND VERTEX COVERING NUMBER ON KRONECKER PRODUCT OF W_n

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Abstract: Let $\alpha(G)$ and $\beta(G)$ be the independent number and vertex covering number, respectively. The Kronecker Product $G_1 \otimes G_2$ of graph of G_1 and G_2 has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2)|u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$. In this paper, let G is a simple graph with order m, we prove that,$\alpha(W_n \otimes G) = \max\left\{n\alpha(G), m\lfloor\frac{n-1}{2}\rfloor\right\}$ and $\beta(W_n \otimes G) = \min\left\{n\beta(G), m(1 + \lceil\frac{n-1}{2}\rceil)\right\}$.

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Key Words: Kronecker product, independent number, vertex covering number

1. Introduction

In this paper, graphs must be simple graphs which can be the trivial graph. Let G_1 and G_2 be graphs. The Kronecker product of graph G_1 and G_2, denote

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by $G_1 \otimes G_2$, is the graph with $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2) | u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$. In [1], there are some properties about Kronecker product of graph. We recall these here.

Proposition 1. Let $H = G_1 \otimes G_2 = (V(H), E(H))$ then
(*i*) $|V(H)| = |V(G_1)||V(G_2)|$
(*ii*) $|E(H)| = 2|E(G_1)||E(G_2)|$
(*iii*) for every $(u, v) \in V(H), d_H((u, v)) = d_{G_1}(u)d_{G_2}(v)$.

Theorem 2. *Let G_1 and G_2 be connected graphs, The graph $H = G_1 \otimes G_2$ is connected if and only if G_1 or G_2 contains an odd cycle.*

Theorem 3. *Let G_1 and G_2 be connected graphs with no odd cycle then $G_1 \otimes G_2$ has exactly two connected components.*

Next we get that general form of graph of Kronecker Product of W_n and a simple graph.

Proposition 4. Let G be a connected graph of order m, the graph of

$$W_n \otimes G \quad is \quad \bigcup_{j=2}^{n} H_{1j} \cup \bigcup_{i=2}^{n-1} H_{i(i+1)} \cup H_{2n}$$

where $V(H_{ij}) = S_i \cup S_j, S_i = \{(i, 1), (i, 2), ..., (i, m)\}$ and $E(H_{ij}) = \{(i, u)(j, v)/uv \in E(G)\}$. Moreover, if G has no odd cycle then each H_{ij} has exactly two connected components isomorphic to G.

Example

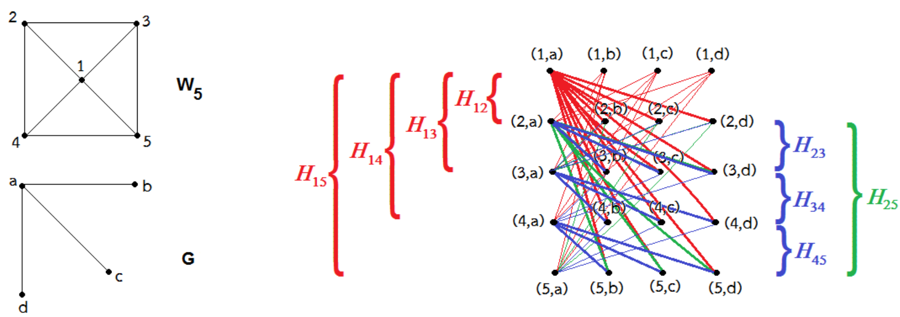


Figure 1: The graph of $W_5 \otimes G$

Next, we give the definitions about some graph parameters. A subset U of the vertex set $V(G)$ of G is said to be an independent set of G if the induced subgraph $G[U]$ is a trivial graph. An independent set of G with maximum number of vertices is called a maximum independent set of G . The number of vertices of a maximum independent set of G is called the independent number of G , denoted by $\alpha(G)$.

A vertex of graph G is said to cover the edges incident with it, and a vertex cover of a graph G is a set of vertices covering all the edges of G . The minimum cardinality of a vertex cover of a graph G is called the vertex covering number of G , denoted by $\beta(G)$.

By definitions of independent number and vertex covering number, clearly that $\alpha(W_n) = \lfloor \frac{n-1}{2} \rfloor$ and $\beta(W_n) = 1 + \lceil \frac{n-1}{2} \rceil$.

2. Independent Number of the Graph of $W_n \otimes G$

We now state proposition and prove lemma before stating our main results. We begin this section by giving the proposition 5 show character of independent set and the lemma 6 show character of independent set for each H_{ij}.

Proposition 5. Let $I(G) = \{v_1, v_2, ..., v_k\}$ is maximum independent set of connected graph G if

(i) v_i is not adjacent with v_j for all $i \neq j$ and $i, j = 1, 2, ..., k$

and (ii) $V(G) - I(G) = \bigcup_{i=1}^{k} N(v_i)$.

Lemma 6. Let $W_n \otimes G = \bigcup_{j=2}^{n} H_{1j} \cup \bigcup_{i=2}^{n-1} H_{i(i+1)} \cup H_{2n}$. For each H_{ij} then $\alpha(H_{ij}) = 2\alpha(G)$

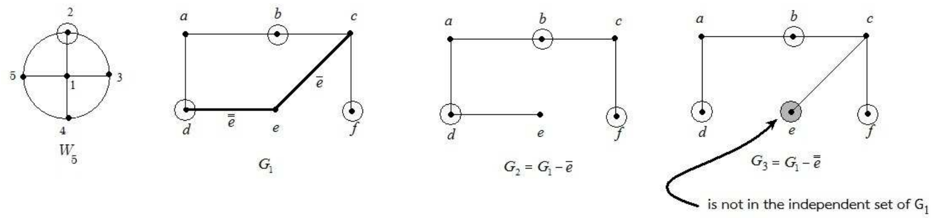
Proof. Suppose G has no odd cycle, by proposition 4, we get H_{ij}=$2G$. So $\alpha(H_{ij}) = 2\alpha(G)$.

If G has odd cycle, for each H_{ij} , vertex $(u_i, v) \in S_i$ and $(u_j, v) \in S_j; i < j$ have $d_{H_{ij}}((u_i, v)) = d_{H_{ij}}(u_j, v)) = d_G(v)$. Let $\overline{W} = \bigcup_{j=2}^{n} \overline{H}_{1j} \cup \bigcup_{i=2}^{n-1} \overline{H}_{i(i+1)} \cup \overline{H}_{2n} = W_n \otimes (G - \overline{e})$ when $\overline{e}$ is an edge in odd cycle, I be the maximum independent set of G . We get $\overline{H_{ij}} = 2(G - \overline{e})$ then

$$\alpha(\overline{H_{ij}}) = 2\alpha(G - \overline{e}) = \begin{cases} 2[\alpha(G) + 1], & \text{if } \overline{e} = xy \text{ then } x \in I,\ y \notin I \\ & \text{and is not adjecent with } z \in I \\ 2\alpha(G), & \text{otherwise.} \end{cases}$$

When we add $\overline{e}$ comeback, in the case $\alpha(G-\overline{e}) = \alpha(G)+1$ be not impossible because the end vertices of edge $\overline{e}$ are in independent set of $G-\overline{e}$, so $\alpha(H_{ij}) = \alpha(\overline{H_{ij}}) - 1$.

Hence $\alpha(H_{ij}) = 2\alpha(G)$. $\square$



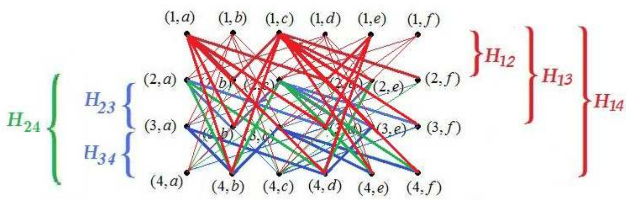


Figure 2: The graph of $W_5 \otimes G_1$

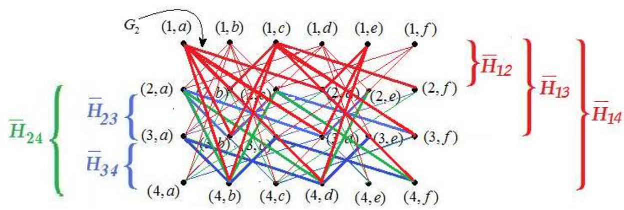


Figure 3: The graph of $W_5 \otimes G_2$

Next, we establish theorem 7 for a maximum independent number of $W_n \otimes G$.

Theorem 7. *Let G be connected graph order m, then $\alpha(W_n \otimes G) = max\{n\alpha(G), m\lfloor\frac{n-1}{2}\rfloor\}$.*

Proof. Let $V(W_n) = \{u_i, i = 1, 2, ..., n\}$, $V(G) = \{v_i, i = 1, 2, ..., m\}$, $S_i = \{(v_i, u_j) \in V(W_n \otimes G)/j = 1, 2, ..., m\}, i = 1, 2, ..., n$ and since $\alpha(W_n) = \lfloor\frac{n-1}{2}\rfloor$.

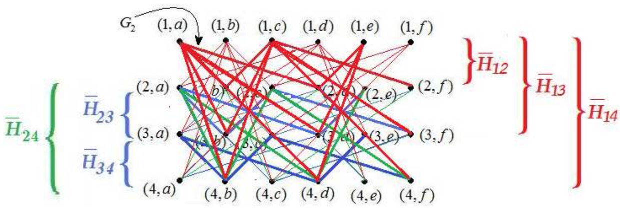


Figure 4: The graph of $W_5 \otimes G_3$

Assume that the maximum independent set of W_n, G be $I_1 = \{u_3, u_5, ..., u_{2\lfloor \frac{n-1}{2} \rfloor +1}\}$, $I_2 = \{v_1, v_2, ..., v_k\}$, respectively.

For H_{1j} , by lemma 6 we have $\alpha(H_{1j}) = 2\alpha(G), j = 1, 2, ..., n$. Since every $H_{1j}, H_{1k}; i \neq k; k = 2, 3, ..., n$ have $\alpha(G)$ common vertices in their independent set which is in S_1. So the independent set of $\bigcup_{j=2}^{n} H_{1j}$ be in $\bigcup_{i=1}^{n} S_i$.

Similarly, for the independent set of $H_{i(i+1)}, i = 2, 3, ..., n-1$ have $\alpha(G)$ common vertices in their independent set which is in $S_2, S_3, ..., S_n$, respectively.

Finally, for the independent set of H_{n2} is in S_n and S_2.

But the independent set of $H_{i(i+1)}$ and H_{n2} are subset of the independent set of H_{1j}, then $\alpha(W_n \otimes G) \geq n\alpha(G)$.

In the author hand, we get another independent set of $W_n \otimes G$ be $\alpha(W_n \otimes G) \geq m\lfloor \frac{n-1}{2} \rfloor$.

Hence $\alpha(W_n \otimes G) \geq max\{n\alpha(G), m\lfloor \frac{n-1}{2} \rfloor\}$.

Suppose that $\alpha(W_n \otimes G) >$max$\{n\alpha(G), m\lfloor \frac{n-1}{2} \rfloor\}$, then there exists $uv_j(or u_h v) \in V(W_n \otimes G) - K(or S_i); j = k+1, k+2, ..., m, h \neq i$, which is not adjacent with another vertices in K (or S_i), $K = \{uv_k / v_k \in I_2\}$ and $S_i = \{u_i v / i = 1, 2, ...$ or $n\}$. It is not true, because for every H_{ij} that has $V(H_{ij}) - K = \bigcup_{j=k+1}^{m} N((u_i, v_j))$.

Hence $\alpha(W_n \otimes G) = max\{n\alpha(G), m\lfloor \frac{n-1}{2} \rfloor\}$. $\square$

3. Vertex Covering Number of the Graph of $W_n \otimes G$

We begin this section by giving the lemma 8 that shows a relation of independent number and vertex covering number.

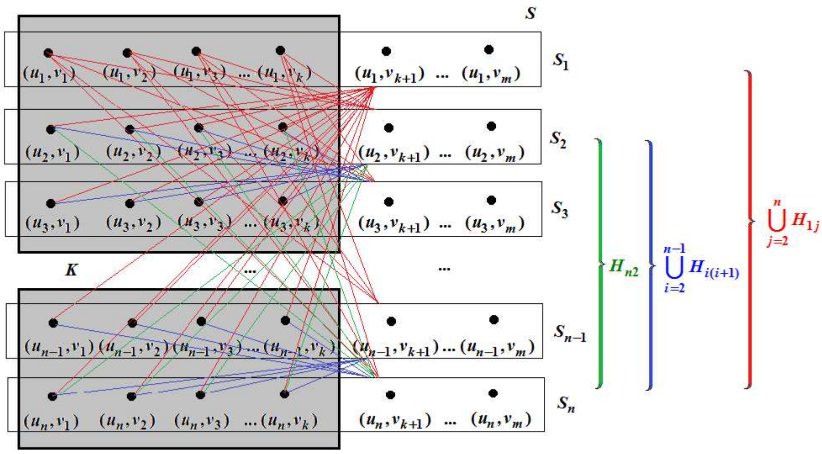


Figure 5: The region of K, S when n is odd

Lemma 8. [2] Let G be a simple graph with order n. Then $\alpha(G) + \beta(G) = n$

Next we establish theorem 9 for a minimum vertex covering number of $W_n \otimes G$.

Theorem 9. *Let G be connected graph order m, then* $\beta(W_n \otimes G) = min\{n\beta(G), m(1 + \lceil \frac{n-1}{2} \rceil)\}$

Proof. By theorem 7 and lemma 8, we can also show that

$$\begin{aligned}
\alpha(W_n \otimes G) + \beta(W_n \otimes G) &= nm \\
max\{n\alpha(G), m\lfloor \frac{n-1}{2} \rfloor\} + \beta(W_n \otimes G) &= nm \\
\beta(W_n \otimes G) &= nm - max\{n\alpha(G), m\lfloor \frac{n-1}{2} \rfloor\} \\
&= nm + min\{-n\alpha(G), -m\lfloor \frac{n-1}{2} \rfloor\} \\
&= min\{n(m - \alpha(G)), m(1 + \lceil \frac{n-1}{2} \rceil)\} \\
&= min\{n\beta(G)), m(1 + \lceil \frac{n-1}{2} \rceil)\}.
\end{aligned}$$

□

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