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# B-SPLINE TOEPLITZ INVERSE UNDER CORNER PERTURBATIONS

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**Abstract:** For Toeplitz matrices associated with degree 3 and 4 uniform B-spline interpolation, the inverse may be analytically known, saving the standard inverse calculations. It generalizes to any degree as a row of the Eulerian numbers triangle.

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## 1. Introduction

Uniform B-spline interpolation is known as a good competitor for convolution interpolation on one dimensional equally spaced data $\{s_i | i = 0, N\}$. In this regard, they play a prominent role in collocation methods for solving partial differential equation. Let an interval $[a, b]$ be uniformly partitioned into knots $a = x_0 < x_1 < \ldots < x_N = b$ with stepsize $h = (b - a)/N$ so that, in each interval, the variable $x = x_i + h\xi$ changes the domain to the unit $\xi \in [0, 1]$. For

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a window on 6 knots, the convolution B-splines kernel leads to:

$$\begin{aligned} 4!\phi_{i-2}(\xi) =&(1-\xi)^4 \\ 4!\phi_{i-1}(\xi) =&(2-\xi)^4 - \binom{5}{1}(1-\xi)^4 \\ 4!\phi_{i}(\xi) =&(2+\xi)^4 - \binom{5}{1}(1+\xi)^4 + \binom{5}{2}\xi^4 \\ 4!\phi_{i+1}(\xi) =&(1+\xi)^4 - \binom{5}{1}\xi^4 \\ 4!\phi_{i+2}(\xi) =&\xi^4 \end{aligned}$$

where the factorial scaling makes the writing of derivatives straightforward:

$$\begin{aligned} 3!\phi^{(1)}_{i-2}(\xi) =& -(1-\xi)^3, & 2!\phi^{(2)}_{i-2}(\xi) =&(1-\xi)^2, \\ 3!\phi^{(1)}_{i-1}(\xi) =& -(2-\xi)^3 + \binom{5}{1}(1-\xi)^3, & 2!\phi^{(2)}_{i-1}(\xi) =&(2-\xi)^2 + \binom{5}{1}(1-\xi)^2, \\ 3!\phi^{(1)}_{i}(\xi) =&(2+\xi)^3 - \binom{5}{1}(1+\xi)^3 + \binom{5}{2}\xi^3, & 2!\phi^{(2)}_{i}(\xi) =&(2+\xi)^2 - \binom{5}{1}(1+\xi)^2 + \binom{5}{2}\xi^2, \\ 3!\phi^{(1)}_{i+1}(\xi) =&(1+\xi)^3 - \binom{5}{1}\xi^3, & 2!\phi^{(2)}_{i+1}(\xi) =&(1+\xi)^2 - \binom{5}{1}\xi^2, \\ 3!\phi^{(1)}_{i+1}(\xi) =&\xi^3, & 2!\phi^{(2)}_{i+1}(\xi) =&\xi^2 \end{aligned}$$

Then interpolation reduces to $S(x,t) = \sum_{i=-2}^{N+1} w_i(t)\phi_i(\xi)$ for $x = x_i + h\xi$ varying in $[x_i, x_{i+1}]$ justifying, a posteriori, the B-splines indexing with respect to the convolution: indeed, $\phi_{i-2}(x)$ stands for $(x_{i+3}-x)^4$, $x \in [x_{i+2}, x_{i+3}]$ whence $(1-\xi)^4$, $\xi \in [0,1]$. On each interval, it simplifies to

$$S_i(\xi,t) = \frac{1}{4!}\begin{bmatrix}1 & \xi & \xi^2 & \xi^3 & \xi^4\end{bmatrix}\begin{bmatrix}1 & 11 & 11 & 1 & 0 \\ -4 & -12 & 12 & 4 & 0 \\ 6 & -6 & -6 & 6 & 0 \\ -4 & 12 & -12 & 4 & 0 \\ 1 & -4 & 6 & -4 & 1\end{bmatrix}\begin{bmatrix}w_{i-2}(t) \\ w_{i-1}(t) \\ w_i(t) \\ w_{i+1}(t) \\ w_{i+2}(t)\end{bmatrix}$$

where $s_i = S_i(0,t) = w_{i-2}(t) + 11w_{i-1}(t) + 11w_i(t) + w_{i+1}(t)$ and $s_{i+1} = S_i(1,t) = w_{i-1}(t) + 11w_i(t) + 11w_{i+1}(t) + w_{i+2}(t) = S_{i+1}(0,t)$ as expected for continuity. The derivatives are given by

$$S_i^{(\xi)}(\xi,t) = \frac{1}{3!}\begin{bmatrix}1 & \xi & \xi^2 & \xi^3 & \xi^4\end{bmatrix}\begin{bmatrix}-1 & -3 & 3 & 1 & 0 \\ 3 & -3 & -3 & 3 & 0 \\ -3 & 9 & -9 & 3 & 0 \\ 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}w_{i-2}(t) \\ w_{i-1}(t) \\ w_i(t) \\ w_{i+1}(t) \\ w_{i+2}(t)\end{bmatrix}$$

$$S_i^{(\xi\xi)}(\xi,t) = \frac{1}{2!}\begin{bmatrix}1 & \xi & \xi^2 & \xi^3 & \xi^4\end{bmatrix}\begin{bmatrix}1 & -1 & -1 & 1 & 0\\ -2 & 6 & -6 & 2 & 0\\ 1 & -4 & 6 & -4 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}w_{i-2}(t)\\ w_{i-1}(t)\\ w_i(t)\\ w_{i+1}(t)\\ w_{i+2}(t)\end{bmatrix}$$

with $S_i^{(\xi)}(1,t) = S_{i+1}^{(\xi)}(0,t)$ for fulfilling smoothness at knots. Overall, the weights $\{w_i\}$ follow a linear Toeplitz structure, in the data $\{s_i\}$, whose length varies with the degree of the spline basis:

$$B_4 = \begin{bmatrix}36 & -12 & 0 & \ldots\ldots\ldots\ldots & 0 & & \\ 13 & 10 & 1 & 0 & \ldots\ldots & 0 & \\ 1 & 11 & 11 & 1 & 0 & \ldots\ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & \ldots\ldots & 0 & 1 & 11 & 11 & 1\\ 0 & \ldots\ldots\ldots & & 0 & 1 & 11 & 11\\ 0 & \ldots\ldots\ldots\ldots & & & 0 & 12 & 12\end{bmatrix}, \quad \frac{1}{4!}B_4\begin{bmatrix}w_0\\ w_1\\ \ldots\\ w_N\end{bmatrix} = \begin{bmatrix}s_0\\ s_1\\ \ldots\\ s_N\end{bmatrix}$$

where $w_{-2}$, $w_{-1}$ (resp. $w_{N+1}$) have been eliminated at either end knot with smoothness conditions $S_0^{(\xi\xi)} = 0$, $S_0^{(\xi\xi\xi)} = 0$ (resp. $S_N^{(\xi\xi)} = 0$), to give $w_{-2} = 3w_0 - 2w_1$, $w_{-1} = 2w_0 - w_1$ (resp. $w_{N+1} = -w_{N-2} + w_{N-1} + w_N$. For cubic case, the recurrence relationship involves a simpler tridiagonal structure, after eliminating $w_{-2} = 2w_{-1} - w_0$ and $w_N = 2w_{N-1} - w_{N-2}$ from smoothness conditions $S_0^{(\xi\xi)} = 0$, $S_N^{(\xi\xi)} = 0$

$$B_3 = \begin{bmatrix}6 & 0 & 0 & \ldots\ldots & & & 0\\ 1 & 4 & 1 & 0 & \ldots & & 0\\ 0 & 1 & 4 & 1 & 0 & . & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & \ldots & & 0 & 1 & 4 & 1\\ 0 & \ldots\ldots & & & 0 & 0 & 6\end{bmatrix}, \quad \frac{1}{3!}B_3\begin{bmatrix}w_{-1}\\ w_0\\ \ldots\\ w_{N-1}\end{bmatrix} = \begin{bmatrix}s_0\\ s_1\\ \ldots\\ s_N\end{bmatrix}$$

Three term recurrences have been thoroughly studied [9, 15, 12, 5, 6] among many authors for eigenvalues computation; in this article, we use the technique of reciprocation of the kernel in order to give the analytical form of each entry in the inverse of both the regular Toeplitz case and the (corners) perturbed case. We refer to the integer sequences obtained during the reciprocation process by their sequence number in the Online Encyclopedia of Integer Sequences [13]

## 2. Uniform Cubic B-Spline: A Symmetric Case

### 2.1. Unperturbed Toeplitz Case

Let us consider the vector
$\phi(z,n) = \begin{bmatrix} z^0 & z & \cdots & z^n \end{bmatrix}^t$ and the inverse definition for the unperturbed $(N+1)\times(N+1)$ Toeplitz matrix $B_3$; stretch it by basis vectors $e_0$, on the left side, and $e_N$ on the right side, such that we get the two different evaluations

$$\begin{aligned} B_3^{-1}\begin{bmatrix} e_0 & B_3 & e_N \end{bmatrix}\phi(z,N+2) =& (1+4z+z^2)B_3^{-1}\phi(z,N) \\ =& \begin{bmatrix} v_0 & I & v_N \end{bmatrix}\phi(z,N+2) \end{aligned}$$

where $v_0$ and $v_N$ are respectively the first and last column of the inverse. In other words, we get the formal inverse as the reciprocation of the kernel $1+4z+z^2 = (1-z)^4\,\mathrm{Li}(-3,z)/z$, a composition with polylogarithm function.

$$B_3^{-1}\phi(z,N) = \frac{v_0 + Iz\phi(z,N) + z^{N+2}v_N}{1+4z+z^2}$$

which formulates any row of the inverse as a convolution between generating functions

$$r_i(z) = \sum_{j\geq 0} r_{ij}z^j = \frac{r_{i0} + z^{i+1} + r_{iN}z^{N+2}}{1+4z+z^2}$$

http://www.research.att.com/ njas/sequences/A001353 with generating function $1/(1-4z+z^2) = \sum a_n z^n$ and coefficients $a_n = \frac{(2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1}}{2\sqrt{3}}$, for $n \geq 0$, leads to an easy convolution between $1/(1+4z+z^2) = \sum(-1)^n a_n z^n$ and $r_{i0}z^0 + z^{i+1} + r_{iN}z^{N+2}$ and to the system in the first column unknowns

$$[z^{N+1}]r_i(z) = 0 = (-1)^{N-i}a_{N-i} + (-1)^{N+1}r_{i0}a_{N+1}$$

From the 3 terms recurrence structure, it is easy to see that the inverse is symmetric and invariant by reversal permutation; therefore, the kernel recurrence applies to the first column and gives all the coefficients

$$r_{ij} = \begin{cases} (-1)^i a_{N-i}/a_{N+1}, & j=0 \\ (-1)^j a_j r_{i0} = (-1)^{i+j}a_i a_{N-i}/a_{N+1}, & i \leq j < N \\ r_{(N-i)0}, & j=N \end{cases}$$

(unperturbed B3−inverse)

### 2.2. B-Spline Corners

Let $\tilde{B}_3$ be the B-spline weighing matrix with perturbations in both diagonal corners; we assume, for the ease of simplifying expressions, that it encompasses the 3! scaling factor, then

$$\begin{aligned}\tilde{B}_3\phi(z,N) &= B_3\phi(z,N) + (2-z)e_0 - z^{N-1}(1-2z)e_N \\ \begin{bmatrix} e_0 & \tilde{B}_3 & e_N \end{bmatrix}\phi(z,N+2) &= (1+4z+z^2)\phi(z,N) + (2-z)ze_0 \\ &\quad - z^N(1-2z)e_N \\ \tilde{B}_3^{-1}\begin{bmatrix} e_0 & \tilde{B}_3 & e_N \end{bmatrix}\phi(z,N+2) &= (1+4z+z^2)\tilde{B}_3^{-1}\phi(z,N) + (2-z)z\tilde{v}_0 \\ &\quad - z^N(1-2z)\tilde{v}_N \\ &= \begin{bmatrix} \tilde{v}_0 & 6I & \tilde{v}_N \end{bmatrix}\phi(z,N+2)\end{aligned}$$

where $\tilde{v}_0$ and $\tilde{v}_N$ are the first and last columns of inverse. Therefore, the kernel method leads to

$$\begin{aligned}\tilde{B}_3^{-1}\phi(z,N) &= \frac{(1-2z+z^2)\tilde{v}_0 + 6Iz\phi(z,N) + z^N(1-2z+z^2)\tilde{v}_N}{1+4z+z^2} \\ \tilde{r}_i(z) = \sum_{j\geq 0}\tilde{r}_{ij}z^j &= \frac{\tilde{r}_{i0}(1-2z+z^2) + 6z^{i+1} + \tilde{r}_{iN}(1-2z+z^2)z^N}{1+4z+z^2}\end{aligned}$$

http://www.research.att.com/ njas/sequences/A001352 whose generating function is $\frac{(1-z)^2}{1+4z+z^2} = \sum(-1)^n\tilde{a}_nz^n$ has long been known to be related with uniform cubic B-spline [7]; however, we give here a more concise proof and an explicit formula for the coefficients $\tilde{a}_i = a_i + 2a_{i-1} + a_{i-2} = 6a_{i-1}$ starting with $\tilde{a}_0 = 1$, as a trivial corollary of http://www.research.att.com/ njas/sequences/A001353 since $\frac{(1-z)^2}{1+4z+z^2} = 1 - \frac{6z}{1+4z+z^2}$, a reason for the scaling trick.

Once more, the fisrt column unknowns are solved after expanding the convolution beyond $N$

$$\begin{aligned}&[z^{N+1}]\tilde{r}_i(z) = 0 = (-1)^{N-i}6a_{N-i} + (-1)^{N+1}r_{i0}6a_N - r_{iN}6a_0 \\ &[z^{N+2}]\tilde{r}_i(z) = 0 = (-1)^{N-i+1}6a_{N-i+1} + (-1)^{N+2}r_{i0}6a_{N+1} + r_{iN}6a_1 \\ &r_{i0} = (-1)^i\frac{4a_{N-i} - a_{N-i+1}}{4a_N - a_{N+1}}, \quad r_{iN} = (-1)^{N-i}\frac{a_Na_{N-i+1} - a_{N+1}a_{N-i}}{4a_N - a_{N+1}}\end{aligned}$$

The reversal invariance, the central symmetry of $(N-1)\times(N-1)$ principal matrix and the 3 terms recurrence gives the closed form of ${B_3}^{-1}$ with corner perturbations, after discarding the 3! scaling technicality

$$r_{ij} = \begin{cases} \frac{(-1)^i}{6}\frac{4a_{N-i}-a_{N-i+1}}{4a_N-a_{N+1}}, & j=0 \\ (-1)^j 6a_{j-1}r_{i0} = (-1)^{i+j}a_{j-1}\frac{4a_{N-i}-a_{N-i+1}}{4a_N-a_{N+1}}, & \begin{cases} i \le j < N \\ j \le i < N \end{cases} \\ r_{(N-i)0}, & j=N \end{cases} \tag{B3−inverse}$$

## 3. Uniform Quartic B-Spline: An Unsymmetric Case

### 3.1. Unperturbed Toeplitz Case

Similarly to cubic case, propagating the Toeplitz structure, towards both ends of main diagonal, we get

$$\begin{aligned} B_4^{-1}\begin{bmatrix} e_0 & 11e_0+e_1 & B_4 & e_N \end{bmatrix}\phi(z,N+3) \\ &= (1+11z+11z^2+z^3)B_4^{-1}\phi(z,N) \\ &= \begin{bmatrix} v_0 & 11v_0+v_1 & I & v_N \end{bmatrix}\phi(z,N+3) \end{aligned}$$

where $v_i$ indexes the columns of the inverse. In other words, we get the formal inverse as the reciprocation of the kernel $(1+z)(1+10z+z^2) = (1-z)^5\operatorname{Li}(-4,z)/z$

$$B_4^{-1}\phi(z,N) = \frac{v_0(1+11z)+v_1z+Iz^2\phi(z,N)+v_Nz^{N+3}}{(1+z)(1+10z+z^2)}$$

$$r_i(z) = \frac{r_{i0}(1+11z)+r_{i1}z+z^{i+2}+r_{iN}z^{N+3}}{(1+z)(1+10z+z^2)}$$

http://www.research.att.com/ njas/sequences/A004189 gives, for $1/(1+10z+z^2) = \sum(-1)^n a_n z^n$, the closed formula $a_n = \frac{(5+2\sqrt{6})^{n+1}-(5-2\sqrt{6})^{n+1}}{4\sqrt{6}}$ in the roots, as is customary for simple roots. Then, the signed partial sums $(1+z)(1+10z+z^2) = \sum(-1)^n\bar{a}_n z^n$ are related to Chebyshev polynomials by the signed version of http://www.research.att.com/ njas/sequences/A097784, namely $\bar{a}_n = \sum_0^n a_j$. so that convolving it with $r_{i0}(1+11z)+r_{i1}z+z^{i+2}+r_{iN}z^{N+3}$ solve the column unknowns.

$$[z^{N+1}]r_i(z) = 0 = (-1)^{1-i}\bar{a}_{N-1-i} + r_{i0}(11\bar{a}_N - \bar{a}_{N+1}) + r_{i1}\bar{a}_N$$

$$[z^{N+2}]r_i(z) = 0 = (-1)^{1-i}\bar{a}_{N-i} + r_{i0}(\bar{a}_{N+1} + \bar{a}_N - 1) + r_{i1}\bar{a}_{N+1}$$

where recurrence $10\bar{a}_N - \bar{a}_{N+1} = \bar{a}_{N-1} - 1$ is used; all the lower Hessenberg entries are solved by the forward 3 terms recurrence starting at $r_{i2} = -11r_{i0} - 11r_{i1}$ with initial solutions
$r_{i0} = (-1)^i\big(\bar{a}_{N+1}\bar{a}_{N-1-i} - \bar{a}_N\bar{a}_{N-i}\big)/\big(\bar{a}_{N+1}\bar{a}_{N-1} - \bar{a}_N^2 - \bar{a}_{N+1} + \bar{a}_N\big)$ and $r_{i1}$

$$r_{ij} = \begin{cases} (-1)^i \frac{\bar{a}_{N+1}\bar{a}_{N-1-i} - \bar{a}_N\bar{a}_{N-i}}{\bar{a}_N(a_{N+1}-a_N)-a_{N+1}(a_N+1)}, j = 0 \\ (-1)^{i+1} \frac{(a_{N+1}+a_N)\bar{a}_{N-1-i} - (\bar{a}_N+\bar{a}_{N-1}-1)a_{N-i}}{\bar{a}_N(a_{N+1}-a_N)-a_{N+1}(a_N+1)}, j = 1 \\ (-1)^{i+2} 11 \frac{a_N\bar{a}_{N-1-i} - (\bar{a}_{N-1}-1)a_{N-i}}{\bar{a}_N(a_{N+1}-a_N)-a_{N+1}(a_N+1)}, j = 2, i \geq 1 \\ -r_{i(j-3)} - 11r_{i(j-2)} - 11r_{i(j-1)}, \quad i \geq j-1 \end{cases}$$

(unperturbed B4−lower inverse)

The antidiagonal symmetry, i.e. $r_{ij} = r_{(N-j)(N-i)}$, further reduces row indices to $j - 1 \leq i < N - j$ for $j$-th column.

Now, let us stretch the matrix columnwise instead, then we get the rules for backward recurrence:

$$\phi^t(z, N+3) \begin{bmatrix} e_0^t \\ B_4 \\ e_{N-1}^t + 11e_N^t \\ e_N^t \end{bmatrix} B_4^{-1} = (1 + 11z + 11z^2 + z^3)\phi^t(z, N)B_4^{-1}$$
$$= \phi^t(z, N+3)\begin{bmatrix} w_0 & I & w_{N-1} + 11w_N & w_N \end{bmatrix}^t$$

for rows $w_i$ of the inverse.

$$\phi^t(z, N)B_4^{-1} = \frac{w_0 + Iz\phi(z, N) + w_{N-1}z^{N+2} + w_N z^{N+2}(11 + z)}{(1 + z)(1 + 10z + z^2)}$$
$$c_j(z) = \sum c_{ij}z^i = \frac{c_{0j} + z^{j+1} + c_{(N-1)j}z^{N+2} + c_{Nj}z^{N+2}(11 + z)}{(1 + z)(1 + 10z + z^2)}$$

whose solving yields the coefficients of upper triangular part, columnwise and, from top to bottom

$$[z^{N+1}]c_j(z) = 0 = c_{0j}\bar{a}_{N+1} + (-1)^{1-j}\bar{a}_{N-j}$$

then, the 3 terms recurrence implies successively

$$c_{ij} = \begin{cases} (-1)^j \frac{\bar{a}_{N-j}}{\bar{a}_{N+1}}, \quad i = 0 \\ (-1)^{j+1} \frac{11\bar{a}_{N-j}}{\bar{a}_{N+1}}, \quad i = 1 \\ (-1)^{j+2} \frac{110\bar{a}_{N-j}}{\bar{a}_{N+1}}, \quad i = 2 \\ (-1)^{j+i} \frac{\bar{a}_i\bar{a}_{N-j}}{\bar{a}_{N+1}}, \quad 2 < i \leq j \end{cases} \qquad \text{(unperturbed B4−upper-inverse)}$$

where, by induction, scaling is nothing but $\bar{a}_n$ sequence.

### 3.2. B-Spline Corners

Once more, for the ease of canceling expressions, we assume the 4! scaling factor included in $\tilde{B}_4$ along with the perturbations in both diagonal corners; it leads to the forward recurrence

$$\begin{aligned}\tilde{B}_4\phi(z,N) &= B_4\phi(z,N) + (25-13z)e_0 + (2-z)e_1 \\ &\quad + z^{N-2}(-1+z+z^2)e_N \\ \tilde{B}_4^{-1}\phi(z,N) &= \big((1+11z-25z^2+13z^3)\tilde{v}_0 + z(1-2z+z^2)\tilde{v}_1 \\ &\quad + 24Iz^2\phi(z,N) + z^N(1-z-z^2+z^3)\tilde{v}_N\big)/(1+11z+11z^2+z^3) \\ &= (1-z)^2\frac{(1+13z)\tilde{v}_0 + z\tilde{v}_1 + (1+z)\tilde{v}_N}{(1+z)(1+10z+z^2)} + \frac{24Iz^2\phi(z,N)}{(1+z)(1+10z+z^2)}\end{aligned}$$

and the backward recurrence

$$\begin{aligned}\phi^t(z,N)\tilde{B}_4 &= \phi^t(z,N)B_4 + (25+2z)e_0^t - (13+z)e_1^t \\ &\quad - z^N e_{N-2}^t + z^N e_{N-1}^t + z^N e_N^t \\ \phi^t(z,N)\tilde{B}_4^{-1} &= \big((1-25z-2z^2)\tilde{w}_0 + z(13+z)\tilde{w}_1 + 24Iz\phi(z,N) \\ &\quad + z^{N+1}\tilde{w}_{N-2} - z^{N+1}(1-z)\tilde{w}_{N-1} - z^{N+1}(1-11z-z^2)\tilde{w}_N\big) \\ &\quad /(1+11z+11z^2+z^3)\end{aligned}$$

Using $\bar{a}_{n+1} = 10\bar{a}_n - a_{n-1} + 1$, the forward system simplifies to

$$12\begin{bmatrix}\bar{a}_{N-2}+3\bar{a}_{N-1} & \bar{a}_{N-1}+1/12 & (-1)^{N-1}1 \\ -(\bar{a}_{N-1}+3\bar{a}_N) & -(\bar{a}_N+1/12) & (-1)^N 10 \\ \bar{a}_N+3\bar{a}_{N+1} & \bar{a}_{N+1}+1/12 & (-1)^{N-1}99\end{bmatrix}\begin{bmatrix}r_{i0}\\ r_{i1}\\ r_{iN}\end{bmatrix} = 24(-1)^i\begin{bmatrix}\bar{a}_{N-1-i}\\ -\bar{a}_{N-i}\\ \bar{a}_{N+1-i}\end{bmatrix}$$

After discarding the 4! scaling factor, we get the lower inverse

$$\begin{aligned}r_{i0} &= (-1)^i\frac{(12\bar{a}_{N-2}(\bar{a}_{N-1-i}-10\bar{a}_{N-i}+\bar{a}_{N+1-i}) - 43\bar{a}_{N-1-i} + 34\bar{a}_{N-i} - 3\bar{a}_{N+1-i}}{48(\bar{a}_{N-1}-\bar{a}_{N-2})} \\ &= (-1)^i\frac{12\bar{a}_{N-2} - 4\bar{a}_{N-2-i} + 1}{48a_{N-1}}, \quad 0 \le i < N \\ r_{i1} &= (-1)^{N+i+1}\frac{12(\bar{a}_{N-1}-13\bar{a}_{N-2}) + 516\bar{a}_{N-1-i} - 408\bar{a}_{N-i} + 36\bar{a}_{N+1-i}}{48(\bar{a}_{N-1}-\bar{a}_{N-2})} \\ &= (-1)^{N+i+1}\frac{12(\bar{a}_{N-1}-13\bar{a}_{N-2}) - 48(\bar{a}_{N-i}-10\bar{a}_{N-1-i}) + 36}{48a_{N-1}}, \quad 1 \le i < N\end{aligned}$$

$$\begin{aligned} r_{i2} =& -36r_{i0} - 13r_{i1}, \quad 2 \le i < N \\ r_{i3} =& 12r_{i0} - 10r_{i1} - 11r_{i2}, \quad 3 \le i < N \\ r_{ij} =& -r_{i(j-3)} - 11r_{i(j-2)} - 11r_{i(j-1)}, \quad j \le i < N, \quad 4 \le j < N \\ r_{Nj} =& -r_{(N-1)j} \\ r_{NN} =& \frac{1 - r_{N(N-1)}}{12} \end{aligned} \qquad \text{(B4−lower inverse)}$$

where induction involves upper left corner, then Toeplitz structure, and finally lower right diagonal to fill last row and diagonal in the inverse. Similarly for backward recurrence, we have a system in the columns that simplifies further according to the linear dependencies for first and last rows respectively, $c_{0j} - 3c_{1j} = 0$ and $c_{(N-1)j} + c_{Nj} = 24\delta_{jN}$.

$$\begin{bmatrix} (\bar{a}_{N+1} + 25\bar{a}_N - 2\bar{a}_{N-1}) & -(13\bar{a}_N - \bar{a}_{N-1}) & -1 & 1 & 1 \\ -(\bar{a}_{N+2} + 25\bar{a}_{N+1} - 2\bar{a}_N) & (13\bar{a}_{N+1} - \bar{a}_N) & 11 & -12 & -22 \\ (\bar{a}_{N+3} + 25\bar{a}_{N+2} - 2\bar{a}_{N+1}) & -(13\bar{a}_{N+2} - \bar{a}_{N+1}) & -110 & 121 & 230 \end{bmatrix} \begin{bmatrix} c_{0j} \\ c_{1j} \\ c_{(N-2)j} \\ c_{(N-1)j} \\ c_{Nj} \end{bmatrix}$$

$$= 24(-1)^j \begin{bmatrix} \bar{a}_{N-j} & -\bar{a}_{N+1-j} & \bar{a}_{N+2-j} \end{bmatrix}^t$$

$$\begin{bmatrix} (\bar{a}_{N+1} - 14\bar{a}_N + \bar{a}_{N-1}) & -1 & 0 \\ (\bar{a}_{N+2} - 14\bar{a}_{N+1} + \bar{a}_N) & -11 & 10 \\ (\bar{a}_{N+3} - 14\bar{a}_{N+2} + \bar{a}_{N+1}) & -110 & 109 \end{bmatrix} \begin{bmatrix} c_{0j} \\ c_{(N-2)j} \\ c_{Nj} \end{bmatrix}$$

$$= 24(-1)^{j-1} \begin{bmatrix} \bar{a}_{N-j} - \delta_{Nj}/12 & \bar{a}_{N+1-j} - \delta_{Nj} & \bar{a}_{N+2-j} - 121\delta_{Nj}/12 \end{bmatrix}^t$$

$$\begin{bmatrix} (1 - 4\bar{a}_N) & -1 & 0 \\ (1 - 4\bar{a}_{N+1}) & -11 & 10 \\ (1 - 4\bar{a}_{N+2}) & -110 & 109 \end{bmatrix} \begin{bmatrix} c_{0j} \\ c_{(N-2)j} \\ c_{Nj} \end{bmatrix}$$

$$= 24(-1)^j \begin{bmatrix} \bar{a}_{N-j} - \delta_{Nj}/12 & \bar{a}_{N+1-j} - \delta_{Nj} & \bar{a}_{N+2-j} - 121\delta_{Nj}/12 \end{bmatrix}^t$$

where the rule $\bar{a}_{n+1} = 10a_n - a_{n-1} + 1$ is used for last equation. Solving for last column entries, we get

$$\begin{bmatrix} c_{0N} \\ c_{(N-2)N} \\ c_{NN} \end{bmatrix} = (-1)^{N-1} \frac{1}{20 + 178\bar{a}_N - 18\bar{a}_{N+1}}$$

$$\begin{bmatrix} -99 & 109 & -10 \\ -139 + 40\bar{a}_N + 36\bar{a}_{N+1} & 109 - 436\bar{a}_N & -10 + 40\bar{a}_N \\ -143 + 44\bar{a}_N & 113 - 444\bar{a}_N + 40\bar{a}_{N+1} & -10 + 44\bar{a}_N - 4\bar{a}_{N+1} \end{bmatrix} \begin{bmatrix} 11 \\ 120 \\ 1199 \end{bmatrix}$$

which agrees with forward recurrence at $c_{NN}$ since $20 + 178\bar{a}_N - 18\bar{a}_{N+1} = 2(\bar{a}_{N-1} - \bar{a}_{N-2}) = 2a_{N-1}$. Notice that $c_{0N} = (-1)^{N+1}/2a_{N-1}$ means that last column numerators are given, upto the alternating signs, by the sequence starting with $[1, 3, 43,$ and 4 terms recurrence $\bar{c}_{i+1} = 11\bar{c}_i - 11\bar{c}_{i-1} + \bar{c}_{i-2}$ of http://www.research.att.com/ njas/sequences/A097784. To summarize, the backward recurrence yields after discarding the 4! scaling factor

$$\begin{aligned} c_{0N} =&(-1)^{N+1}1/48a_{N-1} \\ c_{1N} =&(-1)^{N+1}3/48a_{N-1} \\ c_{2N} =&(-1)^{N}43/48a_{N-1} \\ c_{iN} =&(-1)^{N+i}\big(11\bar{c}_{i-1} - 11\bar{c}_{i-2} + \bar{c}_{i-3}\big)/48a_{N-1} \\ c_{i(N-1)} =&-12c_{iN}, \quad 0 \le i \le N-1 \\ c_{i(N-2)} =&-11c_{i(N-1)} - 12c_{iN} = 120c_{iN}, \quad 0 \le i \le N-2 \\ c_{ij} =&-11c_{i(j+1)} - 11c_{i(j+2)} - c_{i(j+3)}, \quad 0 \le i \le j, \quad 0 \le j \end{aligned} \tag{B4$-$upper inverse}$$

## 4. Discussion

let us note $B_d(z)$ the B-spline kernel of degree $d$, namely the $d-$th row in the Eulerian numbers triangle under conventional indexing for rows and columns; $B_d(z) = (1-z)^{d+1}\,\mathrm{Li}(-d,z)/z = \sum \left\langle {d \atop k} \right\rangle z^{d-k}$. Since the zeros of polylog function are different, reciprocation of the B-spline kernel should simplify as shown in this article for $d = 3, 4$.

### 4.1. Easy Reciprocation

For small degree $d$, we have noticed that $1/B_d(-z) = \sum_{n\ge0} a_n z^n$ follows the simple recurrence $a_{n+1} = \sum_{k=1}^{d-1}(-1)^{k-1}\left\langle {d \atop k} \right\rangle a_{n-k}$ starting with $a_0 = 1$ and $a_k = 0$ for all negative $k$; as a consequence, the kernel reciprocal is just $1/B_d(z) = \sum_{n\ge0}(-1)^n a_n z^n$. This easy recurrence formula is underlied by the known fact that zeros of polylogarithm functions are real and simple in which case reciprocation simplifies to $\prod_i 1/(z-z_i) = \sum_i(\prod_{j\ne i} 1/(z_j - z_i))/(z-z_i)$. Then, either Toeplitz inverse or B-spline matrix inverse amounts to convolve the kernel reciprocal with discrepancies arising at both diagonal corners. Unlike Toeplitz eigenvectors computation [6], it allows reciprocation of higher than 3 terms recurrences; on the other hand, the two sided discrepancies require both vertical and horizontal stretching. In all cases, solving a small sized (less than the degree) system of equations gives the closed form of matrix inverse entries in terms of the coefficients $[z^n]1/B_d(z)$ whose complexity is $o(n)$. It is both accurate and fast ($o(n)$) since all computations may be done in parallel. When $a_n$

coefficient has a closed form, as in the cases dealt with this article, it becomes optimal in $o(1)$.

### 4.2. Harder Reciprocation

Quartic case has been merely solved by decomposition of the kernel into its *conjugate* roots; however, it enters the more general case of reciprocation of a quadratic polynomial adressed in [11, 10]. To handle higher degrees $d = 5, 6 \ldots$, it suggests to use their technique instead; for instance, quintic kernel factorizes as $1+26z+66z^2+26z^3+z^4 = (1+(13+\sqrt{105})z+z^2)(1+(13-\sqrt{105})z+z^2)$ so that convolution of both reciprocations will yield the inverse. It should be stressed here, that there is no need for the general technique since the coefficients are merely given by those of the radical in the reciprocal of $(1+(13-\sqrt{105})z+z^2)$ due to the *conjugation* of both trinomials. For sextic kernel, the same situation arises

$$\frac{1+57z+302z^2+302z^3+57z^4+z^5}{1+z} = (1+(28+\sqrt{540})z+z^2)(1+(28-\sqrt{540})z+z^2)$$

We leave for future work, this powerful extension to higher degrees/diagonals using rational arithmetic only.

### 4.3. Collocation Application

For PDE applications, the closed forms given in this article apply for initial solution of collocation method; for later time step, the interpolation formula keep the diagonal structure but lose the initial Toeplitz structure. It is challenging to carry the double three-term recurrence scheme for tridiagonal case [14], to more diagonals for speeding up inverse computation.

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