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SOME MULTIPLICATIVE PRODUCTS OF n-DIMENSIONAL DISTRIBUTIONS. PART I

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Abstract: This article is the first of a series devoted to the multiplicative product of n-dimensional distributions obtained as a generalization of analogous unidimensional ones by means of a change of variables.

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1. Introduction and Preliminaries Resuls

In theory of distributions is a known fact that there is no definition of the product of two distributions whose validity is as wide as possible.

There is no canonical method to define a product of distributions such that in accordance with the usual product of functions when they are locally summables. A product of distribution compatible with the ordinary product of functions can be found in Schwartz (cf. [8]) and is given for the case be the product of a distribution by a infinitely differentiable function. In this case it is also commutative.

It also seems natural to define the product of two distributions by the integral $\int f(x)T(x)\varphi(x)dx$. However, this definition leads lo difficulties. For

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example, for the case of $|x|^{-1/2}$ that are summable function at the origin and then defined a distribution is such that multiplied by itself is not summable.

To solve the problem of multiplication of one dimensional distributions there are several different methods, among which may be highlight two: a first given by the sequential approach and a second by using Fourier analysis. J. Tysk compare the two definitions and proved that if the product exist by using the Fourier method, it also exist in the sequential sense, and the products are equal (cf. [11]).

The study of distributions concentrated on manifolds, especially those connected with quadratic forms since the definition given by Gelfand, and the products of these has been the subject of deep and varied developments among which we highligh those due to Aguirre who used the Taylor expansion of distributions to give meaning to the product of certain singular distributions cf. [1], [2], and also the ones due Trione cf. [9], [10].

In order to obtain product of n-dimensional distributions we will use an uni-dimensional delta sequences in $\mathbb{R}$ and then by changing variables will be obtained the desired generalizations. The method of the delta sequences that have been actively used by B. Fisher, essentially involves a sequence of functions starting from a fixed function g belonging to the space D of infinitely differentiable functions with compact support as follows.

If S and T are distributions, their multiplicative product is defined by the formula

$$S.T = \lim_{n\to\infty} \{S * g_n(x)\} \{T * g_n(x)\}$$

if the limit exists for every mollifier $g_n(x)$. The symbol $*$ denotes, as usual, convolution.

The functions so defined form a sequence of functions in D which converge to the Dirac delta.

By a mollifier we mean a sequence $g_n(x) = ng(nx)$, where the function has the properties

1. $g(x) \geq 0$
2. $g(x) \in C_0^\infty$
3. $\int_{-\infty}^{\infty} g(x)dx = 1$
4. $g(x) = g(-x)$
5. supp $g(x) = [-1, 1]$
6. $g(x)$ is increasing for $-1 \leq x \leq 0$ and decreasing for $0 \leq x \leq 1$.

In this definition we follow Gonzalez Dominguez (cf. [6]) who entered a slight variant of the definition due to Mikusinski. This method and other analogous also shown in the review article by Li (cf. [7]).

In order to give a sense to all products evaluated in the paragraphs II, or to regularize them, we deal with multidimensional generalizations obtained by means of the following change of variables.

Let ϕ_s be a distribution of one variable s and let $u(x) \in C^\infty(\mathbb{R}^n)$ be such that the $n-1$-dimensional manifold $u(x_1, x_2, ..., x_n) = 0$ has no critical point; $\phi_n(x)$ denotes the distribution defined on $\mathbb{R}^n$ by the formula (called the Leray formula)

$$\int_{\mathbb{R}^n} f(x)\phi_u(x)dx_1...dx_n = \int_{-\infty}^{\infty} \phi_s ds \int_{u(x)=s} f(x)\omega_u(x, dx)$$

here ω_u is an $n-1$-dimensional form on u defined as follows

$$du \wedge d\omega = dx_1 \wedge dx_2 \wedge ... \wedge dx_n$$

the manifold $u(x) = s$ has the orientation such that $\omega_u(x, dx) > 0$.

By this way we extend some unidimensional distributional multiplicative products to certain kinds of n-dimensional distributions called "causal" and "anticausal" distributions.

Definition 1. Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional euclidean space $\mathbb{R}^n$. Let $P(x)$ be a non degenerate quadratic form in n variables of the form

$$P = P(x) = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2 \tag{1.1}$$

where $n = p + q$.

The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\epsilon \to 0} (P \pm i\epsilon |x|^2)^\lambda \tag{1.2}$$

where $\epsilon > 0$, $|x|^2 = x_1^2 + ... + x_n^2$; $\lambda \in \mathbb{C}$.

The distributions $(m^2 + P \pm i0)^\lambda$ are defined in an analogue manner as the distributions $(P \pm i0)^\lambda$. Let us put (cf. [5], p. 289)

$$(m^2 + P \pm i0)^\lambda = \lim_{\epsilon \to 0} (m^2 + P \pm i\epsilon |x|^2)^\lambda \tag{1.3}$$

where ϵ is an arbitrary positive number, and m is a positive real number.

It is useful to state an equivalent definition of the distributions $(m^2 + P \pm i0)^\lambda$. In this definition appear the distributions

$$(m^2 + P)_+^\lambda = \begin{cases} (m^2 + P)^\lambda & \text{if } (m^2 + P) \geq 0 \\ 0 & \text{if } (m^2 + P) < 0 \end{cases} \tag{1.4}$$

$$(m^2 + P)_-^\lambda = \begin{cases} 0 & \text{if } (m^2 + P) > 0 \\ (m^2 + P)^\lambda & \text{if } (m^2 + P) \leq 0 \end{cases} \tag{1.5}$$

We can prove, without difficulty that the following formula is valid (see [3], p. 566)

$$(m^2 + P \pm i0)^\lambda = (m^2 + P)_+^\lambda + e^{\pm i\pi\lambda}(m^2 + P)_-^\lambda \tag{1.6}$$

From this formula we immediately conclude that

$$(m^2 + P \pm i0)^\lambda = (m^2 + P - i0)^\lambda = (m^2 + P)^\lambda \tag{1.7}$$

when $\lambda = k =$ positive integer.

The distribution $(m^2 + P \pm i0)^\lambda$ are entire distributional function of λ. This is the principal difference between the distributions, formally analogue $(P \pm i0)^\lambda$ which have poles at the point $\lambda = -\frac{n}{2} - k$, $k = 0, 1, ...$ It can be proved (cf. [3] p. 573, formula (2.14) and p. 575, formula (3.5)) that

$$(m^2 + P \pm i0)^{-k} = Pf(m^2 + P)^{-k} \mp i\pi \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(m^2 + P), \tag{1.8}$$

$k = 1, 2, ...$

The following formula of multiplicative product is true for every $\lambda, \mu \in \mathbb{C}$, and $m^2 \neq 0$.

$$(m^2 + P \pm i0)^\lambda.(m^2 + P \pm i0)^\mu = (m^2 + P \pm i0)^{\lambda+\mu} \tag{1.9}$$

(cf. [9], p. 23, formula(1,3,16)), and

$$\left[sgn(m^2 + P)\right] \left|m^2 + P\right|^\lambda = (m^2 + P)_+^\lambda - (m^2 + P)_-^\lambda, \tag{1.10}$$

and

$$\left|m^2 + P\right|^\lambda = (m^2 + P)_+^\lambda + (m^2 + P)_-^\lambda, \tag{1.11}$$

and

$$\Big\{(m^2 + P \pm i0)^\lambda \ln^r(m^2 + P + i0)\Big\} . \big\{(m^2 + P \pm i0)^\mu \ln^s(m^2 + P + i0)\big\} = (m^2 + P \pm i0)^{\lambda+\mu} \ln^{r+s}(m^2 + P + i0), \tag{1.12}$$

for all $\lambda, \mu, \lambda+\mu \in \mathbb{C}$, where

$$(m^2+P\pm i0)^\lambda \ln^r(m^2+P+i0) = \frac{\partial^r}{\partial\lambda^r}(m^2+P+i0)^\lambda \tag{1.13}$$

and

$$\ln(m^2+P+i0) = \lim_{\epsilon\to 0}\ln(m^2+P+i\epsilon\,|x|^2) \\ = \ln\left|m^2+P\right| + i\pi H\left(-(m^2+P)\right), \tag{1.14}$$

where

$$H(m^2+P) = \begin{cases} 1 & \text{for } m^2+P>0 \\ 0 & \text{for } m^2+P<0 \end{cases}$$

In particular we have, taking into account (I.7) and (I.14), that

$$(m^2+P+i0)^r\ln(m^2+P+i0) = (m^2+P)^r\ln\left|m^2+P\right| \\ +(-1)^r i\pi(m^2+P)_-^r \tag{1.15}$$

and

$$(m^2+P+i0)^r\ln^2(m^2+P+i0) \\ = (m^2+P)^r\ln^2\left|m^2+P\right| + 2(-1)^r i\pi(m^2+P)_-^r\ln(m^2+P)_- \\ -(-1)^r\pi^2(m^2+P)_-^r. \tag{1.16}$$

From (I.6) we have

$$(m^2+P+i0)^{-r+\frac{1}{2}} = (m^2+P)_+^{-r+\frac{1}{2}} + (-1)^r i(m^2+P)_-^{-r+\frac{1}{2}} \tag{1.17}$$

Then, we obtain

$$(m^2+P+i0)^{-r+\frac{1}{2}}\ln(m^2+P+i0) \\ = (m^2+P)_+^{-r+\frac{1}{2}}\ln(m^2+P)_+ - (-1)^r\pi(m^2+P)_-^{-r+\frac{1}{2}} \\ +(-1)^r i(m^2+P)_-^{-r+\frac{1}{2}}\ln(m^2+P)_- \tag{1.18}$$

and

$$(m^2+P+i0)^{-r+\frac{1}{2}}\ln^2(m^2+P+i0) = (m^2+P)_+^{-r+\frac{1}{2}}\ln^2(m^2+P)_+ -$$

$$-(-1)^r i\pi(m^2+P)_-^{-r+\frac{1}{2}} - 2(-1)^r\pi(m^2+P)_-^{-r+\frac{1}{2}}\ln(m^2+P)_- +$$

$$+(-1)^r i(m^2+P)_-^{-r+\frac{1}{2}}\ln^2(m^2+P) \tag{1.19}$$

By considering the formula ([3], p. 577, form. (4.9))

$$(m^2+P+i0)^{-r}=(m^2+P)^{-r}-\frac{(-1)^{r-1}i\pi}{(r-1)!}\delta^{(r-1)}(m^2+P) \tag{1.20}$$

we have

$$(m^2+P+i0)^{-r}\ln(m^2+P+i0)=(-1)^r i\pi(m^2+P)_-^{-r}-$$
$$-\frac{(-1)^r\pi^2\delta^{(r-1)}(m^2+P)}{2(r-1)!}+(m^2+P)^{-r}\ln\left|m^2+P\right| \tag{1.21}$$

and

$$(m^2+P+i0)^{-r}\ln^2(m^2+P+i0)=\frac{(-1)^{-r-1}i\pi^3\delta^{(r-1)}(m^2+P)}{3(r-1)!}-$$
$$-(-1)^r\pi^2(m^2+P)_-^{-r}+2i\pi(-1)^r(m^2+P)_-^{-r}\ln(m^2+P)_-+$$
$$+(m^2+P)^{-r}\ln^2\left|m^2+P\right|. \tag{1.22}$$

2. Main Results

In this section we will prove some Lemmas that are generalizations of results due by B. Fisher (cf. [4]).

Lemma 1. *For $r,s=1,2,3,...$ we have*

$$(-1)^s\pi(m^2+P)_+^{-r+\frac{1}{2}}(m^2+P)_-^{-s}-\frac{(-1)^{r+s}\pi^2}{2(s-1)!}(m^2+P)^{-r+\frac{1}{2}}.$$
$$.\delta^{(s-1)}+(-1)^r(m^2+P)_-^{-r+\frac{1}{2}}\left[(m^2+P)^{-s}\ln\left|m^2+P\right|\right]$$
$$=(-1)^{r+s}(m^2+P)_-^{-r-s+\frac{1}{2}}\ln(m^2+P)_- \tag{2.1}$$

and

$$(-1)^s\pi(m^2+P)_-^{-r+\frac{1}{2}}(m^2+P)_+^{-s}+\frac{(-1)^r\pi^2}{2(s-1)!}(m^2+P)_+^{-r+\frac{1}{2}}\delta^{(s-1)}(m^2+P)+$$
$$+(-1)^{r+s}(m^2+P)_+^{-r+\frac{1}{2}}\left[(m^2+P)^{-s}\ln\left|m^2+P\right|\right]$$
$$=(-1)^{r+s}(m^2+P)_+^{-r-s+\frac{1}{2}}\ln(m^2+P)_+ \tag{2.2}$$

Proof. From (I.12)we have

$$(m^2+P+i0)^{-r+\frac{1}{2}}\left\{(m^2+P+i0)^{-s}\ln(m^2+P+i0)\right\}$$
$$(m^2+P+i0)^{-r-s-\frac{1}{2}}\ln(m^2+P+i0), \tag{2.3}$$

$r, s = 1, 2, ...$

Equivalently, taking into account the formulae (I.17)(I.21), we have

$$\left\{(m^2+P)_+^{-r+\frac{1}{2}}+(-1)^r i(m^2+P)_-^{-r+\frac{1}{2}}\right\}.$$
$$.\left\{(-1)^s i\pi(m^2+P)_-^{-s}-\frac{(-1)^s\pi^2\delta^{s-1}(m^2+P)}{2(s-1)!}+(m^2+P)^{-s}\ln\left|m^2+P\right|\right\}$$
$$=(m^2+P)^{-r-s+\frac{1}{2}}\ln(m^2+P)_+-(-1)^{r+s}\pi(m^2+P)_-^{-r-s+\frac{1}{2}}+$$
$$+(-1)^{r+s}i(m^2+P)_-^{-r-s+\frac{1}{2}}\ln(m^2+P)_- \tag{2.4}$$

Expanding the equation (II.4) and equating the real parts we obtain

$$\frac{-(-1)^s\pi^2}{2(s-1)!}(m^2+P)_+^{-r+\frac{1}{2}}\delta^{s-1}(m^2+P)+(m^2+P)_+^{-r+\frac{1}{2}}.$$
$$.\left[(m^2+P)^{-s}\ln(m^2+P)\right]-(-1)^{r+s}\pi(m^2+P)_-^{-r+\frac{1}{2}}(m^2+P)_-^{-s}$$
$$=(m^2+P)_+^{-r-s+\frac{1}{2}}\ln(m^2+P)_+-(-1)^{r+s}\pi(m^2+P)_-^{-r-s+\frac{1}{2}} \tag{2.5}$$

Equating the imaginary parts of the equation (II.4) we get

$$(-1)^s\pi(m^2+P)_+^{-r+\frac{1}{2}}(m^2+P)_-^{-s}-\frac{(-1)^{r+s}\pi^2}{2(s-1)!}(m^2+P)^{-r+\frac{1}{2}}.$$
$$.\delta^{(s-1)}+(-1)^r(m^2+P)_-^{-r+\frac{1}{2}}\left[(m^2+P)^{-s}\ln\left|m^2+P\right|\right]$$
$$=(-1)^{r+s}(m^2+P)_-^{-r-s+\frac{1}{2}}\ln(m^2+P)_- \tag{2.6}$$

By replacing (m^2+P) by $-(m^2+P)$ in the equation (II.6) we obtain

$$(-1)^s\pi(m^2+P)_-^{-r+\frac{1}{2}}(m^2+P)_+^{-s}+\frac{(-1)^r\pi^2}{2(s-1)!}(m^2+P)_+^{-r+\frac{1}{2}}\delta^{(s-1)}(m^2+P)+$$
$$+(-1)^{r+s}(m^2+P)_+^{-r+\frac{1}{2}}\left[(m^2+P)^{-s}\ln\left|m^2+P\right|\right]$$
$$=(-1)^{r+s}(m^2+P)_+^{-r-s+\frac{1}{2}}\ln(m^2+P)_+ \tag{2.7}$$

□

Lemma 2. *For* $r, s = 1, 2, 3, ...$ *holds*

$$(-1)^s(m^2+P)_-^{-r+\frac{1}{2}}(m^2+P)^{-s} + \frac{(-1)^r}{(s-1)!}\pi(m^2+P)_+^{-r+\frac{1}{2}}\delta^{(s-1)}(m^2+P)$$

$$= (m^2+P)_-^{-r-s+\frac{1}{2}} \qquad (2.8)$$

and

$$(m^2+P)_+^{-r+\frac{1}{2}}(m^2+P)^{-s} - \frac{(-1)^{r+s}}{(s-1)!}\pi(m^2+P)_-^{-r+\frac{1}{2}}\delta^{(s-1)}(m^2+P)$$

$$= (m^2+P)_+^{-r-s+\frac{1}{2}} \qquad (2.9)$$

Proof. By subtracting $(-1)^{r+s}$ times the equation (II.5) from the equation (II.7), we have (II.8)

And replacing (m^2+P) by $-(m^2+P)$ in the equation (II.8), result (II.9). □

Lemma 3. *For* $r, s = 1, 2, 3, ...$ *is valid*

$$\left|m^2+P\right|^{-r+\frac{1}{2}}(m^2+P)^{-s} + \frac{(-1)^{r+s}\pi}{(s-1)!}\left[sgn(m^2+P)\left|m^2+P\right|^{-r+\frac{1}{2}}\right]\delta^{(s-1)}(m^2+P)$$

$$= (m^2+P)_+^{-r-s+\frac{1}{2}} + (-1)^s(m^2+P)_-^{-r-r+\frac{1}{2}} \qquad (2.10)$$

and

$$\left[sgn(m^2+P)\left|m^2+P\right|^{-r+\frac{1}{2}}\right](m^2+P)^{-s} - \frac{(-1)^{r+s}\pi}{(s-1)!}\left|m^2+P\right|^{-r+\frac{1}{2}}\delta^{(s-1)}(m^2+P)$$

$$= (m^2+P)_+^{-r-s+\frac{1}{2}} - (-1)^s(m^2+P)_-^{-r-s+\frac{1}{2}} \qquad (2.11)$$

Proof. By adding $(-1)^s$ times the equation (II.8) to equation (II.9), we obtain (II.10), and subtracting $(-1)^s$ times the equation (II.8) to the equation (II.9), we have (II.11). □

Lemma 4. *For* $r, s = 1, 2, 3, ...$ *is valid*

$$\pi(m^2+P)_-^{-r+\frac{1}{2}}\left[sgn(m^2+P)(m^2+P)^{-s}\right]$$
$$+ 2(-1)^r(m^2+P)_+^{-r+\frac{1}{2}}\left[(m^2+P)^{-s}\ln\left|m^2+P\right|\right]$$
$$= -(-1)^s\pi(m^2+P)_-^{-r-s+\frac{1}{2}} + 2(-1)^r(m^2+P)_+^{-r-s+\frac{1}{2}}\ln(m^2+P)_+ \qquad (2.12)$$

and

$$\pi(m^2+P)_+^{-r+\frac{1}{2}}\left[sgn(m^2+P)(m^2+P)^{-s}\right]$$
$$+2(-1)^r(m^2+P)_-^{-r+\frac{1}{2}}\left[(m^2+P)^{-s}\ln\left|m^2+P\right|\right]$$
$$=-\pi(m^2+P)_+^{-r-s+\frac{1}{2}}+2(-1)^{r+s}(m^2+P)_-^{-r-s+\frac{1}{2}}\ln(m^2+P)_- \quad (2.13)$$

Proof. Adding $(-1)^{r+s}$ times the equation (II.5) to the equation (II.7) we obtain (II.12) and replacing (m^2+P) by $-(m^2+P)$ in the last equation, it result (II.13). □

Lemma 5. *For* $r, s = 1, 2, 3, ...$ *holds*

$$\pi\left|m^2+P\right|^{-r+\frac{1}{2}}\left[sgn(m^2+P)(m^2+P)^{-s}\right]$$
$$+2(-1)^r\left|m^2+P\right|^{-r+\frac{1}{2}}\left[(m^2+P)^{-s}\ln\left|m^2+P\right|\right]$$
$$=-(-1)^s\pi(m^2+P)_-^{-r-s+\frac{1}{2}}-\pi(m^2+P)_+^{-r-s+\frac{1}{2}}+2(-1)^{r+s}.$$
$$.\left[(m^2+P)_+^{-r-s+\frac{1}{2}}\ln(m^2+P)_+ +(m^2+P)_-^{-r-s+\frac{1}{2}}\ln(m^2+P)_-\right] \quad (2.14)$$

and

$$\pi\left[sgn(m^2+P)\left|m^2+P^{-r+\frac{1}{2}}\right|\right]\left[sgn(m^2+P)(m^2+P)^{-s}\right]+$$
$$+2(-1)^r\left[sgn(m^2+P)\left|m^2+P\right|^{-r+\frac{1}{2}}\right]\left[(m^2+P)^{-s}\ln\left|m^2+P\right|\right]$$
$$=-\pi(m^2+P)_+^{-r-s+\frac{1}{2}}+(-1)^s\pi(m^2+P)_-^{-r-s+\frac{1}{2}}+$$
$$+2(-1)^{r+s}(m^2+P)_-^{-r-s+\frac{1}{2}}\ln(m^2+P)_- -$$
$$-2(-1)^{r+s}(m^2+P)_+^{-r-s+\frac{1}{2}}\ln(m^2+P)_+ \quad (2.15)$$

Proof. Adding the formulas (II.12) and (II.13) we have the following formula(II.14), and subtracting (II.12) to the equation (II.13) we obtain (II.15). □

Lemma 6. *For* $r, s = 1, 2, 3, ...$ *is valid*

$$(m^2+P)_+^{-r+\frac{1}{2}}\left[(m^2+P)^{-s}\ln\left|m^2+P\right|\right]-(-1)^{r+s}\pi(m^2+P)_-^{-r+\frac{1}{2}}(m^2+P)^s$$
$$(m^2+P)_+^{-r+s+\frac{1}{2}}\ln(m^2+P)_+ -(-1)^{r+s}(m^2+P)_-^{-r+s+\frac{1}{2}}\pi \quad (2.16)$$

and

$$(m^2+P)_-^{-r+\frac{1}{2}}\left[(m^2+P)^s \ln\left|m^2+P\right|\right]+(-1)^{r+s}\pi(m^2+P)_-^s(m^2+P)_+^{-r+\frac{1}{2}}$$

$$(-1)^s(m^2+P)_-^{-r+s+\frac{1}{2}}\ln(m^2+P)_- \tag{2.17}$$

Proof. From (I.12) we have

$$(m^2+P+i0)^{-r+\frac{1}{2}}\left\{(m^2+P+i0)^s\ln(m^2+P+i0)\right\}$$

$$=(m^2+P+i0)^{-r+s+\frac{1}{2}}\ln(m^2+P+i0) \tag{2.18}$$

Equivalently, we can write taking into account (I.17), (I.15) and (I.18)

$$\left[(m^2+P)_+^{-r+\frac{1}{2}}+(-1)^r i(m^2+P)_-^{-r+\frac{1}{2}}\right]$$

$$\left[(m^2+P)^s\ln\left|m^2+P\right|+(-1)^s i\pi(m^2+P)_-^s\right]$$

$$(m^2+P)_+^{-r+s+\frac{1}{2}}\ln(m^2+P)_+-(-1)^{r-s}\pi(m^2+P)_-^{-r+s+\frac{1}{2}}+$$

$$+(-1)^{r-s}i(m^2+P)_-^{-r+s+\frac{1}{2}}\ln(m^2+P)_- \tag{2.19}$$

for $r,s=1,2,....$

Expanding the equation (II.19) and equating the real parts and the imaginary parts, we obtain, respectively

$$(m^2+P)_+^{-r+\frac{1}{2}}\left[(m^2+P)^{-s}\ln\left|m^2+P\right|\right]-(-1)^{r+s}\pi(m^2+P)_-^{-r+\frac{1}{2}}(m^2+P)^s$$

$$(m^2+P)_+^{-r+s+\frac{1}{2}}\ln(m^2+P)_+-(-1)^{r+s}(m^2+P)_-^{-r+s+\frac{1}{2}}\pi \tag{2.20}$$

for $r,s=1,2,...,$ and

$$(m^2+P)_-^{-r+\frac{1}{2}}\left[(m^2+P)^s\ln\left|m^2+P\right|\right]+(-1)^{r+s}\pi(m^2+P)_-^s(m^2+P)_+^{-r+\frac{1}{2}}$$

$$(-1)^s(m^2+P)_-^{-r+s+\frac{1}{2}}\ln(m^2+P)_-, \tag{2.21}$$

for $r,s=1,2,....$ □

Lemma 7. *For $r, s = 1, 2, 3, ...,$ we have*

$$(m^2+P)_-^{-r+\frac{1}{2}}\left[(m^2+P)^s \ln\left|m^2+P\right|\right] - (-1)^r \pi (m^2+P)_+^{-r+\frac{1}{2}}(m^2+P)_+^s$$

$$= (-1)^s (m^2+P)_-^{-r+s+\frac{1}{2}} \ln(m^2+P)_- - (-1)^r \pi (m^2+P)_+^{-r+s+\frac{1}{2}} \qquad (2.22)$$

and

$$(m^2+P)_+^{-r+\frac{1}{2}}\left[(m^2+P)^s \ln\left|m^2+P\right|\right] + (-1)^r \pi (m^2+P)_-^{-r+\frac{1}{2}}(m^2+P)_+^s$$

$$(m^2+P)_+^{-r+s+\frac{1}{2}} \ln(m^2+P)_+ \qquad (2.23)$$

Proof. Replacing (m^2+P) by $-(m^2+P)$ in equation (II.20) we obtain (II.22).

And replacing (m^2+P) by $-(m^2+P)$ in equation (II.21) we have (II.23). □

Lemma 8. *For $r, s = 1, 2, 3, ...,$ we have*

$$\left|m^2+P\right|^{-r+\frac{1}{2}}\left[(m^2+P)^s \ln\left|m^2+P\right|\right]$$
$$+ (-1)^{r+s}\pi\left[sgn(m^2+P)\left|m^2+P\right|^{-r+\frac{1}{2}}\right](m^2+P)_-^s$$
$$(m^2+P)_+^{-r+s+\frac{1}{2}} \ln(m^2+P)_+ + (-1)^s (m^2+P)_-^{-r+s+\frac{1}{2}} \ln(m^2+P)_-$$
$$- (-1)^{r+s}\pi(m^2+P)_-^{-r+s+\frac{1}{2}} \qquad (2.24)$$

and

$$\left[sgn(m^2+P)\left|m^2+P\right|^{-r+\frac{1}{2}}\right]\left[(m^2+P)^s \ln\left|m^2+P\right|\right] -$$

$$-(-1)^{r+s}\left|m^2+P\right|^{-r+\frac{1}{2}}(m^2+P)_-^s$$

$$= (m^2+P)_+^{-r+s+\frac{1}{2}} \ln(m^2+P)_+ + (-1)^s (m^2+P)_-^{-r+s+\frac{1}{2}} \ln(m^2+P)_- -$$

$$-(-1)^r \pi (m^2+P)_+^{-r+s+\frac{1}{2}} \qquad (2.25)$$

Proof. Adding (II.20) to (II.21), results (II.24).

Subtracting (II.21) to the equation (II.20), we have (II.25). □

Lemma 9. *For $r, s = 1, 2, 3, ...,$ holds*

$$\left|m^2+P\right|^{-r+\frac{1}{2}}\left[(m^2+P)^s \ln\left|m^2+P\right|\right]+$$

$$+(-1)^r\pi\left[sgn(m^2+P)\left|m^2+P\right|^{-r+\frac{1}{2}}\right](m^2+P)_+^s$$

$$=(m^2+P)_-^{-r+s+\frac{1}{2}}\ln(m^2+P)_- + (-1)^r(m^2+P)_+^{-r+s+\frac{1}{2}}\ln(m^2+P)_+ -$$

$$-\pi(m^2+P)_+^{-r+s+\frac{1}{2}} \tag{2.26}$$

and

$$\left[sgn(m^2+P)\left|m^2+P\right|^{-r+\frac{1}{2}}\right]$$

$$\left[(m^2+P)^s\ln\left|m^2+P\right| - (-1)^r\left|m^2+P\right|^{-r+\frac{1}{1}}(m^2+P)_+^s\right]$$

$$=(m^2+P)_-^{-r+s+\frac{1}{2}}\ln(m^2+P)_- -$$

$$-(-1)^r(m^2+P)_+^{-r+s+\frac{1}{2}}\ln(m^2+P)_+ - \pi(m^2+P)_+^{-r+s+\frac{1}{2}} \tag{2.27}$$

Proof. Replacing (m^2+P) by $-(m^2+P)$ in equation (II.24) we have (II.26). And replacing (m^2+P) by $-(m^2+P)$ in equation (II.25) results (II.27). □

Lemma 10. *For $r, s = 1, 2, 3, ...,$ we have*

$$2(m^2+P)_+^{-r+\frac{1}{2}}\left[(m^2+P)^s\ln(m^2+P)\right] + (-1)^r\pi(m^2+P)_-^{-r+\frac{1}{2}}$$
$$\left[sgn(m^2+P)(m^2+P)^s\right]$$

$$=2(m^2+P)_+^{-r+s+\frac{1}{2}}\ln(m^2+P)_+ - (-1)^{r+s}\pi(m^2+P)_-^{-r+s+\frac{1}{2}} \tag{2.28}$$

and

$$2(m^2+P)_-^{-r+\frac{1}{2}}\left[(m^2+P)^s\ln\left|m^2+P\right|\right] - (-1)^r\pi(m^2+P)_+^{-r+\frac{1}{2}}$$
$$\left[sgn(m^2+P)(m^2+P)^s\right]$$

$$=2(-1)^s(m^2+P)_+^{-r+s+\frac{1}{2}}\ln(m^2+P)_- - (-1)^r\pi(m^2+P)_+^{-r+s+\frac{1}{2}} \tag{2.29}$$

Proof. Adding the equation (II.20) to the equation (II.23) we obtain (II.28). And adding (II.21)to the equation (II.22), it results (II.29). □

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