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ON CONVOLUTIONS OF α-MODIFIED DISTRIBUTIONS

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Abstract: In this paper we give formulae for convolutions of α-modified Poisson distributions, and α-modified binomial distributions. We discuss also convolutions of inflated and truncated of α-modified distributions.

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1. Definitions and Preliminaries

α-modified binomial distribution was introduced by Berg and Jaworski (see [3]) for describing simple characteristics of random mappings. The probability function of the α-modified binomial distribution ($X \sim MB(N, p, \phi)$) is given by

$$P(X = x) = \binom{N}{x} \frac{(p + \alpha\phi)^x q^{N-x}}{(1 + \alpha\phi)^N}, \quad x = 0, 1, \dots, N, \tag{1}$$

where p, $0 < p < 1$, and $\phi \geq 0$ are parameters, $q = 1 - p$, and α refers to Riordan's symbol (see [13]), where in binomial expansions α^k is replaced by $k!$. Thus (1) can also be written as

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$$P(X=x)=\binom{N}{x}\frac{(p+\alpha\phi)^x q^{N-x}}{1+\sum_{k=1}^{N} N!/(N-k)!\phi^k}, \quad x=0,1,\dots,N. \tag{2}$$

A new class of α-modified binomial distributions was introduced by Chakraborty (see [5]). Here $X\sim MB(N,p,q,\phi)$ is defined by

$$P(X=x)=\binom{N}{x}\frac{(p+\alpha\phi)^x q^{N-x}}{(q+p+\alpha\phi)^N}, \quad x=0,1,\dots,N, \tag{3}$$

where $\phi\geq 0,\ p,\ p+\phi\geq 0$, and $q>0$ are parameters and α is defined as above. Here an alternative form of (3) is

$$P(X=x)=\binom{N}{x}\frac{(p+\alpha\phi)^x q^{N-x}}{(q+p)^N+\sum_{k=1}^{N} N!/(N-k)!\phi^k (q+p)^{N-k}} \tag{4}$$

for $x=0,1,\dots,N$.

We see that (3) reduces to (1) when $p+q=1$ and $p>0$. If additionally $\phi=0$ we have the classical binomial distribution with parameters (N,p).

Berg and Jaworski (see [3]) introduced 1-parameter and 2-parameter α-modified Poisson distributions. The probability function of the 2-parameter α-modified Poisson distribution $X\sim MP(\lambda,\psi)$ is given by

$$P(X=x)=\frac{(\lambda+\alpha\psi)^x}{x!}(1-\psi)e^{-\lambda}, \quad x=0,1,\dots, \tag{5}$$

where $\lambda>0$ and ψ are parameters such that $|\psi|<1$ and $\lambda+\psi\geq 0$. The 1-parameter α-modified Poisson distribution can be treated as a special case of (5) when $X\sim MP(-\lambda,\lambda)$ with

$$P(X=x)=\frac{\lambda^x}{x!}(-1+\alpha)^x(1-\lambda)e^{\lambda}, \quad x=0,1,\dots, \tag{6}$$

for $0<\lambda<1$.

We now discuss truncated and inflated distributions, starting from a discrete random variable Y with possible values $0,1,2,\dots$.

Definition 1. X is said to have the zero truncated (truncated at zero) Y-distribution if

$$\begin{aligned}P(X=x)&=P(Y=x|Y>0)\\&=P(Y=x)/(1-P(Y=0)), \quad x=1,2,\dots\ .\end{aligned} \tag{7}$$

Definition 2. For a given integer $s \geq 0$ and given $\delta \in (0, 1\rangle$, X is said to have an inflated Y-distribution if

$$P(X = x) = \begin{cases} \eta + \delta P(Y = x), & x = s, \\ \delta P(Y = x), & x \neq s, \quad x \in \mathbb{N} \cup \{0\}, \end{cases} \tag{8}$$

where $\eta = 1 - \delta$. Thus this inflated distribution is a mixture of the Y-distribution and a degenerate distribution concentrated at $x = s$.

Truncated distributions have been widely studied in the literature, for example by Placket [12], Sund and Vernic [15]. Malik [9] and Ahuja [1] investigated convolutions of truncated binomial distribution. Inflated Poisson distributions have been studied extensively in the literature; see Pandey [11], Grzegórska [6] , Gupta et al. [7], Murat and Szynal [10]. Of particular interest are zero-inflated Poisson models for counts data with extra zeros Hall [8], Angers and Biswas [2], Yip and Yau [17], Chen Xue-Dong [16].

Here we consider zero truncated and inflated α-modified distributions. In the case when $Y \sim MP(\lambda, \psi)$ in (5), the zero truncated X-distribution in (7) is given by

$$P(X = x) = \frac{(\lambda + \alpha\psi)^x}{x!} \frac{(1 - \psi)e^{-\lambda}}{1 - (1 - \psi)e^{-\lambda}}, \quad x = 1, 2, \ldots, \tag{9}$$

and the inflated X-distribution in (8) is given by

$$P(X = x) = \begin{cases} \eta + \delta \frac{(\lambda + \alpha\psi)^x}{x!}(1 - \psi)e^{-\lambda}, \ x = s, \\ \delta \frac{(\lambda + \alpha\psi)^x}{x!}(1 - \psi)e^{-\lambda}, \ x = 0, 1, \ldots, s - 1, s + 1, \ldots \ . \end{cases} \tag{10}$$

In the case when $Y \sim MP(-\lambda, \lambda)$ in (6), the zero truncated X-distribution in (7) is given

$$P(X = x) = \frac{[\lambda(-1 + \alpha)]^x}{x!} \frac{(1 - \lambda)e^{\lambda}}{1 - (1 - \lambda)e^{\lambda}}, \quad x = 1, 2, \ldots \tag{11}$$

and the inflated X-distribution in (8) is given by

$$P(X = x) = \begin{cases} \eta + \delta \frac{[\lambda(-1+\alpha)]^x}{x!}(1 - \lambda)e^{\lambda}, \ x = s, \\ \delta \frac{[\lambda(-1+\alpha)]^x}{x!}(1 - \lambda)e^{\lambda}, \ x = 0, 1, \ldots, s - 1, s + 1, \ldots \ . \end{cases} \tag{12}$$

In the case when $Y \sim MB(N, p, q, \phi)$ in (4), the zero truncated X-distribution in (7) is given by

$$P(X = x) = \frac{\binom{N}{x}(p + \alpha\phi)^x q^{N-x}}{(q + p)^N + \sum\limits_{r=1}^{N} N!/(N - r)!\phi^r (q + p)^{N-r} - q^N} \tag{13}$$

and the inflated X-distribution in (8) is given by

$$P(X=x)=\begin{cases}\eta+\dfrac{\delta\binom{N}{x}(p+\alpha\phi)^x q^{N-x}}{(q+p)^N+\sum\limits_{k=1}^{N}N!/(N-k)!\phi^k(q+p)^{N-k}}, & x=s,\\ \dfrac{\delta\binom{N}{x}(p+\alpha\phi)^x q^{N-x}}{(q+p)^N+\sum\limits_{k=1}^{N}N!/(N-k)!\phi^k(q+p)^{N-k}}, & x=0,...,s-1,s+1,...,N.\end{cases} \tag{14}$$

2. Convolutions of α-Modified Poisson Distributions

Suppose that $X_1, X_2, \ldots, X_m$ are i.i.d. as X. We consider here the distribution of the convolution $Z = X_1 + X_2 + \ldots + X_m$.

In the case of Poisson distributions it is well known that the convolution is also Poisson. Chakraborty (see [5]) showed that there is a similar result for the α-modified Poisson distribution $X \sim MP(-\lambda, \lambda)$, viz. $Z \sim MP_m(-m\lambda, \lambda)$, where from (5)

$$\begin{aligned}P(Z=z)=&\frac{\lambda^z}{z!}(-m+\alpha(m))^z(1-\lambda)^m e^{m\lambda}\\ =&\frac{\lambda^z}{z!}(1-\lambda)^m e^{m\lambda}\left[\sum_{i=0}^{z}\binom{z}{i}(-m)^i\frac{(m+z-i-1)!}{(m-1)!}\right],\\ z=&0,1,2,\ldots,\end{aligned}$$

where $\alpha(m)$ corresponds to α in (1), now with $\alpha^k(m)$ replaced by $(k+m-1)!/(m-1)!$ if $m \geq 1$ and by 0 if $m = 0$ (cf. Chakraborty [5]). We now prove a counterpart of Chakraborty result for $X \sim MP(\lambda, \psi)$.

Theorem 1. *If $X \sim MP(\lambda, \psi)$ then $Z \sim MP_m(\lambda m, \psi)$ with*

$$P(Z=z)=\frac{(\lambda m+\psi\alpha(m))^z}{z!}e^{-\lambda m}(1-\psi)^m, \quad z=0,1,\ldots\ .$$

Proof. From (5) the characteristic function of X is given by

$$\varphi_X(t)=(1-\psi)e^{-\lambda}e^{e^{it}(\lambda+\psi\alpha)},$$

but taking into account the property of the α-Riordan symbol

$$\begin{aligned}\exp(\lambda+\alpha\psi)=&e^{\lambda}\left(1+\alpha\psi+(\alpha\psi)^2/2!+\ldots\right)\\ =&e^{\lambda}(1-\psi)^{-1}, \quad |\psi|<1,\end{aligned}$$

we get the formula

$$\varphi_X(t) = \left((1-\psi)/(1-\psi e^{it})\right) e^{\lambda(e^{it}-1)}.$$

Hence

$$\varphi_Z(t) = \left((1-\psi)/(1-\psi e^{it})\right)^m e^{\lambda m(e^{it}-1)}.$$

Using the inversion formula for characteristic functions (Birnbaum, see [4] p.109) we obtain:

$$\begin{aligned} P(Z=z) &= \lim_{k\to+\infty} \frac{1}{2k}\int_{-k}^{k} e^{-itz}\left((1-\psi)/(1-\psi e^{it})\right)^m e^{\lambda m(e^{it}-1)}dt \\ &= \frac{(\lambda m+\psi\alpha(m))^z}{z!}e^{-\lambda m}(1-\psi)^m, \end{aligned}$$

which completes the proof of Theorem 1. □

Now we consider convolution of zero truncated α-modified Poisson distribution.

Theorem 2. *If X has the distribution in* (9) *then*

$$\begin{aligned} P(Z=z) = &\left(\frac{e^{-\lambda}(1-\psi)}{1-e^{-\lambda}(1-\psi)}\right)^m \sum_{l=1}^{m}(-1)^{m-l}\binom{m}{l}\frac{(\lambda l+\psi\alpha(l))^z}{z!}, \\ &z = m, m+1, \ldots\ . \end{aligned} \tag{15}$$

Proof. From (9) the characteristic function of X is given by

$$\varphi_X(t) = \frac{e^{-\lambda}(1-\psi)}{1-e^{-\lambda}(1-\psi)}\left[e^{e^{it}\lambda}(1-e^{it}\psi)^{-1}-1\right].$$

From this it follows that

$$\varphi_Z(t) = \frac{e^{-\lambda m}(1-\psi)^m}{(1-e^{-\lambda}(1-\psi))^m}\sum_{l=1}^{m}\binom{m}{l}(-1)^{m-l}e^{e^{it}\lambda l}(1-e^{it}\psi)^{-l}.$$

Using the inverse formula for characteristic functions we obtain

$$\begin{aligned} P(Z=z) &= \lim_{k\to\infty}\frac{1}{2k}\int_{-k}^{k} e^{-itz}\varphi_Z(t) \\ &= \frac{e^{-\lambda m}(1-\psi)^m}{(1-e^{-\lambda}(1-\psi))^m}\sum_{l=1}^{m}(-1)^{m-l}\binom{m}{l}\frac{(\lambda l+\psi\alpha(l))^z}{z!}, \end{aligned}$$

which proves (15). □

For 1-parameter α-modified Poisson distribution (11) we have

Theorem 3. *If X has the distribution in* (11) *then*

$$P(Z=z)=\left(\frac{e^{\lambda}(1-\lambda)}{1-e^{\lambda}(1-\lambda)}\right)^m\sum_{l=1}^{m}(-1)^{m-l}\binom{m}{l}\frac{[\lambda(-l+\alpha(l))]^z}{z!},$$
$$z=m,m+1,\dots\ .$$

Now we study convolutions of zero inflated α-modified $MP(\lambda,\psi)$ Poisson distributions in (10) when $s=0$ viz. distribution

$$P(X=x)=\begin{cases}\eta+\delta(1-\psi)e^{-\lambda}, & x=0,\\ \delta\frac{(\lambda+\alpha\psi)^x}{x!}(1-\psi)e^{-\lambda}, & x=1,2,\dots\ .\end{cases} \tag{16}$$

Theorem 4. *If X has the distribution in* (16) *then*

$$P(Z=z)=\begin{cases}\left[\eta+\delta(1-\psi)e^{-\lambda}\right]^m, & z=0,\\ \sum_{l=1}^{m}\binom{m}{l}\eta^{m-l}\delta^l\frac{(\lambda l+\psi\alpha(l))^z}{z!}(1-\psi)^le^{-\lambda l}, & z=1,2,\dots\ .\end{cases} \tag{17}$$

Proof. From (16) the characteristic function of X is given by

$$\varphi_X(t)=\eta+\delta\left((1-\psi)/(1-\psi e^{it})\right)e^{\lambda(e^{it}-1)}.$$

Hence

$$\varphi_Z(t)=\sum_{l=0}^{m}\binom{m}{l}\eta^{m-l}\delta^l\left((1-\psi)/(1-\psi e^{it})\right)^le^{\lambda l(e^{it}-1)}.$$

Using the inversion formula for characteristic functions we obtain:

$$\begin{aligned}P(Z=z)&=\lim_{k\to+\infty}\frac{1}{2k}\int_{-k}^{k}e^{-itz}\varphi_Z(t)dt\\ &=\sum_{l=0}^{m}\binom{m}{l}\eta^{m-l}\delta^l\frac{(\lambda l+\psi\alpha(l))^z}{z!}(1-\psi)^le^{-\lambda l}.\end{aligned}$$

Hence

$$P(Z=0)=\left[\eta+\delta(1-\psi)e^{-\lambda}\right]^m,$$

and

$$P(Z=z)=\sum_{l=1}^{m}\binom{m}{l}\eta^{m-l}\delta^l\frac{(\lambda l+\psi\alpha(l))^z}{z!}(1-\psi)^le^{-\lambda l},$$

which gives (17). □

For the inflated α-modified $MP(-\lambda,\lambda)$ Poisson distribution with

$$P(X=x)=\begin{cases}\eta+\delta(1-\lambda)e^{\lambda}, & x=0,\\ \delta\frac{[\lambda(-1+\alpha)]^{x}}{x!}(1-\lambda)e^{\lambda}, & x=1,2,\dots,\end{cases} \tag{18}$$

we have

Theorem 5. *If X has the distribution in* (18) *then*

$$P(Z=z)=\begin{cases}\left[\eta+\delta(1-\lambda)e^{\lambda}\right]^{m}, & z=0,\\ \sum\limits_{l=1}^{m}\binom{m}{l}\eta^{m-l}\delta^{l}\frac{[\lambda(-l+\alpha(l))]^{z}}{z!}(1-\lambda)^{l}e^{\lambda l}, & z=1,2,\dots\ .\end{cases}$$

Theorem 4 and Theorem 5 are generalizations a result given in Grzegórska [6].

Remark 1. One can observe that the probability function of the zero truncated α-modified $MP(\lambda,\psi)$ Poisson distribution inflated at 0

$$P(X=x)=\frac{\frac{\delta(\lambda+\psi\alpha)^{x}}{x!}(1-\psi)e^{-\lambda}}{1-[1-\delta+\delta(1-\psi)e^{-\lambda}]}=\frac{\frac{(\lambda+\psi\alpha)^{x}}{x!}(1-\psi)e^{-\lambda}}{1-(1-\psi)e^{-\lambda}}$$

is the same as the probability function of the zero truncated α-modified $MP(\lambda,\psi)$ Poisson distribution. Hence, the distribution of their sums is as given in Theorem 2. Similarly, the probability function of the zero truncated α-modified $MP(-\lambda,\lambda)$ Poisson distribution inflated at 0 is given in Theorem 3.

Now we are going to discuss convolution of zero truncated α-modified $MP(\lambda,\psi)$ Poisson distribution inflated at the point 1

$$P(X=x)=\begin{cases}\eta+\delta\frac{(\lambda+\alpha\psi)(1-\psi)e^{-\lambda}}{1-e^{-\lambda}(1-\psi)}, & x=1,\\ \delta\frac{\frac{(\lambda+\alpha\psi)^{x}}{x!}(1-\psi)e^{-\lambda}}{1-e^{-\lambda}(1-\psi)}, & x=2,3,\dots\ .\end{cases} \tag{19}$$

Theorem 6. *If X has the distribution in* (19) *then*

$$P(Z=z)=\begin{cases}\sum\limits_{l=0}^{m}\sum\limits_{r=0}^{l}\binom{m}{l}\binom{l}{r}\delta^{l}\eta^{m-l}(-1)^{l-r}(1-\psi)^{l}e^{-\lambda l}\frac{(\lambda r+\psi\alpha(r))^{l}}{l!}\\ \times\left(1-e^{-\lambda}(1-\psi)\right)^{-l}, \quad z=m,\\ \sum\limits_{l=1}^{m}\sum\limits_{r=0}^{l}\binom{m}{l}\binom{l}{r}\delta^{l}\eta^{m-l}(-1)^{l-r}(1-\psi)^{l}e^{-\lambda l}\frac{(\lambda r+\psi\alpha(r))^{z-m+l}}{(z-m+l)!}\\ \times\left(1-e^{-\lambda}(1-\psi)\right)^{-l}, \quad z=m+1,m+2,\dots\ .\end{cases}$$

Proof. From (19) the characteristic function of X is given by

$$\varphi_X(t) = e^{it}\eta + \delta\frac{e^{-\lambda}(1-\psi)}{1-e^{-\lambda}(1-\psi)}\left[e^{e^{it}\lambda}(1-e^{it}\psi)^{-1}-1\right].$$

Hence

$$\varphi_Z(t) = \left\{e^{it}\eta + \delta\frac{e^{-\lambda}(1-\psi)}{1-e^{-\lambda}(1-\psi)}\left[e^{e^{it}\lambda}(1-e^{it}\psi)^{-1}-1\right]\right\}^m.$$

Using the inverse formula for characteristic functions we obtain

$$\begin{aligned}P(Z=z) &= \lim_{k\to\infty}\frac{1}{2k}\int_{-k}^{k} e^{-itz}\varphi_Z(t)dt \\ &= \sum_{l=0}^{m}\sum_{r=0}^{l}\binom{m}{l}\binom{l}{r}(-1)^{l-r}\eta^{m-l}\delta^l\frac{e^{-\lambda l}(1-\psi)^l}{[1-e^{-\lambda}(1-\psi)]^l} \\ &\times \lim_{k\to\infty}\frac{1}{2k}\int_{-k}^{k} e^{-it(z-m+l)}e^{e^{it}(\lambda r+\psi\alpha(r))}dt,\end{aligned}$$

but

$$\lim_{k\to\infty}\frac{1}{2k}\int_{-k}^{k} e^{-it(z-m+l)}e^{e^{it}(\lambda r+\psi\alpha(r))}dt = \begin{cases}\frac{(\lambda r+\psi\alpha(r))^l}{l!}, & z=m,\\ \frac{(\lambda r+\psi\alpha(r))^{z-m+l}}{(z-m+l)!}, & z=m+1,\dots,\end{cases}$$

which completes the proof. □

For X having the zero truncated α-modified $MP(-\lambda,\lambda)$ Poisson distribution inflated at the point 1 with

$$P(X=x) = \begin{cases}\eta+\delta\frac{\lambda(-1+\alpha)(1-\lambda)e^{\lambda}}{1-e^{\lambda}(1-\lambda)}, & x=1,\\ \delta\frac{\frac{[\lambda(-1+\alpha)]^x}{x!}(1-\lambda)e^{\lambda}}{1-e^{\lambda}(1-\lambda)}, & x=2,3,\dots,\end{cases} \tag{20}$$

we have

Theorem 7. *If X has the distribution in* (20) *then*

$$P(Z=z) = \begin{cases}\sum_{l=0}^{m}\sum_{r=0}^{l}\binom{m}{l}\binom{l}{r}\delta^l\eta^{m-l}(-1)^{l-r}(1-\lambda)^l e^{\lambda l}\frac{[\lambda(-r+\alpha(r))]^l}{l!} \\ \times\left(1-e^{\lambda}(1-\lambda)\right)^{-l}, \quad z=m,\\ \sum_{l=1}^{m}\sum_{r=0}^{l}\binom{m}{l}\binom{l}{r}\delta^l\eta^{m-l}(-1)^{l-r}(1-\lambda)^l e^{\lambda l}\frac{[\lambda(-r+\alpha(r))]^{z-m+l}}{(z-m+l)!} \\ \times\left(1-e^{\lambda}(1-\lambda)\right)^{-l}, \quad z=m+1,m+2,\dots\,.\end{cases}$$

Theorem 6 and Theorem 7 extend a result given in Grzegórska [6].

Suppose that $Z_1, Z_2, \ldots, Z_n$ are i.i.d as Z. We consider here the distribution of the convolution $Y = Z_1 + Z_2 + \ldots + Z_n$ in the case when Z is the truncated sum of zero inflated $MP(\lambda, \psi)$ variates with (Theorem 4)

$$P(Z = z) = \sum_{l=1}^{m} \binom{m}{l} \delta^l \eta^{m-l} \frac{(\lambda l + \psi\alpha(l))^z}{z!} (1-\psi)^l e^{-\lambda l} \times \Big[1 - (\eta + \delta(1-\psi)e^{-\lambda})^m\Big]^{-1}, \quad z = 1, 2, \ldots \, . \tag{21}$$

The following theorem is a generalization of a result given in Grzegórska [6] (cf. formula (10)).

Theorem 8. *If Z has the distribution in* (21) *then*

$$P(Y = y) = \Big[1 - (\eta + \delta(1-\psi)e^{-\lambda})^m\Big]^{-n} \times \sum_{r=1}^{n} \sum_{s=1}^{mr} \binom{n}{r} \binom{mr}{s} (-1)^{n-r} \delta^s \eta^{mr-s} \times (\eta + \delta(1-\psi)e^{-\lambda})^{m(n-r)} e^{-\lambda s} (1-\psi)^s \frac{(\lambda s + \psi\alpha(s))^y}{y!},$$
$$y = n, n+1, \ldots \, .$$

Proof. From (21) the characteristic function of Z is given by

$$\begin{aligned}\varphi_Z(t) &= \Big[1 - (\eta + \delta(1-\psi)e^{-\lambda})^m\Big]^{-1} \\ &\times \sum_{l=1}^{m} \binom{m}{l} \delta^l \eta^{m-l} (1-\psi)^l e^{-\lambda l} \sum_{z=1}^{\infty} e^{itz} \frac{(\lambda l + \psi\alpha(l))^z}{z!} \\ &= \Big[1 - (\eta + \delta(1-\psi)e^{-\lambda})^m\Big]^{-1} \\ &\times \left\{ \left(\eta + \delta \frac{1-\psi}{1-e^{it}\psi} e^{\lambda(e^{it}-1)}\right)^m - \left(\eta + \delta(1-\psi)e^{-\lambda}\right)^m \right\}.\end{aligned}$$

Hence

$$\begin{aligned}\varphi_Y(t) &= \Big[1 - (\eta + \delta(1-\psi)e^{-\lambda})^m\Big]^{-n} \sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} \\ &\times \left(\eta + \delta(1-\psi)e^{-\lambda}\right)^{m(n-r)} \left(\eta + \delta \frac{1-\psi}{1-e^{it}\psi} e^{\lambda(e^{it}-1)}\right)^{mr}.\end{aligned}$$

Using the inverse formula for characteristic functions, we obtain

$$P(Y=y)=\lim_{k\to+\infty}\frac{1}{2k}\int_{-k}^{k}e^{-ity}\varphi_Y(t)dt$$
$$=\Big[1-(\eta+\delta(1-\psi)e^{-\lambda})^m\Big]^{-n}\sum_{r=1}^{n}\sum_{s=1}^{mr}\binom{n}{r}\binom{mr}{s}(-1)^{n-r}\delta^s\eta^{mr-s}$$
$$\times(\eta+\delta(1-\psi)e^{-\lambda})^{m(n-r)}e^{-\lambda s}(1-\psi)^s\frac{(\lambda s+\psi\alpha(s))^y}{y!},$$

which ends the proof. □

For Z being the truncated sum of zero inflated α-modified $MP(-\lambda,\lambda)$ Poisson distribution we have

Theorem 9. *If Z has the distribution*

$$P(Z=z)=\sum_{l=1}^{m}\binom{m}{l}\delta^l\eta^{m-l}\frac{[\lambda(-l+\alpha(l))]^z}{z!}(1-\lambda)^l e^{\lambda l}$$
$$\times\Big[1-(\eta+\delta(1-\lambda)e^{\lambda})^m\Big]^{-1},\quad z=1,2,\ldots$$

then

$$P(Y=y)=\Big[1-(\eta+\delta(1-\lambda)e^{\lambda})^m\Big]^{-n}$$
$$\times\sum_{r=1}^{n}\sum_{s=1}^{mr}\binom{n}{r}\binom{mr}{s}(-1)^{n-r}\delta^s\eta^{mr-s}$$
$$\times(\eta+\delta(1-\lambda)e^{\lambda})^{m(n-r)}e^{\lambda s}(1-\lambda)^s\frac{[\lambda(-s+\alpha(s))]^y}{y!},$$
$$y=n,n+1,\ldots\ .$$

3. Convolutions of α-Modified Binomial Distributions

The sum of the independent and identically distributed random variables having the binomial probability function is also binomially distributed. We consider here convolution of α-modified $MB(N,p,q,\phi)$ binomial distribution in (4).

Theorem 10. *If X_1, X_2 are i.i.d as $X \sim MB(N,p,q,\phi)$ in (4) and if $Z = X_1 + X_2$ then*

$$P(Z=n) = \frac{q^{2N-n}\sum_{k=0}^{n}\binom{N}{k}\binom{N}{n-k}(p+\alpha\phi)^k(p+\alpha\phi)^{n-k}}{\left((q+p)^N + \sum_{j=1}^{N} N!/(N-j)!(q+p)^{N-j}\phi^j\right)^2}, \qquad n = 0,1,\ldots,2N. \tag{22}$$

Proof. From (4) the probability generating function of X is given by

$$G_X(s) = \frac{\sum_{x=0}^{N} s^x\binom{N}{x}(p+\alpha\phi)^x q^{N-x}}{(q+p)^N + \sum_{j=1}^{N} N!/(N-j)!\phi^j(q+p)^{N-j}} = \frac{\sum_{x=0}^{N} s^x\binom{N}{x}q^{N-x}\sum_{r=0}^{x} x!/(x-r)!\phi^r p^{x-r}}{(q+p)^N + \sum_{j=1}^{N} N!/(N-j)!\phi^j(q+p)^{N-j}}.$$

It is easy to verify that

$$G_X^{(k)}(s) = \frac{\sum_{i=k}^{N}\binom{N}{i}\frac{i!}{(i-k)!}s^{i-k}q^{N-i}(p+\alpha\phi)^i}{(q+p)^N + \sum_{j=1}^{N} N!/(N-j)!\phi^j(q+p)^{N-j}}$$
$$= \frac{\binom{N}{k}k!q^{N-k}(p+\alpha\phi)^k + \sum_{i=k+1}^{N}\binom{N}{i}\frac{i!}{(i-k)!}s^{i-k}q^{N-i}(p+\alpha\phi)^i}{(q+p)^N + \sum_{j=1}^{N} N!/(N-j)!\phi^j(q+p)^{N-j}}.$$

Using the Leibniz formula for n-th derivative of a product of two factors we get

$$G_Z^{(n)}(s) = \sum_{k=0}^{n} \binom{n}{k} G_{X_1}^{(k)}(s) G_{X_2}^{(n-k)}(s)$$

$$= \frac{\sum\limits_{k=0}^{n} \binom{n}{k} \left(\frac{N! q^{N-k}}{(N-k)!} (p+\alpha\phi)^k + \sum\limits_{i=k+1}^{N} \binom{N}{i} \frac{i! s^{i-k} q^{N-i}}{(i-k)!} (p+\alpha\phi)^i \right)}{\left((q+p)^N + \sum\limits_{j=1}^{N} N!/(N-j)! \phi^j (q+p)^{N-j} \right)^2}$$

$$\times \left(\frac{N! q^{N-n+k}}{(N-n+k)!} (p+\alpha\phi)^{n-k} + \sum_{i=n-k+1}^{N} \binom{N}{i} \frac{i! s^{i-n+k} q^{N-i}}{(i-n+k)!} (p+\alpha\phi)^i \right).$$

But

$$P(Z=n) = \frac{G_Z^{(n)}(0)}{n!},$$

and

$$G_Z^{(n)}(0) = \frac{q^{2N-n} n! \sum\limits_{k=0}^{n} \binom{N}{k} \binom{N}{n-k} (p+\alpha\phi)^k (p+\alpha\phi)^{n-k}}{\left((q+p)^N + \sum\limits_{j=1}^{N} N!/(N-j)! \phi^j (q+p)^{N-j} \right)^2}$$

so we obtain the formula (22). □

Corollary 1. *In the case $q+p=1$ the formula (22) has the form*

$$P(Z=n) = \frac{q^{2N-n} \sum\limits_{k=0}^{n} \binom{N}{k} \binom{N}{n-k} (p+\alpha\phi)^k (p+\alpha\phi)^{n-k}}{\left(1 + \sum\limits_{j=1}^{N} N!/(N-j)! \phi^j \right)^2}.$$

Now we extend Theorem 10 to $Z = X_1 + X_2 + \ldots + X_m$.

Theorem 11. *If $X \sim MB(N, p, q, \phi)$ in (4) then*

$$P(Z=n) = \frac{q^{Nm-n} \sum\limits_{k_1+k_2+\ldots+k_m=n} \binom{N}{k_1} \binom{N}{k_2} \cdot \ldots \cdot \binom{N}{k_m}}{\left((q+p)^N + \sum\limits_{j=1}^{N} N!/(N-j)! \phi^j (q+p)^{N-j} \right)^m} \tag{23}$$

$$\times (p+\alpha\phi)^{k_1} (p+\alpha\phi)^{k_2} \cdot \ldots \cdot (p+\alpha\phi)^{k_m}, \; n = 0, 1, \ldots, mN.$$

Proof. The proof is similar to the proof of the Theorem 10. We use the generalized Leibniz formula for n-th derivative of a product of m factors. Then

$$G_Z^{(n)}(s) = \sum_{k_1+\ldots+k_m=n} \frac{n!}{k_1!\cdot\ldots\cdot k_m!} G_{X_1}^{(k_1)}(s)\cdot\ldots\cdot G_{X_m}^{(k_m)}(s)$$

with

$$G_{X_t}^{(k_t)}(s) = \frac{\frac{N!q^{N-k_t}}{(N-k_t)!}(p+\alpha\phi)^{k_t} + \sum\limits_{i=k_t+1}^{N}\binom{N}{i}\frac{i!s^{i-k_t}q^{N-i}}{(i-k_t)!}(p+\alpha\phi)^i}{(q+p)^N + \sum\limits_{j=1}^{N} N!/(N-j)!\phi^j(q+p)^{N-j}},$$

$$t = 1, 2, \ldots, m,$$

leads to the formula

$$G_Z^{(n)}(s) = \frac{\sum\limits_{k_1+\ldots+k_m=n}\frac{n!}{k_1!\cdot\ldots\cdot k_m!}}{\left((q+p)^N + \sum\limits_{j=1}^{N} N!/(N-j)!\phi^j(q+p)^{N-j}\right)^m}$$

$$\times\left(\frac{N!q^{N-k_1}}{(N-k_1)!}(p+\alpha\phi)^{k_1} + \sum_{i=k_1+1}^{N}\binom{N}{i}\frac{i!s^{i-k_1}q^{N-i}}{(i-k_1)!}(p+\alpha\phi)^i\right)$$

$$\cdot\ldots\cdot\left(\frac{N!q^{N-k_m}}{(N-k_m)!}(p+\alpha\phi)^{k_m} + \sum_{i=k_m+1}^{N}\binom{N}{i}\frac{i!s^{i-k_m}q^{N-i}}{(i-k_m)!}(p+\alpha\phi)^i\right).$$

Hence

$$G_Z^{(n)}(0) = \frac{n!\sum\limits_{k_1+\ldots+k_m=n}\binom{N}{k_1}q^{N-k_1}(p+\alpha\phi)^{k_1}\cdot\ldots\cdot\binom{N}{k_m}q^{N-k_m}(p+\alpha\phi)^{k_m}}{\left((q+p)^N + \sum\limits_{j=1}^{N} N!/(N-j)!\phi^j(q+p)^{N-j}\right)^m},$$

which completes the proof of (23). □

Corollary 2. *In the case $p+q=1$ the formula* (23) *has the form*

$$P(Z=n)=\frac{q^{Nm-n}\sum\limits_{k_1+k_2+\ldots+k_m=n}\binom{N}{k_1}\binom{N}{k_2}\cdot\ldots\cdot\binom{N}{k_m}}{\left(1+\sum\limits_{j=1}^{N}N!/(N-j)!\phi^j\right)^m}$$
$$\times(p+\alpha\phi)^{k_1}(p+\alpha\phi)^{k_2}\cdot\ldots\cdot(p+\alpha\phi)^{k_m},\ n=0,1,\ldots,mN.$$

Malik (1969) and Ahuja (1970) investigated the distribution of the sum of zero truncated binomial variables. We generalize their result for α-modified $MB(N,p,q,r)$ binomial distributions truncated at zero.

Theorem 12. *If X has the distribution in* (13) *then for $Z=X_1+X_2+\ldots+X_m$*

$$P(Z=n)=\frac{\sum\limits_{\nu=1}^{m}\binom{m}{\nu}(-1)^{m-\nu}q^{Nm-n}\sum\limits_{k_1+\ldots+k_\nu=n}\binom{N}{k_1}\cdot\ldots\cdot\binom{N}{k_\nu}}{\left((q+p)^N+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r(q+p)^{N-r}-q^N\right)^m}$$
$$\times(p+\alpha\phi)^{k_1}\cdot\ldots\cdot(p+\alpha\phi)^{k_\nu},\quad n=m,m+1,\ldots,mN. \tag{24}$$

Proof. From (13) the probability generating function of X is given by

$$G_X(s)=\frac{\sum\limits_{x=0}^{N}s^x\binom{N}{x}q^{N-x}\sum\limits_{r=0}^{x}x!/(x-r)!\phi^r p^{x-r}-q^N}{(q+p)^N+\sum\limits_{j=1}^{N}N!/(N-j)!\phi^j(q+p)^{N-j}-q^N}.$$

Hence

$$G_Z(s)=\frac{\sum\limits_{\nu=0}^{m}\binom{m}{\nu}(-1)^{m-\nu}q^{N(m-\nu)}\left(\sum\limits_{x=0}^{N}s^x\binom{N}{x}q^{N-x}\sum\limits_{r=0}^{x}x!/(x-r)!\phi^r p^{x-r}\right)^\nu}{\left((q+p)^N+\sum\limits_{j=1}^{N}N!/(N-j)!\phi^j(q+p)^{N-j}-q^N\right)^m}.$$

Therefore

$$G_Z^{(n)}(s) = \frac{\sum_{\nu=0}^{m} \binom{m}{\nu} (-1)^{m-\nu} q^{N(m-\nu)}}{\left((q+p)^N + \sum_{j=1}^{N} N!/(N-j)!\phi^j (q+p)^{N-j} - q^N \right)^m} \times \left[\left(\sum_{x=0}^{N} s^x \binom{N}{x} q^{N-x} \sum_{r=0}^{x} x!/(x-r)!\phi^r p^{x-r} \right)^{\nu} \right]^{(n)}$$

and using the generalized Leibniz formula for n-th derivative of a product of ν factors we get

$$G_Z^{(n)}(s) = \frac{\sum_{\nu=0}^{m} \binom{m}{\nu} (-1)^{m-\nu} q^{N(m-\nu)} \sum_{k_1+\ldots+k_\nu=n} \frac{n!}{k_1!\cdot\ldots\cdot k_\nu!}}{\left((q+p)^N + \sum_{j=1}^{N} N!/(N-j)!\phi^j (q+p)^{N-j} - q^N \right)^m}$$
$$\times \left(\frac{N! q^{N-k_1}}{(N-k_1)!}(p+\alpha\phi)^{k_1} + \sum_{i=k_1+1}^{N} \binom{N}{i} \frac{i! s^{i-k_1} q^{N-i}}{(i-k_1)!}(p+\alpha\phi)^i \right)$$
$$\cdot \ldots \cdot \left(\frac{N! q^{N-k_\nu}}{(N-k_\nu)!}(p+\alpha\phi)^{k_\nu} + \sum_{i=k_\nu+1}^{N} \binom{N}{i} \frac{i! s^{i-k_\nu} q^{N-i}}{(i-k_\nu)!}(p+\alpha\phi)^i \right).$$

Therefore

$$G_Z^{(n)}(0) = \frac{n! \sum_{\nu=1}^{m} \binom{m}{\nu} (-1)^{m-\nu} q^{Nm-n} \sum_{k_1+\ldots+k_\nu=n} \binom{N}{k_1} \cdot \ldots \cdot \binom{N}{k_\nu}}{\left((q+p)^N + \sum_{r=1}^{N} N!/(N-r)!\phi^r (q+p)^{N-r} - q^N \right)^m} \times (p+\alpha\phi)^{k_1} \cdot \ldots \cdot (p+\alpha\phi)^{k_\nu},$$

which completes the proof of Theorem 12. □

Corollary 3. *In the case $p+q=1$ the formula (24) has the form*

$$P(Z=n) = \frac{\sum_{\nu=1}^{m} \binom{m}{\nu} (-1)^{m-\nu} q^{Nm-n} \sum_{k_1+\ldots+k_\nu=n} \binom{N}{k_1} \cdot \ldots \cdot \binom{N}{k_\nu}}{\left(1 + \sum_{r=1}^{N} N!/(N-r)!\phi^r - q^N \right)^m} \times (p+\alpha\phi)^{k_1} \cdot \ldots \cdot (p+\alpha\phi)^{k_\nu}, \quad n = m, m+1, ..., mN. \tag{25}$$

Now we study convolution of zero inflated α-modified $MB(N,p,q,\phi)$ binomial distribution in (14) when $s=0$ viz. distribution

$$P(X=x)=\begin{cases}\eta+\delta\dfrac{q^N}{(q+p)^N+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r(q+p)^{N-r}}, & x=0,\\ \delta\binom{N}{x}\dfrac{(p+\alpha\phi)^x q^{N-x}}{(q+p)^N+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r(q+p)^{N-r}}, & x=1,2,\dots,N.\end{cases} \tag{26}$$

Theorem 13. *If X has the distribution in* (26) *then*

$$P(Z=n)=\begin{cases}\left(\eta+\delta\dfrac{q^N}{(q+p)^N+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r(q+p)^{N-r}}\right)^m, \quad n=0,\\ \sum\limits_{\nu=1}^{m}\dfrac{\binom{m}{\nu}\delta^\nu\eta^{m-\nu}q^{N\nu-n}\sum\limits_{k_1+\ldots+k_v=n}\binom{N}{k_1}\cdot\ldots\cdot\binom{N}{k_\nu}(p+\alpha\phi)^{k_1}\cdot\ldots\cdot(p+\alpha\phi)^{k_\nu}}{\left((q+p)^N+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r(q+p)^{N-r}\right)^\nu},\\ n=1,2,\dots,mN.\end{cases} \tag{27}$$

Proof. From (26) the probability generating function of X is given by

$$G_X(s)=\eta+\delta\frac{\sum\limits_{x=0}^{N}s^x\binom{N}{x}q^{N-x}\sum\limits_{r=0}^{x}x!/(x-r)!\phi^r p^{x-r}}{(q+p)^N+\sum\limits_{j=1}^{N}N!/(N-j)!\phi^j(q+p)^{N-j}}.$$

Hence

$$G_Z(s)=\sum_{\nu=0}^{m}\binom{m}{\nu}\delta^\nu\eta^{m-\nu}\left(\frac{\sum\limits_{x=0}^{N}s^x\binom{N}{x}q^{N-x}\sum\limits_{r=0}^{x}x!/(x-r)!\phi^r p^{x-r}}{(q+p)^N+\sum\limits_{j=1}^{N}N!/(N-j)!\phi^j(q+p)^{N-j}}\right)^\nu,$$

which leads us to (27). □

For $\phi=0$ in Theorem 13 we get a result for inflated binomial distribution given in Sobich and Szynal [14].

Corollary 4. *In the case $q+p=1$ the formula (27) has the form*

$$P(Z=n)=\begin{cases}\left(\eta+\delta\frac{q^N}{1+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r}\right)^m, \quad n=0,\\ \sum\limits_{\nu=1}^{m}\frac{\binom{m}{\nu}\delta^\nu\eta^{m-\nu}q^{N\nu-n}\sum\limits_{k_1+\ldots+k_\nu=n}\binom{N}{k_1}\cdot\ldots\cdot\binom{N}{k_\nu}(p+\alpha\phi)^{k_1}\cdot\ldots\cdot(p+\alpha\phi)^{k_\nu}}{\left(1+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r\right)^\nu},\\ n=1,2,\ldots,mN.\end{cases}$$

Remark 2. One can observe that the probability function of the zero truncated α-modified $MB(N,p,q,\phi)$ binomial distribution inflated at 0

$$P(X=x)=\frac{\delta\frac{\binom{N}{x}(p+\alpha\phi)^x q^{N-x}}{(q+p)^N+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r(q+p)^{N-r}-q^N}}{1-\left(1-\delta+\delta\frac{q^N}{(q+p)^N+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r(q+p)^{N-r}}\right)}$$
$$=\frac{\frac{\binom{N}{x}(p+\alpha\phi)^x q^{N-x}}{(q+p)^N+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r(q+p)^{N-r}-q^N}}{1-\frac{q^N}{(q+p)^N+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r(q+p)^{N-r}}} \tag{28}$$

is the same as the probability function of the zero truncated α-modified $MB(N,$ $p,q,\phi)$ binomial distribution. Hence, the distribution of their sums is as given in Theorem 12.

Corollary 5. *In the case $q+p=1$ the formula (28) has the form*

$$P(X=x)=\frac{\delta\frac{\binom{N}{x}(p+\alpha\phi)^x q^{N-x}}{1+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r-q^N}}{1-\left(1-\delta+\delta\frac{q^N}{1+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r}\right)}=\frac{\frac{\binom{N}{x}(p+\alpha\phi)^x q^{N-x}}{1+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r-q^N}}{1-\frac{q^N}{1+\sum\limits_{r=1}^{N}N!/(N-r)!\phi^r}}$$

and $P(Z=n)$ is given in (25).

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