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ON EWMA PROCEDURE FOR AR(1) OBSERVATIONS WITH EXPONENTIAL WHITE NOISE

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Abstract: In this paper, we use Fredholm second kind integral equations method to solve the corresponding Average Run Length (ARL), when the observations of a random process are serially-correlated. We derive explicit expressions for the ARL of an EWMA control chart, or its corresponding AR(1) process, when the observations follow an exponential distribution white noise. The analytical expressions derived, are easy to implement in any computer packages, and as a consequence, it reduces considerably the computational time comparable with the traditional numerical methods used to solve integral equations.

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1. Introduction

The Exponentially Weighted Moving Average (EWMA) control chart was first introduced by Roberts [1] in quality control, in order to detect a small shift in the mean of a production process as soon as it occurs. In practice, EWMA control charts are widely used in statistical process control, to detect changes in the performances of a stochastic system, e.g., mean or variance. For an introduction to EWMA control charts and their applications, see Barger et al. [3], Sibel et al. [4] and Serel [5]. Usually, traditional control chart methodology is based on the standard assumption that random observations are statistically independent and uniformly distributed. However, for the random data of interest in practical applications the observations are usually serially correlation. In many practical processes such as in chemical processes, the random variables are always serially-correlated. In the case of daily flow of a river, wind speeds, or the amount of dissolved oxygen in a river, most process data are autocorrelated.

The research related to control charts for serially-correlated processes has been proposed in the work of Wardell et al. [6], William and Frederick [7], Lu and Reynolds [8], Zhang [9], Chen and Elsayed [10], Daniel et al. [11], Rosolowski and Schmid [12], Vermaat et al. [13] and Torng et al. [14].

Frequently, the major criterion for measuring the characteristics of a control chart is the Average Run Length (ARL), defined as the expectation of an alarm time taken to signal about a possible change. It is always desirable to have a large ARL when a chart is in control. In the paper we adopt the notation $\mathrm{ARL} = \mathbb{E}_\infty(\tau) = T$ where $\mathbb{E}_\infty(\cdot)$ is the expectation corresponding to the target value and T is given as large enough. The ARL when the process is out-of-control, is called the Average Delay time (AD) , defined as the expectation of delay for true alarm time. This time should minimize the quantity

$$\mathrm{AD} = \hat{\mathbb{E}}_\nu\left(\tau - \nu + 1 | \tau \geq \nu\right)$$

where $\hat{\mathbb{E}}_\nu(\cdot)$ is the expectation under the assumption that a change-point occurs in time. In practice, the condition is usually calculated when $\nu = 1$.

In literature several methods for evaluating ARL when the process observations are autocorrelated have been studied, Goldsmith and Whitfield [14] studied ARL for V-mask control schemes based on cumulative deviation charts with normally distributed observation, and serially correlated class. Yashchin [15], found that the presence of serial correlation destroys the Markov property of the monitoring statistic of the CUSUM scheme and the exact analysis becomes practically infeasible. Schmid [16] introduced the exact ARL of a

Shewhart chart for an autoregressive processes (AR). Atienza et al. [17] used Monte Carlo experiments to analyze the ARL properties of the sample autocorrelation chart (SACC). VanBrackle and Reynolds [18] presented EWMA and CUSUM control charts for the process mean when the observations are from an AR(1) process with additional random error, the performance of the control charts were evaluated numerically using an integral equation approach, and a Markov chain approach. Lu and Reynolds [19] evaluated the ARL of EWMA control chart which the observations as an AR(1) process plus a random error. Using a Monte Carlo simulation, Canan and Xianzhe [20] evaluated the average run length performance of autocorrelated process control charts. An exact evaluation for the ARL with Markov chain approach for monitoring a Poisson INAR(1) process, was presented by Christian [21]. Chang and Wu [22] used the finite Markov chain imbedding (FMCI) technique, introduced by Fu and Koutras [23] to investigate the run length properties for various control charts when the process observations are autocorrelated. Han and Tsung [24], considered the detection performance of the CUSUM and EWMA charts in monitoring the mean shifts in such autocorrelated processes. Both methods, theoretical analysis and numerical simulation were used to evaluate the ARL. Recently Areepong and Novikov [26] derived the explicit formulas for the expectation of stopping time for a one-sided EWMA process by distribution from an exponential distribution. Later, Suriyakat et al [25] derived the explicit formulas of ARL for EWMA control chart when AR(1) observations are Exp(1) distributed. In this article, our objective is to derive the explicit formulas for detect changes in the mean of EWMA control charts for AR(1) process, with exponential white noise.

This article is organized as follows. In Section 2, we introduce the model for the process observations and discuss the EWMA control chart. The solution for the ARL of EWMA control charts for AR(1) process with exponential white noise is presented in Section 3. Further, numerical and comparison results given in Section 4, and finally, in Section 5, we state our conclusions.

2. EWMA Control Chart and a Model for the Process Observations

In this section we consider the EWMA chart have serially-correlated as a first-order autoregressive process, AR(1). The recursive equation for EWMA chart designed to detect an increase in the mean of observed sequence of an AR(1) process with random variable ξ_t is defined as

$$Y_t = (1-\lambda)Y_{t-1} + \lambda X_t, t = 1, 2, 3, ..., Y_0 = u$$

where $0 \leq \lambda \leq 1$ and $\{X_t\}_{t\geq 1}$ is a sequence of an AR(1) random process. We consider the AR(1) process defined by

$$X_t = \phi X_{t-1} + \xi_t$$

where $0 \leq \phi \leq 1$ and $\{\xi_t\}_{t\geq 1}$ is a sequence of independent and identically distributed random variable with $X_0 = v$.

Control limits for monitoring the sequence Y_t are of the form

$$UCL = \mu + L\sigma_Y = H,$$

$$LCL = \mu - L\sigma_Y = 0,$$

where UCL is upper control limit, LCL is lower control limit. μ is mean, σ is the standard deviation of the distribution and L is a constant. L is usually chosen as 3 for the case of standard Gaussian distribution.

The problem is to calculate the expectation of the stopping time

$$\tau_H = inf\{t \geq 0 : Y_t \geq H\}, u \leq H.$$

3. Explicit Formulas for ARL of EWMA Control Charts for AR(1) Process with Exponential white Noise

In this section we analyze the case of EWMA control chart where $\xi_t \sim Exp(\alpha)$. Recall that, the probability density function of ξ_t is given by $f(x) = \frac{1}{\alpha}e^{-\frac{x}{\alpha}}$.

Using a method similar to VanBrackle and Reynolds [18] and Suriyakat et al [25]. Let $L(u)$ denote the ARL of one-sided EWMA control chart when the initial value is u and substitution $w = \xi_1$. Since $\xi_t \geq 0$ we can assume the lower limit and the upper limit are $H_L = 0$ and $H_U = H$, respectively. $L(u)$ is given by the integral equation

$$L(u) = 1 + \int_{0\leq(1-\lambda)u+\lambda(\phi v+w)\leq H} L((1-\lambda)u + \lambda v + w)f(w)dw.$$

A change of variable gives

$$L(u) = 1 + \frac{1}{\lambda}\int_0^H L(w)f(\frac{w-(1-\lambda)u}{\lambda} - \phi v)dw. \qquad (1)$$

Equation (1) is a Fredholm integral equation of the second kind.

In equation (1) if we consider $w_t \sim Exp(\alpha)$ then we have $f(w) = \frac{1}{\alpha}e^{-\frac{w}{\alpha}}, w \geq 0$

$$f(\frac{w-(1-\lambda)u}{\lambda} - \phi v) = \frac{1}{\alpha}e^{-\frac{w}{\lambda\alpha}}e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\phi v}{\alpha}}$$

and

$$L(u) = 1 + \frac{1}{\lambda\alpha}\int_0^H L(w)e^{-\frac{w}{\lambda\alpha}}e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\phi v}{\alpha}}dw.$$

Let

$$C(u) = e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\phi v}{\alpha}}; 0 \leq u \leq H$$

so, we have

$$L(u) = 1 + \frac{C(u)}{\lambda\alpha}\int_0^H L(w)e^{-\frac{w}{\lambda\alpha}}dw; 0 \leq u \leq H.$$

Here we have taking into account that

$$d = \int_0^H L(w)e^{-\frac{w}{\lambda\alpha}}dw$$

then we obtain

$$L(u) = 1 + \frac{C(u)}{\alpha\lambda}d. \tag{2}$$

Now we can express the constant d as

$$d = \int_0^H L(w)e^{-\frac{w}{\lambda\alpha}} = \frac{-\lambda\alpha(e^{-\frac{H}{\lambda\alpha}}-1)}{1+(e^{-\frac{H}{\alpha}}-1)\frac{e^{\frac{-\phi v}{\alpha}}-1}{\lambda}}, \tag{3}$$

where substitute (3) in (2) the solution for the integral equation (1) is

$$L(u) = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha}}(e^{-\frac{H}{\alpha\lambda}}-1)}{\lambda e^{-\frac{\phi v}{\alpha}} + e^{-\frac{H}{\alpha}} - 1} \tag{4}$$

4. Numerical Comparisons with the Analytical Results

In this section we present a numerical method to evaluate solution of the integral equations. Firstly, from equation (1)

$$L(u) = 1 + \frac{1}{\lambda}\int_0^H L(w)f(\frac{w-(1-\lambda)u}{\lambda} - \phi v)dw.$$

Now, in general, we can approximate the integral $\int_0^H f(z)dz$ by the sum of rectangles with bases H/m with heights chosen as the values of f at the midpoints of intervals of length H/m beginning at zero, i.e. on the interval $[0, H]$ with the division points $0 \leq a_1 \leq a_2 \leq a_3 \leq ... \leq a_m \leq H$ and weights $w_j = H/m$, then

$$\int_0^H f(z)dz \approx \sum_{j=1}^{m} w_j f(a_j),$$

where $a_j = \frac{H}{m}(j - \frac{1}{2})$, $k = 1, 2, 3, ..., m$.

The integral equation (1) can be approximated by:

$$L(a_i) \approx 1 + \frac{1}{\lambda} \sum_{j=1}^{m} w_j L(a_j) f(\frac{a_j - (1-\lambda)a_i}{\lambda} - \phi v), i = 1, 2, 3, ..., m$$

That is

$$L(a_1) \approx 1 + \frac{1}{\lambda} \sum_{j=1}^{m} w_j L(a_j) f(\frac{a_j - (1-\lambda)a_1}{\lambda} - \phi v)$$

$$L(a_2) \approx 1 + \frac{1}{\lambda} \sum_{j=1}^{m} w_j L(a_j) f(\frac{a_j - (1-\lambda)a_2}{\lambda} - \phi v)$$

$$L(a_3) \approx 1 + \frac{1}{\lambda} \sum_{j=1}^{m} w_j L(a_j) f(\frac{a_j - (1-\lambda)a_3}{\lambda} - \phi v)$$

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$$L(a_m) \approx 1 + \frac{1}{\lambda} \sum_{j=1}^{m} w_j L(a_j) f(\frac{a_j - (1-\lambda)a_m}{\lambda} - \phi v),$$

or in matrix form as

$$L_{m\times 1} = 1_{m\times 1} + R_{m\times m} L_{m\times 1} or (I_m - R_{m\times m}) L_{m\times 1} = 1_{m\times 1}, \tag{5}$$

where

$$L_{m\times 1} = \begin{pmatrix} L(a_1) \\ L(a_2 \\ . \\ . \\ . \\ L(a_m) \end{pmatrix}, 1_{m\times 1} = \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{pmatrix}, R_{m\times m} = \frac{1}{\lambda} w_j (\frac{a_j - (1-\lambda)a_i}{\lambda} - \phi v)$$

and $I_m = diag(1, 1, ..., 1)$ is the unit matrix of order m. If there exists $(I_m - R_{m\times m})^{-1}$, then the solution of the matrix equation ()

$$L_{m\times 1} = (I_m - R_{m\times m})^{-1} 1_{m\times 1}.$$

We may approximate the function $L(u)$ as

$$\tilde{L}(u) \approx 1 + \frac{1}{\lambda} \sum_{j=1}^{m} w_j L(a_j) f\left(\frac{a_j - (1-\lambda)u}{\lambda} - \phi v\right), \tag{6}$$

where $w_j = \frac{H}{m}$ and $a_j = \frac{H}{m}(j - 0.5)$.

First, we implement the numerical scheme given by equation (6) for an AR(1) process with exponential white noise when the number of divisions is $m = 1500$. The solution for explicit formulas was obtained in equation (4) for $0 \leq u \leq H$. The explicit formulas and the integral equations for several values of H, λ and α are presented in Table 1.

We obtain an excellent agreement between suggested formulas and numerical approximation for the integral equations. The calculations using the explicit form given by equation (4) are much faster. For example, when λ is 0.2, H is 0.22 and α is 1, computing time based on our technique take less than 1 second while CPU time required for approximation integral equations for EWMA run 42.2609 seconds as shown inside the parentheses.

In Table 2, we compare the results for ARL and AD with equation (4) and (6) for EWMA control charts for AR(1) process with exponential white noise with ϕ is 0.3, λ is 0.3, H is 0.34 and α is 1, computing time based on our technique take less than 1 second while CPU time required for approximation integral equations for EWMA control chart run 42.4041 seconds as shown inside the brackets. When λ is 0.4, H is 0.49 and α is 1, computing time based on our technique take less than 1 second while CPU time required for approximation integral equations for EWMA run 42.1813 seconds as shown inside the brackets.

5. Conclusion

We derived explicit formulas for the ARL and AD of one-sided EWMA control charts, in the case of an AR(1) process with exponential white noise distributed observations. We have shown that suggested formulas are considerably faster and requires less computational time than the numerical methods for integral equations. The performance of the control chart has been compared based on ARL and AD criteria.

λ	H	α	Explicit form	Integral equation
0.2	0.22	1.0	377.43941	377.43911 (42.2609)
		1.1	11.74815	11.74816 (78.9848)
		1.2	6.43336	6.43337 (96.6410)
		1.3	4.63071	4.63072 (89.000)
		1.4	3.72461	3.72461 (81.5000)
		1.5	3.17987	3.17987 (71.4690)
0.3	0.35	1.0	154.17815	154.17808 (42.3941)
		1.1	10.79457	10.79457 (53.9485)
		1.2	6.08812	6.08812 (49.7791)
		1.3	4.44839	4.44839 (51.7666)
		1.4	3.61373	3.61373 (43.0790)
		1.5	3.10762	3.10762 (57.9609)

Table 1: Comparison of the ARL and AD by the explicit formulas and the numerical integral equations when ϕ is 0.1. The entries inside the parentheses are CPU times in seconds.

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λ	H	α	Explicit form	Integral equation
0.3	0.34	1.0	89.45431	89.45405 (42.4041)
		1.1	10.15875	10.15880 (109.109)
		1.2	5.85549	5.85549 (137.984)
		1.3	4.31398	4.31398 (107.719)
		1.4	3.52068	3.52068 (109.719)
		1.5	3.03684	3.03678 (112.656)
0.4	0.49	1.0	408.91783	408.91909 (42.1813)
		1.1	11.34231	11.34230 (108.954)
		1.2	6.28327	6.28327 (108.844)
		1.3	4.56522	4.56522 (111.234)
		1.4	3.69869	3.69869 (118.296)
		1.5	3.17557	3.17557 (131.703)

Table 2: Comparison of the ARL and AD by the explicit formulas and the numerical integral equations when ϕ is 0.3. The entries inside the parentheses are CPU times in seconds

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