A NOTE ON INVERSIVE LOCALIZATION IN NOETHERIAN REGULAR DELTA NEAR RINGS

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Abstract: The main objective and aim of this paper is to discuss some existing examples related to torsion theory, introduce inverse localization in Noetherian regular $\delta$-near-rings of rings, and derived a relation between localizing and forming factor of regular $\delta$-near-rings over square matrices which are invertible.

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Key Words: near-ring, semi prime ideal, prime ideal, proper kernel functor, insulated ideal, proper torsion radical, right ideal containing a finite set, (right) annihilator, Noetherian near-ring, regular near-ring, Noetherian regular $\delta$-near-ring (NR-$\delta$NR)

1. Introduction

In 1979 [17] discussed about localization of near-rings over invertible matrices. Regular near-rings were introduced by J.C.Bieldman [6]. Delta near-rings later studied and introduced by Y.V.Reddy and VLN Murthy [26]. And Later many others studied about the ...... in 2010, a note on Noetherian regular $\delta$-near-rings by N.V. Nagendram, T.V. Pradeep Kumar, and Y.V. Reddy introduced Noetherian regular $\delta$-near-rings ( NR- $\delta$-NR) in [13 ]. We know about injective

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and inverse methods already. If instead of inverting matrices, we merely make them right invertible, we obtain an intermediate method, which may be called *semi-inversive*. It is natural to try to apply these methods to obtain a non-commutative localization process. For a Noetherian regular δ-near-ring. Ore’s method can always be applied to yield a semi-simple Artinian quotient ring (Goldie’s theorem), but this is no longer so when we try to localize at a prime ideal of a Noetherian regular δ-near-ring ring. A number of ways of performing such localization, the object of this note is to show how the inversive method may be applied to obtain localization at a semi-prime ideal of a Noetherian regular δ-near ring N. The Noetherian regular delta near ring thus obtained is always a semi-local Noetherian regular delta near ring whose residue class Noetherian regular delta near ring of a ring N is the classical quotient ring of N/n.

For existing literature about near rings refer to Gunter Pilz [5].

**Definition 1.1.** A near-ring N is called a δ-near-ring if it is left simple and N0 is the smallest non-zero ideal of N and a δ-near-ring is a non-constant near ring.

**Example.** (B, +, ∗) be a Near-ring with 1. Let a′ = a + 1; a ∨ b = (a′ ∨ b′)′ if x ∈ N define for a, b ∈ B a ∗x b = a ∗(b ν x) Then (B, +, ∗x) is a δ-near-ring with (P3) is R if x =0.

And also M0 ( . ) δ-near ring.

**Definition 1.2.** A set N is a near-ring together with two binary operations “+” and “.” Such that:

(i) (N, +) is a Group not necessarily Abelian, ( ii ) (N, .) is a semi group, and

(iii) for all n1, n2, n3 ∈ N, (n1 + n2). n3 = (n1 . n3 + n2 . n3) i.e. right distributive law

**Examples 1.3.** Let M2x2 ={(aij) / Z ; Z is treated as a near-ring}. M2x2 under the operation of matrix addition ’+’ and matrix multiplication ’∗’.

**Example 1.4.** Z be the set of positive and negative integers with 0. (Z, +) is a group. Define ’∗’ on Z by a . b = a for all a, b ∈ Z. Clearly (Z, +, .) is a near-ring.

**Example 1.5.** Let Z12 = {0, 1, 2,. . . , 11}. (Z12, +) is a group under ‘+’ modulo 12. Define ’∗’ on Z12 by a . b = a for all a∈ Z12. Clearly (Z12, +, .) is a near-ring.
**Definition 1.6.** A near-ring \( N \) is Regular Near-Ring if each element \( a \in N \) then there exists an element \( x \) in \( N \) such that \( a = axa \).

**Definition 1.7.** A \( \delta \)-Near-Ring \( N \) is isomorphic to \( \delta \)-Near-Ring and is called a Regular \( \delta \)-Near-Ring if every \( \delta \)-Near-Ring \( N \) can be expressed as sub-direct product of near-rings \( \{N_i\} \), \( N_i \) is a non-constant near-ring or a \( \delta \)-Near-Ring \( N \) is sub-directly irreducible \( \delta \)-Near-Rings \( N_i \).

**Example 1.8.** An \( M_0(.) \) \( \delta \)-Near Ring \( M_c(.) \) direct product of Near Fields Every Near – Ring \( \in \gamma_0 \) can be embedded into a Regular \( \delta \)-Near Ring.

### 2. Preliminaries

**Definition 2.1.** If \( N \) be a Noetherian Regular \( \delta \)-Near ring, then a set \( C \) of square matrices over \( N \) is said to be “multiplicative” if \( 1 \in C \) and, for any \( A, B \in C \) and any matrix \( C \) of the right \( AC \) size, \( a( B) \in C \), where \( C \) contains square matrices of all orders; and \( C \), for the set of \( nx tz \)matrices in \( C \).

**Definition 2.2.** If \( C \) is multiplicative and moreover admits all elementary (row and column) transformations, it is called “admissible”.

**Definition 2.3.** A homomorphism \( f : N \rightarrow S \) is said to be \( C \)-inverting if each matrix of \( C \) is mapped to an invertible matrix by \( f \). It is easily seen that for any set \( C \) of square matrices over \( N \) there exists a ring \( N \), and a \( C \)-inverting homomorphism \( A : N \rightarrow N \), which is universal in the sense that every \( C \)-inverting homomorphism \( f : N \rightarrow S \) can be factored uniquely by \( A \), i.e., there exists a unique homomorphism \( A : N \rightarrow N \).

**Definition 2.4.** Let us define \( fI : N \rightarrow S \) such that the accompanying triangle commutes.

We observe that \( I \) always an epi morphism in the category of rings. The Noetherian regular delta near ring \( N \), is called the universal \( I \)-inverting ring.

**Definition 2.5.** A Noetherian regular \( \delta \)-near-ring \( N \) with 1 is called an Inverse Localization in Noetherian regular \( \delta \)-near-ring (IL-NR-\( \delta \)-NR) if localization is a most useful tool in commutative algebra: With every prime ideal \( p \) of a commutative ring \( N \) one associates a local ring \( N \), whose residue class field is \( Q(N/p) \), the field of fractions of the integral domain \( N/p \), together with a homomorphism \( I : N \rightarrow N \), such that the accompanying diagram commutes. \( N \) is Noetherian, kernel \( I \) may also be characterized as the intersection of all the primary components of 0 associated with prime ideals contained in \( p \) now the
process of forming fractions has been generalized to non-commutative rings in a number of ways.

**Example 2.6.** If $I$: is multiplicative, the elements of $N$ may be obtained as the components of the solutions of the matrix equations $uA + a = O,(1)$ where $a$ is a row over $N$ and $A \in X, u \in N$.

**Note 2.7.** The advantage of inverting matrices rather than elements is that the solutions of the equations (1) actually form (and not merely generate) the inverting ring. As is well known, in the case of elements this is so only for the denominator sets (sets satisfying the Ore condition). We recall that a right denominator set in a Noetherian Regular delta Near ring $N$ is a subset $S$ which includes 1, is multiplicatively closed, and is such that

(i) $\forall a \in R, s \in S$, $as \in S$ and $sa \neq a$, and

(ii) $\forall a \in R, s \in S$, if $sa = 0$, then $at = 0$ for some $t \in S$.

Let $S$ be a right denominator set and let $I;_I$ be the multiplicative set of matrices generated by $S$; thus $I;_I$ consists of all upper triangular matrices with elements of $S$ on the main diagonal. Then we can form the ring of fractions $N$, and the universal $E$-inverting Noetherian regular delta near ring $N$ is a subset $S$ which includes 1, is multiplicatively closed, and is such that

(i) $\forall a \in R, s \in S$, $as \in S$ and $sa \neq a$, and

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(ii) $\forall a \in R, s \in S$, if $sa = 0$, then $at = 0$ for some $t \in S$.

**Definition 2.8.** Given any set $I;_I$ of square matrices over $R$, let be the set of matrices of $N$ that are inverted by the canonical mapping $A : N \rightarrow N$. This set is called the saturation of $I;_I$; and if $I;_I = x$, we call $I;_I$ saturated. From the universal properties one sees that, for any $S$, (4) $N,NI$. It is clear that the set $I;_I$ is admissible (for the multiplicative case); hence the multiplicative closure of $I;_I$; and the admissible closure of $I;_I$.

**Proposition 2.9.** Let $N$ be a Noetherian Regular Delta Near ring and $C$ a multiplicative set of matrices over $N$. For any right $N$-module $M$ denote by $t(M)$ the set of elements of $M$ occurring as component in a row $u$ such that $uA = 0$ for some $A \in C$. Then the correspondence $M_i \rightarrow t(t(M))$ is an idempotent radical. Further, both $C$ and its admissible closure give rise to the same radical. The first assertion means that $t(M)$ is a submodule of $M$ and the assignment $MH t(M)$ is an idempotent sub functor of the identity such that $t(M / t(M)) = 0$. 
Proof. Refer to Proposition 2.1 in [23]. □

Note 2.10. When we take $M = R$, $t(R)$ is just the 'left Z component of $\sigma t$, i.e., the set of elements of $R$ occurring as component in some row $u$ such that $UA = 0$ for some $A \in C$.

3. Inversive Localization in Noetherian Rings

Definition 3.1. A Noetherian regular $\delta$-near-ring $N$ with 1 is called an Inverse Localization in Noetherian regular $\delta$-near-ring (IL-NR-$\delta$-NR) if localization is a most useful tool in commutative algebra: With every prime ideal $p$ of a commutative ring $N$ one associates a local ring $N_p$, whose residue class field is $Q(N/p)$, the field of fractions of the integral domain $N/p$, together with a homomorphism $I: N \rightarrow N$, such that the accompanying diagram commutes. $N$ is Noetherian, kernel of $I$ may also be characterized as the intersection of all the primary components of 0 associated with prime ideals contained in $p$.

Now the process of forming fractions has been generalized to non-commutative rings in a number of ways.

Example 3.2. Thus although itself semi local, $N$ is not embedded in the localization constructed in an example in which $l(r) + r(r) \neq \text{Ker}$. Let $N$ be an algebra over a commutative field, on $8$ generators $a_{ij}$, $b_{ij}$, $i, j = 1, 2$, with defining relations (in matrix form) $A = (a_{ij})$, $B = (b_{ij})$.

This is essentially Malcev’s example of an integral domain not embeddable in a field (cf. [lo]). If $I$ is the multiplicative set generated by $a_{ij}$, and $b_{ij}$, then $r$ consists of triangular matrices, and since $R$ is an integral domain, it follows that $l(r) = r(r) = 0$, but it may be verified that $c = a_{ii} b_{ij} + a_{ij} \in \text{kernel of } A$.

By adjoining another element $t$ with the relation $ta_{11} = 0$, we obtain an example in which $l(r) \neq r(r) = 0$. Of course, $N$ is not Noetherian in this example, but it is not even known whether, for a Noetherian ring $R$ and a multiplicative set $Z$ of matrices, the localization $N$, is necessarily Noetherian.

To get a better idea of the size of kernel of $A$ one may have to use something like Cramer’s rule, but it is far from clear whether this would provide a usable criterion.

Let $f I: R \rightarrow S$ such that the accompanying triangle commutes. We observe that $I$ is always an epi-morphism in the category of rings. The ring $R$, is called the universal $I; \text{-inverting ring}$. If $I$: is multiplicative, the elements of $R$ may
be obtained as the components of the solutions of the matrix equations (1) 
\[ uA + a = O, \]
where \( a \) is a row over \( R \) and \( A \in X \).

The advantage of inverting matrices rather than elements is that the solutions of the equations (2) actually form (and not merely generate) the inverting ring. As is well known, in the case of elements this is so only for the denominator sets (sets satisfying the Ore condition). We recall that a right denominator set in a ring \( R \) is a subset \( S \) which includes 1, is multiplicatively closed, and is such that:

(i) for all \( a \in R, s \in S \), \( aS \cap sR \neq a \),

(ii) for all \( a \in R, s \in S \), if \( sa = 0 \), then \( at = 0 \) for some \( t \in S \).

Let \( S \) be a right denominator set and let \( I; \) be the multiplicative set of matrices generated by \( S \); thus \( I; \) consists of all upper triangular matrices with elements of \( S \) on the main diagonal. Then we can form the ring of fractions \( R; \) and the universal \( I; \)-inverting ring \( RI; \) we claim that these rings are isomorphic.

More precisely, there is an isomorphism \( w: N \to N, \) such that the triangle shown commutes, where \( I; \), \( I; \) are the canonical mappings. This follows easily from the universal properties of these mappings; thus \( I;: N \to N, \) is \( Z \)-inverting and \( I;: N \to NI(=N) \) is \( S \)-inverting.

We also recall that kernel of \( I; \) is the left \( S \)-component of 0,

\[ \{ x \in N \mid xs = 0 \text{ for some } s \in S \} \]

Given any set \( I; \) of square matrices over \( R; \), let be the set of matrices of \( R; \) that are inverted by the canonical mapping \( A : N \to N; \). This set is called the saturation of \( I; \); and if \( I; = x \), we call \( I; \) saturated. From the universal properties one sees that, for any \( S, \) (4) \( N, NI \).

**Proposition 5.1.** Let \( N \) be a ring and \( C \) a multiplicative set of matrices over \( N \). For any right \( N \)-module \( M \) denote by \( t( M ) \) the set of elements of \( M \) occurring as component in a row \( u \) such that \( uA = 0 \) for some \( A \in C \). Then the correspondence \( M_k \to t( t( M ) ) \) is an idempotent radical.

Further, both \( C \) and its admissible closure give rise to the same radical.

The first assertion means that \( t( M ) \) is a sub-module of \( M \) and the assignment \( M H t( M ) \) is an idempotent sub-functor of the identity such that \( t( M / t( M ) ) = 0 \).

**Proof.** Let \( x, y \in t( M ) \), say \( x = ul, y = ul \), where \( uA = 0 \), \( vB = 0 \) and \( A, B \in X \). Denote the first row of \( B \) by \( b \); and write \( v^t \) for the row \( v \) with its first element removed, then hence \( x - y \in t( M ) \), and a similar argument applies if \( x, y \) occur in places other than the first. Secondly, if \( UA = 0 \), then, for any \( c \)
\[ \in \mathbb{N}, \text{ and this shows that if } u_1 \in t(M), \text{ then } u_1c \in t(M), \text{ so that } t(M) \text{ is a sub-module.} \]

Clearly any homomorphism \( M \to N \) maps \( t(M) \) into \( t(N) \), and the correspondence \( M \to t(M) \) is easily seen to be a functor. It is also clear that \( t(t(M)) = t(M) \).

Next we show that \( t(M) \) is unchanged if we replace \( C \) by its ‘admissible closure’, i.e., the set of all matrices \( PAQ \), where \( A \in C \) and \( P, Q \) are products of elementary matrices. For, given \( c \in \mathbb{N} \), the pair of elements \( u_1 + u_1c \) lies in \( t(M) \) if and only if \( u_1 \) and \( u_2 \) lie in \( t(M) \); Thus \( u \cdot A = 0 \) is equivalent to \( u \cdot P - 1 \cdot PA = 0 \), where \( P \) is an elementary matrix, and by induction this holds for a product of elementary matrices.

\[ \square \]

### 4. Multiplicative Matrix Sets and Their Torsion Theories

Throughout, all rings are associative with a unit element which is preserved by homomorphism’s, inherited by sub-rings, and acts Unitally on modules.

**Definition 4.1.** Let \( N \) be a Noetherian ring, then a set \( C \) of square matrices over \( N \) is said to be multiplicative if \( 1 \in C \) and, for any \( A, B \in C \) and any matrix \( C \) of the right \( A \cdot C \) size, \( o(B) \in C \). Thus \( C \) contains square matrices of all orders; we write \( C_n \) for the set of \( n \times n \) matrices in \( C \). If \( C \) is multiplicative and moreover admits all elementary (row and column) transformations, it is called admissible.

**Definition 4.2.** A homomorphism \( f: R \to S \) is said to be \( C \)-inverting if each matrix of \( C \) is mapped to an invertible matrix by \( f \). It is easily seen that for any set \( C \) of square matrices over \( R \) there exists a ring \( R \), and a \( C \)-inverting homomorphism \( A: N \to N \) which is universal in the sense that every \( C \)-inverting homomorphism \( f: N \to S \) can be factored uniquely by \( A \), i.e., there exists a unique homomorphism \( ANR \to R \).

**Note 4.3.** It is clear that the set \( I \) is admissible for the multiplicative case; hence the multiplicative closure of \( I \) and the admissible closure of \( I \);

**Note 4.4.** Both have the same saturation as \( C \) itself, and by (4) all have isomorphic universal inverting rings.

We now come to torsion theories. As is well known, with any right denominator set a torsion theory may be associated. Instead of a denominator set we can also start with a multiplicative matrix set, and obtain a torsion theory as before:
Let $A$ by $Q$ presents no difficulty. Thus in what we follows may take $C$ to be admissible. It remains to prove the radical property: $t \left( \frac{M}{t(M)} \right) = 0$. If $uA = 0 \pmod{t(M)}$, where $A \in I$, write $uA = u$ and let $u = (u', u'')$ be a row including all components of $u'$ such that $uB = 0$ for some $B \in C$. We can take $u$ in this special form because $C$ is admissible. Then this shows that $uA0 \pmod{t(M)}$. Hence $t \left( \frac{M}{t(M)} \right) = 0$, and the proof is complete.

When we take $M = N$, $t(N)$ is just the ‘left $Z$ component of $0'$, i.e., the set of elements of $R$ occurring as component in some row $u$ such that $UA = 0$ for some $A \in C$. So that this is a right ideal, and it is clearly also a left ideal.

**Corollary 4.5.:** The left $\sum$-component of 0 is a two-sided ideal of $N$.

By the symmetry of the construction the same holds for the right $C$-component of 0. The functor $t(M)$ can now be used to define a torsion theory, as described e.g. in [24]; we refer to this as the $C$-torsion theory. In general this need not be hereditary, i.e., a sub-module of a $C$-torsion module need not be a $C$-torsion module.

However, let us assume that $X$ is such that the $\sum$-torsion theory is hereditary. Then for any $x \in t(M)$ there exist $u_2, \ldots, u \in N$ and $A \in C$, such that $(x, xu_2, \ldots, xu)A = 0$, since this expresses the fact that $xR$ is a $C$-torsion module containing $x$. More generally, let us define a $mi$-modular row over $R$ as a row $u$ of elements of $R$ such that for a suitable column $u'$ over $N$ we have $uu' = 1$.

**Proposition 4.6.** Let $N$ be a Noetherian ring and $C$ a multiplicative matrix set such that the $C$-torsion theory is hereditary. Then, for any right $N$-module $M$, the $C$-torsion sub-module $t(M)$ consists of all $x \in M$ such that $xuA = 0$ for some $A \in C$ and some $mi$-modular row $u$ over $N$.

Proof. Let $x \in t(M)$; then $xN$ is a $C$-torsion module containing $x$ and this means that $xuA = 0$ holds for some $A \in Z$ and a row $u$ one of whose components is $I$; thus it is certainly unimodular. Conversely, if $xuA = 0$ holds with a unimodular row $u$, then $xu_1 I t(M)$ and if $UU' = 1$, then $t(M)$ also contains $XUU' = x$.

The hereditary torsion theories so obtained form a special class; they are the theories obtainable by flat epi-morphisms. Given a ring homomorphism $f : K \rightarrow S$, we shall call a row $u$ over $R$ $f$-uni modular if its image under $f$ is unimodular in $S$. A homomorphism $f : N \rightarrow S$ is a left flat epi-morphism (i.e., $f$ is a ring epi-morphism such that $NS$ is flat) if and only if for each a kernel of $f$ there exists an $f$-uni-modular row $u$ such that $a^* u = 0$ and for each $b \in S$ there is an $f$-uni-modular row $u$ such that $b^* uf \in \text{im } f$. 
Now consider a ring $R$ and a family $\mathcal{I}$ of rows over $N$ such that, for any right $N$-module $M$, the set $t(M)$ of elements annihilated by some row of $\mathcal{I}$ forms a sub-module and that $t$ is an idempotent radical sub-functor of the identity.

Such a $t$ then defines a torsion theory, this time necessarily hereditary, and the quotient $Q(M)$ of any module $M$ may be constructed as follows: form $M, = M/t(M)$ and define $Q(M)$ by the equation where $I(M)$ is the injective hull of $M$. Regarded as $N$-module, $Q(M)$ is closed; we recall that an $N$-module $N$ is closed if $t(N) = 0$ and $t(Z(N)/N) = 0$ and such modules are called ‘torsion free divisible’.

In particular, $Q(N)$ is the ring obtained by first dividing out by $t(N)$, so as to get $N, = N / t(N)$, and then taking the set of those elements of $I(N)$ that are mapped to a row over $N$, by right multiplication by a row of $\mathcal{I}$.

If we apply this method to a non-hereditary torsion theory, we can again define $Q(M)$ by (6), but there will be no guarantee now that $Q(M)$ is closed (clearly this will be so provided that every essential extension of a torsion free module is torsion free). It is not known whether the torsion theory associated with every multiplicative matrix set has this property, but even if it does not, we can form the module $M@N$. Before comparing this with $Q(M)$ (when this is closed), we introduce an intermediate notion, the semi-inversive theory.

For any set $\mathcal{2}$ of matrices over $N$ we can form the right $Z$-inverting ring $N_{11}$, with canonical homomorphism $p : N \rightarrow N_{11}$, . This is the universal ring over $N$ in which every matrix of $Z$ has a right inverse; it is obtained by taking a presentation of $N$ and for every matrix $A$ of order $n$ adjoining $n^2$ in determinates $a$: $\mathcal{8}$, written as an $n \times n$ matrix $A^\prime = ( a : \neq )$ with defining relations

$$(\text{in matrix form}) \quad AA^\prime = I.$$

Since every matrix of $Z$ has a unique inverse in $N$, the canonical mapping $A : N \rightarrow N_1$ can be factored uniquely by $p$. The passage from $M$ to $M@N$, for some set $\mathcal{C}$ of matrices, is a form of localization which we shall call the semi-inversive method.

Let us compare the ring $( N, )$ with the ring $Q(N)$ obtained by the injective method (in case this has closed quotients). Any right $N_{11}$-module $P$ may be regarded as a right $N$-module, by pullback along $p$. We claim that this module $P$, is closed in the $C$-torsion theory: if $u$ is a row in $P$ such that $uA = 0$ for some $A \in X$, then $u = uAA^\prime = 0$; hence $P$, is torsion free. Now let $u$ be a row in $I(P)$ such that $u = uA$ is a row in $P$, for some $A \in Z$. Then $u = uAA^\prime = uA^\prime$ is a row in $P$; hence $t ( I(P) /P) = 0$, and $P$ is indeed closed or define $v = (v, v^\prime)$ so that $u = uA^\prime = uA = vA^\prime = 0$.

In particular, this applies to any module of the form $M@((R, ))$, where $M$ is a right $R$-module. By the universal property of $Q(M)$, this leads to the
commutative triangle shown: \(\triangle M\)

In particular, for \(M = N (=R)\) we obtain a canonical homomorphism by interpreting the elements of \(Q(N)\) and \(N\) as left multiplications. we see that is in fact a ring homomorphism. If we combine the above triangle with the canonical homomorphism from \(M \oplus Q(N)\) to \(Q(M)\) we obtain the commutative diagram shown: When is an isomorphism, the bottom arrow is an isomorphism for all \(R\)-modules \(M\). Conversely, if this arrow is an isomorphism for all \(M\), then by taking \(M = R = N\) we see that the mapping is an isomorphism.

Note 4.7. (Inverse Localization at a Factor Ring) Let \(N\) be a Noetherian ring and “a” be an ideal in \(N\); our object in this section is to lift a localization of \(N / a\) to one of \(N\). We begin with some general properties of multiplicative matrix sets. We recall that the set of all matrices inverted under a homomorphism is a saturated, hence admissible, and set. To ensure that only square matrices occur we shall assume our rings to be weakly finite; by definition, a ring is weakly finite if every invertible matrix over it is square. Let \(f: N \rightarrow fS\) be any homomorphism and let \(E\), be the set of all square matrices over \(N\) mapped to invertible matrices by \(J\): We shall write \(K\), instead of \(NZ\), and call this the localization associated with the homomorphism \(J\): Thus with every homomorphism \(f: N \rightarrow fS\) there is associated a ring \(R\), and a commutative triangle as shown:

**Theorem 4.8.** Let \(N\) be a \(\delta\)-near-ring. \(S\) be a sub near ring of \(N\). If \(f: N \rightarrow S\) be a homomorphism, and \(N\), the associated localization; then any square matrix over \(N\), mapped to an invertible matrix over \(S\) which is already invertible over \(N\).

Proof. Any \(x \in N\), occurs as a component \(x = u_1\), say, in an equation where \(A \in C\) and \(a\) is a column over \(N\). Write then \(x3\) is invertible in \(S\) if and only if \(A, f\) is invertible over \(S\) and this is so if and only if \(A \in C\); but then \(x\) is invertible in \(N\). This proves the result for \(1 \times 1\) matrices. In the general case, of an \(n \times n\) matrix, consider the corresponding triangle of matrix rings: Any matrix over \(N\), that becomes invertible over \(S\), lies in \(C\) and so becomes invertible over \(K\), hence any element of \((Nf)\), whose image is invertible in \(S\), is already invertible in \((Nj)n\).

We recall that the rational closure of \(N\) in \(S\) (under the mapping \(j\)) is the set of all solutions of equations (8) with \(A \in C\); in particular, if this is the whole of \(S\), the latter is said to be matrix-rational over \(N\). Below we denote the Jacobson radical of a Noetherian ring \(N\) by \(J(N)\).

**Corollary 4.9.** Let \(f: N \rightarrow PS\) be a homomorphism such that \(S\) is matrix-rational over \(N\); then \(f: N \rightarrow S\) is surjective and \(J(N) = f^{-1}(J(S))\).
Proof. The surjectivity is clear, and it implies that $J(N) \simeq f^{-1}(J(S))$. Conversely, let $u \in J(S)$; then, for any $x \in N$, $1 - ax$ has an invertible image in $S$ and so is invertible in $N$, and hence $a \in J(N)$.

Hence, we proved the corollary.

References


