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TRANSIENT AND RELIABILITY ANALYSIS OF M/M/1 FEEDBACK QUEUE SUBJECT TO CATASTROPHES, SERVER FAILURES AND REPAIRS

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Abstract: Continued Fractions method has been used to obtain a transient solution for the system size probabilities in an M/M/1 queue with instantaneous Bernoulli feedback subject to Catastrophes, Server Failures and Repairs. The steady state analysis of system size probabilities and some performance measures of the system are deduced. Further, Busy Period, Reliability and Availability of the model are analyzed. Finally Numerical Illustrations are provided to see the effect of parameters on system performance measures.

AMS Subject Classification: 60K25, 90B25
Key Words: M/M/1 queue with Bernoulli feedback, catastrophes, transient analysis, steady state analysis, busy period and reliability analysis

1. Introduction

In communication systems, the messages which are turned down as errors by the receiver are retransmitted back to the transmitter. The transmitter re sends the original message again. This can be modeled as feedback queues. To study the system (like communication and computer networks) behaviour

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over a time horizon, we require time dependent analysis of queues, since the steady state results are inappropriate in situations where the time horizon of operations is finite. Transient state measures are very important to track down the functioning of the system at any instant of time. In telecommunication systems, especially in wide area network, data are combined into a signal which is then divided by the network management called packets and is sent through the network path. The routing is more complex and dynamic. At each node the processor detects the packet address and sends it through the best available path to the next node. As the transmission channel is an electrical medium(e.g. wires, coaxial cable and optical fiber)packets are often subjected to distortion. We have "repeaters"in the transmission system which rebuild or regenerate the packet(signal) into its original form. This is a situation which demands the time dependent analysis of feedback queues. The detection, decision and path routing are done for each packet in real time and so the packets do not pile up for service. Even if there is a reasonable size buffer or queue, they would soon overflow resulting the packet(message) loss or refusal. At times, packets are held up in nodes due to unavailability of paths in the network. This causes delay of several seconds or minutes for the complete message to reach the intended receiver. Finally this ends up in server service degradation.

By knowing the mean current buffer(queue) size and hence the waiting time of a packet at a node, the incoming packet may be diverted to another feasible node. Time dependent analysis of queueing models will be helpful in perturbing the parameters involved in the system so that we can have control over the system (in areas like flow congestion, short duration failures of nodes), which results in optimum solutions(server service upgradation) of the system.

There are methods that have been derived for obtaining transient solution of queues: generating function method by Bailey [1], spectral method by Lederman and Reuter [13], combinatorial method by Champernowene [2], difference equation method by Conolly [3], alternative approach of generating functions by Parthasarathy [19]. The theory of continued fractions can be found in Jones and Thron [8]. Its application to the study of birth and death process was initiated by Murphy and O'Donohoe [15]. Flajolet and Guillemin [7], Parthasarathy et.al., [16], [17] and [18], Thangaraj and Vanitha [22] have applied continued fraction technique to study the transient behaviour of the queueing systems. We have used the continued fraction technique to obtain the transient solution of the M/M/1 queue with feed back subject to catastrophe ,server failures and repairs.

Feedback queues play a vital role in the areas of Computer networks, Production systems subject to rework, Hospital management, Super markets and

Banking business etc. Takacs [20] introduced the concept of feedback queues. The queueing systems which include the possibility for a customer to return to the server for additional service are called queues with feedback. Disney, McNickle and Simmon [5], D'Avignon and Disney [4], Krishnakumar [10], Thangaraj and Vanitha [21] and [22] are a few to be mentioned for their contribution.

The rest of the paper is organized as follows: In Section 2, the mathematical description of the considered queueing system is given. In Section 3, the transient state probabilities of the system are derived. In Section 4, steady state probabilities of the system are deduced. In Section 5, we obtained several performance measures of the system. In Section 6, reliability analysis of the system has been carried. Finally, in Section 7 some Numerical examples are provided to illustrate the effect of parameters on some performance characteristics.

2. Description of the Model

arrival: Customers arrive at the server according to *poisson process* with parameter λ.

server: Single server *M/M/1* queueing system.

service: Service times of the customers are *iid* exponential random variable with parameter μ and service is *FCFS*.

feedback: After completion of each service the customer either join at the *end* of the queue with probability p or leave the system with probability $q = 1 - p$.

catastrophes: The catastrophes occur at the server as a independent *poisson process* with parameter ν and inactivate the server upon arrival.

repairs: The repair times of the failed server are *iid* exponential random variable with parameter η.

Our system behaves as a standard M/M/1 queue. Whenever a catastrophe occur at a server, immediately the server is inactivated and all the customers in the system are flushed out. Customers arrive during the repair time are considered to be lost for ever. The customers both newly arrived and feedback from the queue are served in the order in which they join the tail of the original queue. we do not consider any distinction between regular and feedback arrival.

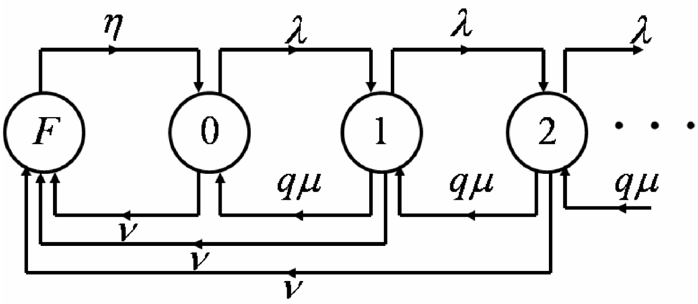

Figure 1: State Transition Diagram

3. Transient Analysis of System Size Probabilities

We consider this queueing system as a model of Continuous Time Markov Chain (CTMC). Let $\{X(t) : t \in \Re^{+}\}$ be the number of customers in the system at time t. Let $P_n(t) = P\big(X(t) = n\big), n = 0, 1, 2, ...$ be the state probabilities that there are n customers in the system at time t. Let $Q(t)$ be the probability that the server is repair at time t. Based on the assumptions the Chapman-Kolmogorov's forward differential difference equations for the state probabilities are given by

$$Q'(t) = -(\eta + \nu)Q(t) + \nu \tag{1}$$

$$P_0'(t) = \eta Q(t) - (\lambda + \nu)P_0(t) + q\mu P_1(t) \tag{2}$$

$$P_n'(t) = \lambda P_{n-1}(t) - (\lambda + q\mu + \nu)P_n(t) + q\mu P_{n+1}(t), n = 1, 2, 3, ... \tag{3}$$

Without loss of generality, we assume that initially there is no customers in the system and the server is up, i.e., $P_0(0) = 1, Q(0) = 0 \quad \text{and} \quad P_n(0) = 0.$

In the following, for any function $f(.)$, let $f^*(z)$ denote its Laplace Transform. By taking Laplace Transform, the above system of equations are transformed into the following system of equations.

$$(z + \eta + \nu)Q^*(z) = \nu \frac{1}{z} \tag{4}$$

$$(z + \lambda + \nu)P_0^*(z) = 1 + \eta Q^*(z) + q\mu P_1^*(z) \tag{5}$$

$$(z + \lambda + q\mu + \nu)P_n^*(z) = \lambda P_{n-1}^*(z) + q\mu P_{n+1}^*(z), n = 1, 2, ... \tag{6}$$

from equation (5), we have

$$(z + \lambda + \nu) = \frac{1}{P_0^*(z)} + \eta \frac{Q^*(z)}{P_0^*(z)} + q\mu \frac{P_1^*(z)}{P_0^*(z)}$$

$$P_0^*(z) = \frac{1}{(z+\lambda+\nu) - \eta\frac{Q^*(z)}{P_0^*(z)} - q\mu\frac{P_1^*(z)}{P_0^*(z)}} \tag{7}$$

from equation(6), we have

$$(z+\lambda+q\mu+\nu) = \lambda\frac{P_{n-1}^*(z)}{P_n^*(z)} + q\mu\frac{P_{n+1}^*(z)}{P_n^*(z)}$$

$$\lambda\frac{P_{n-1}^*(z)}{P_n^*(z)} = (z+\lambda+q\mu+\nu) - q\mu\frac{P_{n+1}^*(z)}{P_n^*(z)}$$

$$\frac{P_n^*(z)}{P_{n-1}^*(z)} = \frac{\lambda}{(z+\lambda+q\mu+\nu) - q\mu\frac{P_{\mathsf{n}+1}^*(z)}{P_{\mathsf{n}}^*(z)}}, \quad n = 1, 2, 3, ... \tag{8}$$

from equation(4), we have

$$Q^*(z) = \frac{\nu}{z(z+\eta+\nu)} \tag{9}$$

substituting $Q^*(z)$ in (7),

$$(z+\lambda+\nu) = \frac{1}{P_0^*(z)}\left(1 + \frac{\eta\nu}{z(z+\eta+\nu)}\right) + q\mu\frac{P_1^*(z)}{P_0^*(z)}$$

from this,we get

$$P_0^*(z) = \frac{1 + \frac{\eta\nu}{z(z+\eta+\nu)}}{(z+\lambda+\nu) - \frac{q\mu P_1^*(z)}{P_0^*(z)}} \tag{10}$$

using (8) iteratively in (10) we express $P_0^*(z)$ as a continued fraction

$$P_0^*(z) = \cfrac{1 + \frac{\eta\nu}{z(z+\eta+\nu)}}{(z+\lambda+\nu) - \cfrac{q\mu\lambda}{(z+\lambda+q\mu+\nu) - \cfrac{q\mu P_2^*(z)}{P_1^*(z)}}}$$

$$P_0^*(z) = \cfrac{1 + \frac{\eta\nu}{z(z+\eta+\nu)}}{(z+\lambda+\nu) - \cfrac{\lambda q\mu}{(z+\lambda+q\mu+\nu) - \cfrac{\lambda q\mu}{(z+\lambda+q\mu+\nu) - \cdots}}}$$

The above equation can be written as,

$$P_0^*(z) = \frac{1 + \frac{\eta\nu}{z(z+\eta+\nu)}}{(z+\lambda+\nu) - \Phi(z)} \tag{11}$$

where

$$\Phi(z) = \cfrac{\lambda q\mu}{(z+\lambda+q\mu+\nu) - \cfrac{\lambda q\mu}{(z+\lambda+q\mu+\nu) - \cfrac{\lambda q\mu}{(z+\lambda+q\mu+\nu) - \cdots}}}$$

$$= \frac{\lambda q\mu}{(z+\lambda+q\mu+\nu) - \Phi(z)}$$

this implies that

$$\Phi(z)^2 - (z+\lambda+q\mu+\nu)\Phi(z) + \lambda q\mu = 0$$

the roots of this quadratic equation are

$$\alpha(z), \beta(z) = \frac{w \pm \sqrt{w^2 - 4\lambda q\mu}}{2}$$

where $w = z+\lambda+q\mu+\nu$. It is seen that $\beta(z)$ is a unique real root within $[0,1)$ and $0 \le z < 1$, [8]. So we consider only $\beta(z)$ for our further discussion substituting $\beta(z)$ for $\Phi(z)$ in(11), we get

$$P_0^*(z) = \frac{1 + \frac{\eta\nu}{z(z+\eta+\nu)}}{(z+\lambda+\nu) - \left(\frac{w-\sqrt{w^2-4\lambda q\mu}}{2}\right)} \tag{12}$$

after some algebra $P_0^*(z)$ reduces to

$$\frac{\left(1 + \frac{\eta\nu}{z(z+\eta+\nu)}\right) \times \frac{w-\sqrt{w^2-4\lambda q\mu}}{2\lambda q\mu}}{1 - \frac{q\mu(w-\sqrt{w^2-4\lambda q\mu})}{2\lambda q\mu}}$$

Expanding binomially

$$P_0^*(z) = \left(1 + \frac{\eta\nu}{z(z+\eta+\nu)}\right)\left(\sum_{n=0}^{\infty}(q\mu)^n\left(\frac{w-\sqrt{w^2-4\lambda q\mu}}{2\lambda q\mu}\right)^{n+1}\right)$$

By Inverting $P_0^*(z)$,we get the explicit expression for $P_0(t)$

$$P_0(t) = \sum_{n=0}^{\infty}\frac{(n+1)(q\mu)^n}{(\lambda q\mu)^{\frac{n+1}{2}}}e^{-(\lambda+q\mu+\nu)t}\frac{I_{n+1}(2\sqrt{\lambda q\mu}t)}{t}$$
$$+ (\eta\nu)\sum_{n=0}^{\infty}\frac{(n+1)(q\mu)^n}{(\lambda q\mu)^{\frac{n+1}{2}}}\int_0^t\int_0^u e^{-(\lambda+\nu)v}e^{-(\lambda+q\mu+\nu)(u-v)}\frac{I_{n+1}(2\sqrt{\lambda q\mu}(u-v))}{(u-v)}dvdu \tag{13}$$

where $I_n(.)$is the modified Bessel function of the first kinds of order n.

We get the other transient probabilities $P_n(t)$ for $n = 1, 2, 3, ...$, in terms of $P_0(t)$ and $Q(t)$. From (8),

$$\frac{P_n^*(z)}{P_{n-1}^*(z)} = \frac{\lambda}{(z+\lambda+q\mu+\nu) - \frac{q\mu P_{n+1}^*(z)}{P_n^*(z)}}, n = 1, 2, 3, ...$$

$$= \frac{\lambda}{(z+\lambda+q\mu+\nu) - \cfrac{\lambda q\mu}{(z+\lambda+q\mu+\nu) - \cdots}} \tag{14}$$

$$= \frac{\lambda}{(z+\lambda+q\mu+\nu) - \Phi(z)}$$

where $\Phi(z) = \dfrac{\lambda q\mu}{(z+\lambda+q\mu+\nu) - \Phi(z)}$
this implies that

$$\Phi(z)^2 - (z+\lambda+q\mu+\nu)\Phi(z) + \lambda q\mu = 0$$

the roots of this quadratic equation are

$$\alpha(z), \beta(z) = \frac{w \pm \sqrt{w^2 - 4\lambda q\mu}}{2}$$

where $w = z + \lambda + q\mu + \nu$. It is seen that $\beta(z)$ is a unique real root within $[0, 1)$ and $0 \le z < 1$, [8]. So we consider only $\beta(z)$ for our further discussion substituting $\beta(z)$ for $\Phi(z)$ in (14), we get

$$\frac{P_n^*(z)}{P_{n-1}^*(z)} = \frac{\lambda}{(z+\lambda+q\mu+\nu) - \frac{w-\sqrt{w^2-4\lambda q\mu}}{2}}$$

$$= \frac{2\lambda(w - \sqrt{w^2 - 4\lambda q\mu})}{4\lambda q\mu}, n = 1, 2, 3, ...$$

Hence, we have

$$P_n^*(z) = \left(\frac{w - \sqrt{w^2 - 4\lambda q\mu}}{2q\mu}\right) P_{n-1}^*(z), n = 1, 2, 3, ...$$

$$= \left(\frac{w - \sqrt{w^2 - 4\lambda q\mu}}{2q\mu}\right)^n P_0^*(z)$$

On Inversion,we get

$$P_n(t) = \int_0^t P_0(u) \left(\frac{\lambda}{q\mu}\right)^{\frac{n}{2}} e^{-(\lambda+q\mu+\nu)(t-u)} n \frac{I_n(2\sqrt{\lambda q\mu}(t-u))}{(t-u)} du \tag{15}$$

We have

$$Q^*(z) = \frac{\nu}{z(z+\eta+\nu)}$$

On Inversion, we get

$$\begin{aligned} Q(t) &= \nu \int_0^t e^{-(\eta+\nu)u} du \\ &= \frac{\nu}{\eta+\nu}\left(1 - e^{-(\eta+\nu)t}\right) \end{aligned} \tag{16}$$

Thus equations (16),(13) and (15) completely determine all the transient state probabilities $Q(t), P_0(t)$ and $P_n(t), n = 1, 2, 3, ...$

Remark 1. If we consider η tends to ∞, then $Q(t) = 0$ and the remaining transient state probabilities $P_n(t), n = 0, 1, 2, 3...$ are all coincides with the results of Thangaraj and Vanitha [22]

Theorem 2. *If $\nu > 0$ and $\eta > 0$, then the asymptotic behaviour of the probability of the server being idle is given by*

$$P_0(t) \to \frac{\eta\nu}{\eta+\nu} \sum_{k=0}^{\infty} \frac{1}{(\lambda+\nu)^{k+1}} \left(\frac{w_1 - \sqrt{w_1^2 - 4\lambda q\mu}}{2}\right)^k \quad as \qquad t \to \infty$$

where $w_1 = \lambda + q\mu + \nu$

Proof. Multiplying the equation (12) by z and taking limit $z \to 0$,

$$\begin{aligned} \lim_{z\to 0} zP_0^*(z) &= \frac{\frac{\eta\nu}{\eta+\nu}}{(\lambda+\nu)\left(1 - \frac{w_1 - \sqrt{w_1^2 - 4\lambda q\mu}}{2(\lambda+\nu)}\right)} \\ &= \frac{\eta\nu}{\eta+\nu} \sum_{k=0}^{\infty} \frac{1}{(\lambda+\nu)^{k+1}} \left(\frac{w_1 - \sqrt{w_1^2 - 4\lambda q\mu}}{2}\right)^k \end{aligned} \tag{17}$$

where $w_1 = \lambda + q\mu + \nu$

By using Tauberian Theorem [23] the result follows. □

Theorem 3. *If $\nu > 0$ and $\eta > 0$, then the asymptotic behaviour of the mean system size $m(t)$ is given by*

$$m(t) \to \frac{\eta(\lambda - q\mu)}{\nu(\eta+\nu)} + \frac{2\eta q\mu}{(\eta+\nu)(2(\lambda+\nu) - (w_1 - \sqrt{w_1^2 - 4\lambda q\mu}))}$$

as $t \to \infty$

Proof. Consider the Kolmogorov's differential difference equations with initial conditions. Let $P(z,t) = Q(t) + \sum_{n=0}^{\infty} P_n(t)z^n$ be the probability generating function. It can be seen that the pgf $P(z,t)$ satisfies the partial differential equation

$$\frac{\partial P(z,t)}{\partial t} = [\lambda z + \tfrac{q\mu}{z} - (\lambda + q\mu + \nu)]P(z,t) + q\mu(1 - \tfrac{1}{z})\big(P_0(t) + Q(t)\big) + \lambda(1-z)Q(t) + \nu$$

The mean system size is $m(t) = \sum_{n=1}^{\infty} nP_n(t) = \frac{\partial P(z,t)}{\partial z}\Big|_{z=1}$

Differentiating the above equation with respect to z and evaluating at $z = 1$, we get

$$\frac{dm(t)}{dt} + \nu m(t) = \lambda - q\mu(1 - P_0(t)) - (\lambda - q\mu)Q(t)$$

Solving the differential equation for $m(t)$ with $m(0) = \sum_{n=1}^{\infty} nP_n(t) = 0$, we have

$$m(t) = \int_0^t \lambda e^{-\nu(t-u)}du - q\mu \int_0^t (1 - P_0(t))e^{-\nu(t-u)}du - (\lambda - q\mu)\int_0^t Q(t)e^{-\nu(t-u)}du$$

Let us denote the Laplace Transform of $m(t)$ by $m^*(z)$, then

$$m^*(z) = \frac{\lambda}{z(z+\nu)} - \frac{q\mu}{z+\nu}\left(\frac{1}{z} - P_0^*(z)\right) - \frac{\lambda - q\mu}{z+\nu}Q^*(z)$$

We obtain the result by multiplying the above equation by z and taking limit $z \to 0$ and using Tauberian Theorem [23]. □

Remark 4. The measure mean $m(t)$, is useful in transient stage as it addresses the buffer requirements for packets in telecommunication switching network, and give an indication of the possible waiting time of a packet before exiting a node in a network. The more the packets in a node it inflicts the delay in receiving the packets at destination.

4. Steady State Analysis

In this section, we derive the stationary distribution of queue length and failure state probability of our queueing model. To define the stationary probabilities we apply the Tauberian Theorem [23],

$$\lim_{s \to 0} sf^*(s) = \lim_{t \to \infty} f(t)$$

Theorem 5. *For $\nu > 0$ and $\eta > 0$, the steady state distributions $\{Q, \pi_n : n \geq 0\}$ of the $M/M/1$ feedback queue subject catastrophes, failures and repairs corresponds to*

$$Q = \frac{\nu}{\eta + \nu} \tag{18}$$

$$\pi_0 = (1 - Q)(1 - \rho) \tag{19}$$

$$\pi_n = (1 - Q)\rho^n(1 - \rho) \quad n = 1, 2, ... \tag{20}$$

$$\textit{where} \quad \rho = \frac{\lambda + \nu + q\mu - \sqrt{(\lambda + q\mu + \nu)^2 - 4\lambda q\mu}}{2q\mu} \tag{21}$$

Proof. We have used the Tauberian Theorem to extract the steady state probabilities from the transient state probabilities. First the failure state probability Q is given by

$$Q = \lim_{z \to 0} zQ^*(z) = \frac{\nu}{\eta + \nu}$$

Now,

$$\pi_0 = \lim_{z \to 0} zP_0^*(z)$$
$$= \frac{\frac{\eta\nu}{\eta+\nu}}{\lambda + \nu - \left(\frac{\lambda+q\mu+\nu-\sqrt{(\lambda+q\mu+\nu)^2-4\lambda q\mu}}{2}\right)}$$

after some algebra π_0 simplifies to $(1 - Q)(1 - \rho)$

similarly,

$$\pi_n = \lim_{z \to 0} zP_n^*(z)$$
$$= \lim_{z \to 0}\left(\frac{w - \sqrt{w^2 - 4\lambda q\mu}}{2q\mu}\right)^n \lim_{z \to 0} zP_0^*(z)$$
$$= (1 - Q)\rho^n(1 - \rho)$$

It is observed that stationary probabilities exists if and only if $\rho < 1$. □

Remark 6. If we set η tends to ∞, then $Q = 0$ and the remaining steady state probabilities $\pi_n, n = 0, 1, 2, 3...$ are all coincides with the results of Thangaraj and Vanitha [22]. Further if we assume $\nu = 0$ then $\rho = \frac{\lambda}{q\mu} < 1$, which is the steady state condition for an $M/M/1$ feedback queue (see[20])

5. Some Probability Measures

In this section we obtain the moments related to the steady state system size probabilities.

Theorem 7. *For $\nu > 0$ and $\eta > 0$, the steady state probability generating function $\Pi(s)$ is given by*

$$\Pi(s) = \frac{Q\rho(1-s) + (1-\rho)}{(1-\rho s)}.$$

The mean, variance of the system size namely $E(N), V(N)$ and the mean of the queue size $E(N_q)$ are obtained as

$$E(N) = \frac{\rho(1-Q)}{1-\rho} \tag{22}$$

$$V(N) = \frac{\rho(1-Q)(1+\rho Q)}{(1-\rho)^2} \tag{23}$$

$$\text{and} \quad E(N_q) = \frac{(1-Q)\rho^2}{1-\rho} \tag{24}$$

We define the steady state probability generating function $\Pi(s)$ as

$$\Pi(s) = (Q + \pi_0) + \sum_{n=1}^{\infty} \pi_n s^n$$

Substituting the values of Q, π_0 and π_n from equations (19),(20) and (21) and after simplification we get

$$\Pi(s) = \frac{Q\rho(1-s) + (1-\rho)}{(1-\rho s)}$$

Let $E(N), V(N)$ denote the mean and variance of the steady state system size. They are obtained by taking the derivatives of $\Pi(s)$ with respect to s and setting $s = 1$. Now

$$E(N) = \lim_{s\to 1} \Pi'(s) = \frac{\rho(1-Q)}{1-\rho}$$

$$\begin{aligned} E(N^2) &= \lim_{s\to 1} \Pi''(s) + \lim_{s\to 1} \Pi'(s) \\ &= \frac{2\rho^2(1-Q)}{(1-\rho)^2} + \frac{\rho(1-Q)}{1-\rho} \end{aligned}$$

Since $V(N) = E(N^2) - E(N)^2$, substituting the values of $E(N)$, $E(N^2)$ and after some simplification we get

$$V(N) = \frac{\rho(1-Q)(1+\rho Q)}{(1-\rho)^2}$$

Let $E(N_q)$ denotes mean of the stationary queue size. Since

$$E(N_q) = \sum_{n=1}^{\infty}(n-1)\pi_n$$

substituting the values of $\pi_n, n = 1, 2, ...$ and after some algebra we see that

$$E(N_q) = \frac{(1-Q)\rho^2}{1-\rho}$$

Note 1. The mean number of customers in the system $E(N)$ includes the repair time and idle time when there are no customers in the system.

$$P(\text{Server is busy}) = \sum_{n=1}^{\infty}\pi_n = (1-Q)\rho$$

$$P(\text{Server is Idle or Under Repair}) = \pi_0 + Q = 1 - \rho(1-Q)$$

$$P(\text{Server is Busy/Server is up}) = \frac{(1-Q)\rho}{(1-Q)} = \rho$$

$$P(\text{Server is Idle/Server is up}) = \frac{\pi_0}{1-Q} = 1-\rho$$

Theorem 8. *For $\nu > 0$ and $\eta > 0$, under steady state, the system throughput U is given as*

$$U = \frac{\eta}{\eta+\nu}\rho q\mu$$

where ρ is given by the equation (21)

Proof. The system throughput U, is the rate at which customers exit the queue whenever there are one or more customers in the system. With the exit rate $q\mu$, we have

$$U = \left[1 - (\pi_0 + Q)\right] q\mu$$

by using equations (19) and (20) we obtain the result. □

6. Busy Period Analysis

According to our model, we define a busy period as the interval of time commencing at the instant 0 when a customer arrives at an empty system and terminating at the instant when the server becomes free for the first time or the interval of time commencing at the instant 0 when a customer arrives at an empty system and terminating at the instant when the server breaks down while in operation. Let the length of the interval which is a random variable be T and $\{N^*(t)\}$ be the stochastic process denoting the number of customers present at the instant t during the busy period. We have $\{N^*(0) = 1\}$ and the duration of the busy period is the first passage time from state 1 to state 0 or from state 1 to the repair state. Let $q_n(t) = P\{N^*(t) = n/N^*(0) = 1\}, \quad n = 1, 2, ...$ be the zero avoiding state probabilities. Then $q_1(0) = 1,\ q_n(0) = 0 \quad n = 2, 3, ...$ Now $q_n(t)$ will satisfy the following differential difference equations.

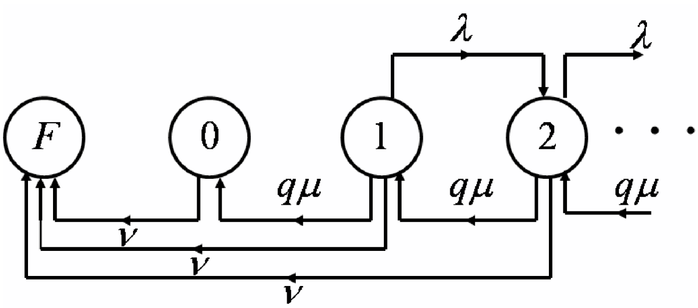

Figure 2: State Transition Diagram for Busy Period

$$q_1'(t) = -(\lambda + q\mu + \nu)q_1(t) + q\mu q_2(t)$$
$$q_n'(t) = -(\lambda + q\mu + \nu)q_n(t) + \lambda q_{n-1}(t) + q\mu q_{n+1}(t) \qquad n = 2, 3, \ldots$$

We solve the above system of equations using continued fraction technique as discussed in the earlier section, the solutions for the above equations are given by

$$q_n(t) = \left(\frac{\lambda}{q\mu}\right)^{\frac{n}{2}} \frac{1}{\lambda} e^{-(\lambda+q\mu+\nu)t} n \frac{I_n(2\sqrt{\lambda q\mu}t)}{t} \qquad n = 1, 2, 3, ...$$

We shall now obtain the probability density function $b(t)$ of the busy period. Conditioning on the number of customers present at the instant t, we consider two cases

Case I: Considering the termination of busy period due to service completion results in

$$
\begin{aligned}
b(t)dt &= P\{t \le T < t+dt\} \\
&= \sum_{n=1}^{\infty} P\{t \le T < t+dt/N^*(t)=n\}P\{N^*(t)=n\} \\
&= P\{t \le T < t+dt/N^*(t)=1\}P\{N^*(t)=1\} \\
&\qquad + \sum_{n=2}^{\infty} P\{t \le T < t+dt/N^*(t)=n\}P\{N^*(t)=n\}
\end{aligned}
$$

The first term implies that there is only one customer at the instant t whose service is completed between $(t, t+dt)$,the probability of this event being $q\mu dt + \circ(dt)$. The second term implies that service completion of two or more customers in $(t, t+dt)$ and the probability of this event is $\circ(dt)$.
Case II: Considering the termination of busy period due to the occurrence of catastrophe results in

$$
\begin{aligned}
b(t)dt &= P\{t \le T < t+dt\} \\
&= \sum_{n=1}^{\infty} P\{t \le T < t+dt/N^*(t)=n\}P\{N^*(t)=n\} \\
&= \sum_{n=1}^{\infty} (\nu dt + \circ(dt))P\{N^*(t)=n\}
\end{aligned}
$$

This is because the occurrence of catastrophe at the server is independent of the number of customers in the system and the probability of occurrence of a catastrophe in$(t, t+dt)$ is $\nu dt + \circ(dt)$.

Since completion of busy period is due to service completion of customers at time t or due to the occurrence of catastrophe at time t. Thus taking limit as $dt \to 0$,we have

$$
\begin{aligned}
b(t) &= q\mu q_1(t) + \sum_{n=1}^{\infty} \nu q_n(t) \\
&= q\mu \frac{1}{\sqrt{\lambda q\mu}} e^{-(\lambda+q\mu+\nu)t} \frac{I_1(2\sqrt{\lambda q\mu}t)}{t} + \nu \sum_{n=1}^{\infty} \left(\frac{\lambda}{q\mu}\right)^{\frac{n}{2}} \frac{1}{\lambda} e^{-(\lambda+q\mu+\nu)t} \\
&\quad \times \frac{nI_n(2\sqrt{\lambda q\mu}t)}{t}
\end{aligned}
$$

Taking Laplace transform

$$b^*(z) = \left(\frac{w - \sqrt{w^2 - 4\lambda q\mu}}{2\lambda}\right) + \frac{\nu}{\lambda}\sum_{n=1}^{\infty}\left(\frac{w - \sqrt{w^2 - 4\lambda q\mu}}{2q\mu}\right)^n$$

$$\text{where} \quad w = z + \lambda + q\mu + \nu$$

Taking the derivatives and evaluating at $z = 0$, we get mean of T

$$E(T) = -\frac{db^*(z)}{dz}\Big|_{z=0} = \frac{1}{2\lambda}\left(\frac{w_1 - \sqrt{w_1^2 - 4\lambda q\mu}}{\sqrt{w_1^2 - 4\lambda q\mu}}\right)$$
$$+ \frac{\nu}{\lambda}\sum_{n=1}^{\infty}\left(\frac{1}{2q\mu}\right)^n n\left(w_1 - \sqrt{w_1^2 - 4\lambda q\mu}\right)^n \frac{1}{\sqrt{w_1^2 - 4\lambda q\mu}}$$
$$\text{where} \quad w_1 = \lambda + q\mu + \nu$$

$$E(T^2) = \frac{d^2b^*(z)}{dz^2}\Big|_{z=0} = \frac{1}{2\lambda}\frac{4\lambda q\mu}{(w_1^2 - 4\lambda q\mu)^{\frac{3}{2}}}$$
$$+ \frac{\nu}{\lambda}\sum_{n=1}^{\infty}\left(\frac{1}{2q\mu}\right)^n n\left(w_1 - \sqrt{w_1^2 - 4\lambda q\mu}\right)^n \frac{(n\sqrt{w_1^2 - 4\lambda q\mu} + w_1)}{\left(w_1^2 - 4\lambda q\mu\right)^{\frac{3}{2}}}$$
$$\text{where} \quad w_1 = \lambda + q\mu + \nu$$

Now variance of T is

$$V(T) = E(T^2) - E(T)^2$$
$$= \frac{8\lambda^2 q\mu - \left(w_1 - \sqrt{w_1^2 - 4\lambda q\mu}\right)^2\sqrt{w_1^2 - 4\lambda q\mu}}{4\lambda^2(w_1^2 - 4\lambda q\mu)^{\frac{3}{2}}}$$
$$+ \frac{\nu}{\lambda}\sum_{n=1}^{\infty}\left(\frac{1}{2q\mu}\right)^n n\left(w_1 - \sqrt{w_1^2 - 4\lambda q\mu}\right)^n \frac{(n\sqrt{w_1^2 - 4\lambda q\mu} + w_1)}{\left(w_1^2 - 4\lambda q\mu\right)^{\frac{3}{2}}}$$
$$- \left(\frac{\nu}{\lambda}\right)^2 \frac{1}{w_1^2 - 4\lambda q\mu}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} nk\left(\frac{1}{2q\mu}\right)^{n+k}\left(w_1 - \sqrt{w_1^2 - 4\lambda q\mu}\right)^{n+k}$$
$$- \frac{\nu}{\lambda^2}\frac{1}{w_1^2 - 4\lambda q\mu}\sum_{n=1}^{\infty} n\left(\frac{1}{2q\mu}\right)^n\left(w_1 - \sqrt{w_1^2 - 4\lambda q\mu}\right)^{n+1}$$
$$\text{where} \quad w_1 = \lambda + q\mu + \nu$$

7. Reliability Analysis

The reliability of the system is defined as the probability that the system continues to work with efficiently, over a given period of time, subject to the given environmental conditions. The availability may be interpreted as the probability that the system is operational at a given point of time or the fraction of the test interval in which it is performing. Let $A(t)$ be the probability that the system is available at time t. Then, from (16) we obtain the point availability as

$$\begin{aligned} A(t) &= 1 - Q(t) \\ &= \frac{\eta}{\eta+\nu} + \frac{\nu}{\eta+\nu} e^{-(\eta+\nu)t} \end{aligned} \tag{25}$$

The average availability of the system in $[0, t]$ is

$$\begin{aligned} \bar{A}(t) &= \frac{1}{t}\int_0^t A(u)du \\ &= \frac{\eta}{\eta+\nu} + \frac{\nu}{(\eta+\nu)^2}\frac{1}{t}\left(1 - e^{-(\eta+\nu)t}\right) \end{aligned} \tag{26}$$

If $\eta = 0$,then from (16) we get

$$Q(t) = 1 - e^{-\nu t}$$

So the Reliability function $R(t)$ of the system is given by

$$R(t) = 1 - Q(t) = e^{-\nu t}$$

the mean time to system failure (MTTF) is given by

$$MTTF = \int_0^\infty R(t)dt = \frac{1}{\nu}$$

Maintainability $M(t)$ of the system is the probability that the system will be repaired within time t. If T is a random variable representing the repair time, then

$$M(t) = P(T \leq t)$$

Since repair time is exponentially distributed with parameter η, then the repair density function is $g(t) = \eta e^{-\eta t}$ and therefore

$$M(t) = P(T \leq t) = \int_0^t \eta e^{-\eta u} du = 1 - e^{-\eta t}$$

The expected value of repair time is called the mean time to repair (MTTR) and is given by

$$MTTR = \int_0^\infty tg(t)dt = \int_0^\infty \eta te^{-\eta t} = \frac{1}{\eta}$$

From equation (25) the steady state or long run availability of the system is

$$A = A(\infty) = \lim_{t\to\infty} A(t) = \frac{\eta}{\eta + \nu}$$

Also from equation (26)

$$\lim_{t\to\infty} \bar{A}(t) = \frac{\eta}{\eta + \nu}$$

Hence

$$A = \frac{\eta}{\eta + \nu} = \frac{\frac{1}{\nu}}{\frac{1}{\nu} + \frac{1}{\eta}} = \frac{MTTF}{MTTF + MTTR} = \lim_{t\to\infty} \bar{A}(t)$$

i.e., in long run the point availability is equivalent to average availability.

8. Numerical Illustrations

In this section we visualize how the system parameters influence the results we have derived in Sections 4 and 5. Fig.3 shows that as λ increases the empty probability of the system π_0 decreases as expected.It is also evident that when the probability q increases, π_0 correspondingly have higher values. Fig.4 describes that as catastrophe rate ν increases the probability of the system becoming empty also increases. In this situation we can also note that as the arrival rate λ increases, π_0 relatively diminishes. Fig.5 reads that as the incoming of catastrophe increases in the system, the average number of customers $E(N)$ in the system dwindles. At the same time as λ increases relatively $E(N)$ also increases. Fig.6 demonstrates as the arrival rate increases the mean number in the system also increases. This figure also reflects that as ν decreases $E(N)$ increases, as expected. Figures 7,8 and 9 give the details of the system performance, namely throughput, against the parameters, ν, λ and μ. Clearly it is evident that throughput U is the increasing function of λ and μ and it is the decreasing function of ν.

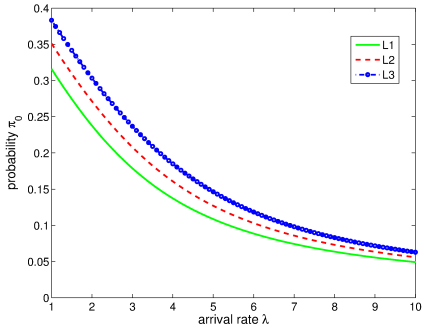

Figure 3: The effect of λ on π_0

L1: $\mu = 5; \nu = 1; q = 0.6; \eta = 0.7;$
L2: $\mu = 5; \nu = 1; q = 0.7; \eta = 0.8;$
L3: $\mu = 5; \nu = 1; q = 0.8; \eta = 0.9;$

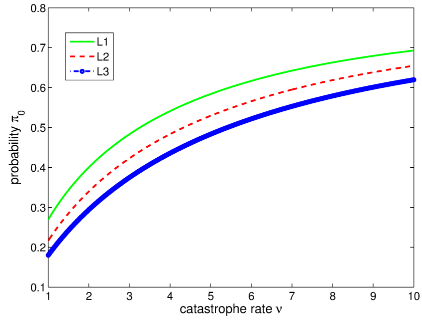

Figure 4: The effect of ν on π_0

L1:$\lambda = 4; \mu = 6; q = 0.3; \eta = 150;$
L2:$\lambda = 5; \mu = 6; q = 0.3; \eta = 175;$
L3:$\lambda = 6; \mu = 6; q = 0.3; \eta = 200;$

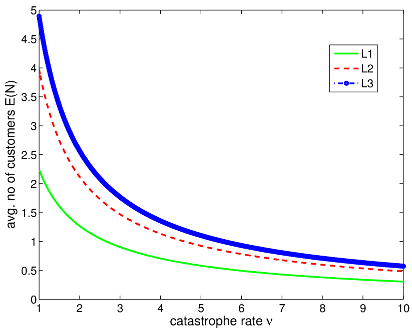

Figure 5: The Influence of ν on $E(N)$

L1:$\lambda = 4; \mu = 5; q = 0.5; \eta = 100;$
L2:$\lambda = 6; \mu = 5; q = 0.5; \eta = 150;$
L3:$\lambda = 7; \mu = 5; q = 0.5; \eta = 175;$

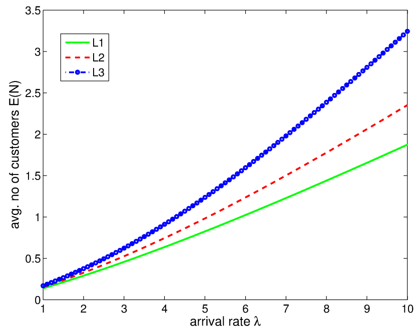

Figure 6: The Influence of λ on $E(N)$

L1: $\mu = 5; \nu = 4; q = 0.7; \eta = 150;$
L2:$\mu = 5; \nu = 3; q = 0.8; \eta = 175;$
L3:$\mu = 5; \nu = 2; q = 0.9; \eta = 200;$

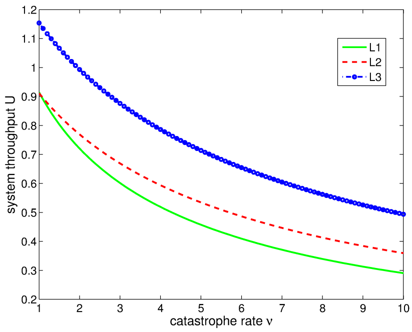

Figure 7: The Influence of ν on the system throughput U
L1:$\lambda = 3; \mu = 5; q = 0.3; \eta = 60;$
L2:$\lambda = 5; \mu = 4; q = 0.3; \eta = 100;$
L3:$\lambda = 6; \mu = 5; q = 0.3; \eta = 75;$

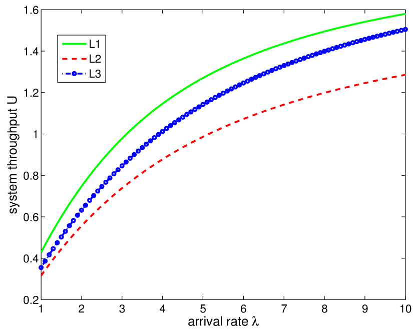

Figure 8: The Influence of λ on the system throughput U
L1: $\nu = 2; \mu = 5; q = 0.4; \eta = 90;$
L2: $\nu = 3; \mu = 6; q = 0.3; \eta = 75;$
L3: $\nu = 3; \mu = 7; q = 0.3; \eta = 100;$

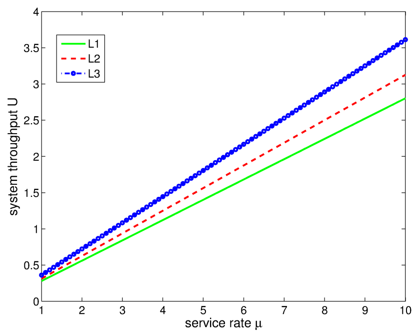

Figure 9: The Influence of μ on the system throughput U
L1:$\lambda = 5; \nu = 2; q = 0.5; \eta = 50;$
L2:$\lambda = 6; \nu = 3; q = 0.6; \eta = 75;$
L3:$\lambda = 6; \nu = 2; q = 0.6; \eta = 90;$

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