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ERROR ESTIMATION OF EMPTY URNS MODEL BY STEIN-CHEN METHOD FOR POISSON APPROXIMATION

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Abstract: In this paper, we give an error estimation of the number of empty urns with arbitrary probability by Poisson approximation, into which throwing balls independently, via Stein-Chen coupling method.

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1. Introduction

The classical combinatorial problems in mathematics are formulated and best understood by using urn models having certain number of urns, into which throwing balls, and interested in some parameter of the model, such as the

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total number of balls, or the fraction of urns satisfying some property. The urn models can be contributed to several fields of studies ([10], [11] and [12]), since many problems in the area of physical sciences, biological sciences, social sciences, computer science, and the others can be described in terms of distributing ball (objects) into specified urns (locations). In computer science, urn models are used for database performance evaluations and for modeling and analyzing algorithms. In medical science, urn model is applied to study cone ratios in human and macaque retinas. In economics, urn models are used to capture the mechanism of reinforcement learning. In communication theory, some transmission channels can be described in terms of contagion urn models. Among the most commonly encountered urn models in physics are the so called Maxwell-Boltzman, Bose-Eistein and Fermi-Dirac model.

One of the most well known in urn models is the occupancy urn models, where having a sequence of urns and trow balls at random into them; often the parameter under study is the number of urns satisfying some property. Many of the problems related to occupancy of a sequence of urn classified into one of the following three types :

- *Static models* : We throw a given number of balls into the sequence of urns, and look at the final configuration, characterized by the probability distribution of some random variable;
- *Waiting time models* : We throw the balls one by one, and wait for the first appearance of a specified configuration;
- *Dynamic models* : We throw the balls one by one, and consider the sequence of configuration.

General references on the urn models are Pólya (1930), Friedman (1949), classical Feller (1957), Johnson and Kotz (1977), Kolchin, Sevasťyanov and Chistyakov (1978).

The urn model which has been the basic of many studies appears when the urns are distinguishable and the balls are indistinguishable. Throughout the paper, we assume that m indistinguishable balls are thrown independently into n urns that have arbitrary probability and compute the probability of the urns that are empty after the balls have been thrown.

(We throw m balls into n urns)

$$\circledcirc \circledcirc \circledcirc \circledcirc \circledcirc \ldots \circledcirc \circledcirc \circledcirc \circledcirc\circledcirc \rightarrow m \text{ balls}$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$$\cup\cup\cup\cup\cup\cup \cdots\cdots \cup\cup\cup \rightarrow n \text{ urns}$$

Let W be the number of empty urns. We will investigate the bound of the probability approximation of W by Poisson distribution via Stein-Chen coupling method.

For each $i \in \{1, 2, 3, \ldots, n\}$, we define the indicator random variable X_i, as follows:

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th urn is empty,} \\ 0 & \text{otherwise.} \end{cases}$$

Let p_i be the probability of each ball falling into the i-th urn where $i \in \{1, 2, 3, \ldots, n\}$ such that $\sum\limits_{i=1}^{n} p_i = 1$. Therefore,

$$\mathrm{E}(X_i) = (1 - p_i)^m = {q_i}^m.$$

And set

$$W = \sum_{i=1}^{n} X_i.$$

Then W is the number of empty urns and we have

$$\lambda = \mathrm{E}(W) = \sum_{i=1}^{n} {q_i}^m.$$

Sevasťyanov ([11], p.323-327) showed that the distribution of W can be approximated by Poisson distribution with parameter λ. Here is their result.

Theorem 1.1. *Let W be the number of empty urns. If $m, n \to \infty$ and satisfying*

(i) $\lim\limits_{n,m\to\infty} \max\limits_{1\le i\le n} (1 - p_i)^m = 0,$

(ii) $\lim\limits_{n,m\to\infty} \sum\limits_{i=1}^{n} (1-p_i)^m = \lambda.$

Then we have

$$\lim_{n,m\to\infty} \mathrm{P}(W=r) = \frac{e^{-\lambda}\lambda^r}{r!}$$

where $r \in \mathbb{N}\cup\{0\}$ *that is the limiting distribution of* W *is Poisson distribution with parameter* λ.

In this paper, we give the error estimation on Poisson approximation of the number of empty urns with arbitrary probability by using Stein-Chen coupling method which is introduced in Section 2. The following are our main results.

Theorem 1.2. *Let* W *be the number of empty urns. Then*

$$|P(W\in A) - Poi_\lambda(A)| \leq (n-1)C_{\lambda,m,A}\left(\frac{p}{q}\right)^2$$

where $p = \max\limits_{1\leq i\leq n} p_i$ *and* $C_{\lambda,m,A} = \max\{\binom{m}{\frac{m-1}{2}}, \binom{m}{\frac{m}{2}}\}\min\{1,\lambda,\frac{\triangle(\lambda)}{M_A+1}\}$,

$$\triangle(\lambda) = \begin{cases} e^\lambda+\lambda-1 & \text{if } \lambda^{-1}(e^\lambda-1)\leq M_A, \\ 2(e^\lambda-1) & \text{if } \lambda^{-1}(e^\lambda-1) > M_A. \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w\subseteq A\} & \text{if } 0\in A, \\ \min\{w \mid w\in A\} & \text{if } 0\notin A. \end{cases}$$

when $C_w = \{0,1,2,\ldots,w\}$.

Corollary 1.3. *Let* W *be the number of empty urns. Then*

$$\sup_{A\subseteq\mathbb{N}} |P(W\in A) - Poi_\lambda(A)| \leq (n-1)(1-e^{-\lambda})\left(\frac{p}{q}\right)^2.$$

Corollary 1.4. *Let* W *be the number of empty urns. Suppose that* $\max\limits_{1\leq i\leq n} p_i = \dfrac{1}{n^\delta}$. *Then*

1. $|P(W\in A) - Poi_\lambda(A)| \leq \dfrac{C_{\lambda,m,A}}{n^{2\delta-1}-2}$ *when* $\delta\in(\frac{1}{2},1)$,
2. $|P(W\in A) - Poi_\lambda(A)| \leq \dfrac{C_{\lambda,m,A}}{n^{\delta}-1}$ *when* $\delta\geq 1$.

2. Poisson Approximation via Stein-Chen Method

The Stein's method is an interesting method in probability theory to obtain bounds on the distance between two probability distributions. The method was originally formulated for standard normal distribution of sums of dependent random variables by Charles Stein in 1972 [4]. Further more, his basic idea was applied for other studies.

In 1975, Louis Chen Hsiao Yun [9] modified Stein's method so as to obtain approximation results for the Poisson distribution, therefore the method is often referred to as Stein-Chen method. In 1992, The Barbour, Holst and Janson [2] were developed the Stein-Chen coupling method and give the fundamental result, as follow.

Theorem 2.1. *If* $W = \sum_{i=1}^{n} X_i$, $p_i = \mathrm{E}(X_i) = \mathrm{P}(X_i = 1)$, $\lambda = \mathrm{E}(W)$ *and for each* i, W_i *be the random variable on the same probability space as* W *such that the distribution* $\mathcal{L}(W)$ *equals to the conditional distribution* $\mathcal{L}(W - X_i \mid X_i = 1)$*. Then*

$$|\mathrm{P}(W \in A) - Poi_\lambda(A)| \leq \parallel g_{\lambda,A} \parallel \sum_{i=1}^{n} p_i \mathrm{E}(|W - W_i|) \tag{2.1}$$

where $\parallel g_{\lambda,A} \parallel := \sup_{w\in A}[g_{\lambda,A}(w+1) - g_{\lambda,A}(w)]$.

The conception in the Stein-Chen method for the Poisson distribution with parameter λ, is given by, for each $\lambda > 0$ and $A \subseteq \mathbb{N}$ there exists a function $g_{\lambda,A}$ on $\mathbb{N} \cup \{0\}$ satisfying Stein's equation

$$I_A(j) - Poi_\lambda(A) = \lambda g_{\lambda,A}(j+1) - j g_{\lambda,A}(j), \quad j \geq 0 \tag{2.2}$$

where indicator function $I_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$.

By substituting j and λ in (2.2) by any integer-valued random variable $W = \sum_{i=1}^{n} X_i$ and $\lambda = \mathrm{E}(W)$, we have

$$\mathrm{P}(W \in A) - Poi_\lambda(A) = \mathrm{E}(\lambda g_{\lambda,A}(W+1)) - \mathrm{E}(W g_{\lambda,A}(W)) \tag{2.3}$$

The well known solution $g_{\lambda,A}$ of (2.2) is of the form

$$g_{\lambda,A}(w) = \begin{cases} (w-1)!\lambda^{-w}e^{\lambda}[\mathcal{P}_\lambda(I_{A\cap C_{w-1}}) - \mathcal{P}_\lambda(I_A)\mathcal{P}_\lambda(I_{C_{w-1}})] & ; w \geq 1, \\ 0 & ; w = 0 \end{cases}$$

where

$$\mathcal{P}_\lambda(I_A) = e^{-\lambda}\sum_{l=0}^{\infty} I_A(l)\frac{\lambda^l}{l!}$$

and

$$C_{w-1} = \{0, 1, 2, \ldots, w-1\}.$$

Several authors determined a bound of $\| g_{\lambda,A} \|$. For $A \subseteq \mathbb{N} \cup \{0\}$, in 1975, Chen [9] showed that

$$\| g_{\lambda,A} \| \leq \min\{1, \lambda^{-1}\}$$

and, in 1994, Janson [14] showed that

$$\| g_{\lambda,A} \| \leq \lambda^{-1}(1-e^{-\lambda}) \leq \min\{1, \lambda^{-1}\}. \tag{2.4}$$

In case of non-uniform bound, in 2003, Neammanee [7] showed that

$$\| g_{\lambda,A} \| \leq \min\left\{\frac{1}{w_0}, \lambda^{-1}\right\}$$

and, in 2005, Teerapabolarn and Neammanee [8] gave bound of $\| g_{\lambda,A} \|$ where $A = \{0, 1, \ldots, w_0\}$ in the terms of

$$\| g_{\lambda,A} \| \leq \lambda^{-1}(1-e^{-\lambda})\min\left\{1, \frac{e^{\lambda}}{w_0+1}\right\}.$$

In general case for each subset A of $\{0, 1, \ldots, n\}$, in 2006, Santiwipanont and Teerapabolarn [15] gave a bound in the form of

$$\| g_{\lambda,A} \| \leq \lambda^{-1}\min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\} \tag{2.5}$$

where

$$\triangle(\lambda) = \begin{cases} e^{\lambda}+\lambda-1 & \text{if } \lambda^{-1}(e^{\lambda}-1) \leq M_A, \\ 2(e^{\lambda}-1) & \text{if } \lambda^{-1}(e^{\lambda}-1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A. \end{cases}$$

The another part in apply Theorem 2.1 is to construct distribution of W_i equals to the conditional distribution of $W - X_i \mid X_i = 1$ and make $\mathrm{E}|W - W_i|$ small. For the case of $X_1, \ldots, X_n$ are independent, we let $W_i = W - X_i$. Then $\mathrm{E}|W - W_i| = p_i$ and from (2.1), we have $|P(W \in A) - Poi_\lambda(A)| \leq \| g_{\lambda,A} \| \sum_{i=1}^{n} p_i^2$.

In next section, we will use Theorem 2.1 to prove our main results by constructing the random variable W_i.

3. Proof of Main Results

Proof of Theorem 1.2. For each $j \in \{1, 2, 3, \ldots, n\}$ such that $j \neq i$, we define X_{ij} in the following way. Take those balls which have landed in the i-th urn, remove the i-th urn and throw them independently into other urns, the indicator random variable X_{ij} is defined as follow,

$$X_{ij} = \begin{cases} 1 & \text{if the } j\text{-th urn is empty after removing} \\ & \text{the } i\text{-th urn which is empty,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $W_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} X_{ij}$ be the number of empty urns after we removing the i-th urn which is empty.

Suppose that $\{j_s | s = 1, 2, \ldots, r\}$ be the set of r empty urns, $r < n$, so for each $r \in \{0, 1, 2, \ldots, n-1\}$, we get

$$\begin{aligned} \mathrm{P}(W_i = r) = \left(1 - \frac{\sum_{s=1}^{r} p_{j_s}}{1 - p_i}\right)^m &= \left(\frac{1 - p_i - \sum_{s=1}^{r} p_{j_s}}{1 - p_i}\right)^m \\ &= \frac{\left(1 - \left(p_i + \sum_{s=1}^{r} p_{j_s}\right)\right)^m}{(1 - p_i)^m}, \end{aligned}$$

and

$$\begin{aligned} P(W - X_i = r \mid X_i = 1) &= \frac{P(W - X_i = r, X_i = 1)}{P(X_i = 1)} \\ &= \frac{P(W = r + 1, X_i = 1)}{P(X_i = 1)} \\ &= \frac{\left(1 - \left(p_i + \sum_{s=1}^{r} p_{j_s}\right)\right)^m}{(1 - p_i)^m}. \end{aligned}$$

Then the distribution of W_i equals to the conditional distribution $(W - X_i \mid X_i = 1)$.

We observe that in case $X_i = 1$, so we have the i-th urn is empty. Thus the number of urns that are empty after removing the i-th urn, equals to the number of the empty urns minus 1, that is

$$W_i = W - 1. \tag{3.1}$$

For each $j \in \{1, 2, 3, \ldots, n\}$ such that $j \neq i$, we define the indicator random variable Y_{ij}, as follow,

$$Y_{ij} = \begin{cases} 1 & \text{if the } j\text{-th urn is not empty after we throw the balls again,} \\ & \text{in which these balls exactly used to land the } i\text{-th urn before,} \\ 0 & \text{otherwise.} \end{cases}$$

In case $X_i = 0$, the number of the urns that are empty after removing the i-th urn and we throw them again as defined, equals to the number of the empty urns minus the sum of number of the j-th urn, where $j \neq i$, is empty in the first-throw and they are not empty after we throw them again, that is

$$W_i = W - \sum_{\substack{j=1 \\ j \neq i}}^{n} X_j Y_{ij}. \tag{3.2}$$

We know that

$$E|W - W_i| = E(W - W_i)^+ + E(W - W_i)^-.$$

where $(W-W_i)^+ = \max\{W-W_i, 0\}$ and $(W-W_i)^- = -\min\{W-W_i, 0\}$.

Form (3.1) and (3.2),

- case $X_i = 1$ we have $(W-W_i)^+ = 1$ and $(W-W_i)^- = 0$,
- case $X_i = 0$ we have $(W-W_i)^+ = \sum_{\substack{j=1 \\ j\neq i}}^{n} X_j Y_{ij}$ and $(W-W_i)^- = 0$.

Therefore,

$$(W-W_i)^+ \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} X_j Y_{ij} \quad \text{and} \quad (W-W_i)^- = 0.$$

By the fact that

$$\begin{aligned}
\mathrm{E}(W-W_i)^+ &\leq \mathrm{E}\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} X_j Y_{ij}\right) \\
&= \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{E}\left(X_j Y_{ij}\right) \\
&= \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{P}(X_j = 1, Y_{ij} = 1) \\
&= \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{P}(X_j = 1)\mathrm{P}(Y_{ij} = 1) \\
&= \sum_{\substack{j=1 \\ j\neq i}}^{n} (1-p_j)^m \left(1 - \left(1 - \frac{p_i p_j}{q_i}\right)^{b_i}\right) \\
&\leq \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(1 - \left(\frac{q_i - p_i p_j}{q_i}\right)^{b_i}\right) \\
&\leq \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(1 - \left(\frac{q_i - p_i p_j}{q_i}\right)^{m}\right)
\end{aligned}$$

where b_i is the number of balls contained in the i-th urn.

Suppose that $p = \max_{1 \le i \le n} p_i$.

From the fact that $p_j \le 1 - p_i = q_i$ because $p_i + p_j + \sum_{\substack{k=1 \\ k \ne i,j}}^{n} p_k = 1$, we have

$$\begin{aligned}
\mathrm{E}(|W - W_i|) &\le \sum_{\substack{j=1 \\ j \ne i}}^{n} \left(1 - \left(\frac{q_i - p_i p_j}{q_i}\right)^m\right) \\
&= (n-1)\left(1 - \sum_{k=0}^{m} \binom{m}{k} (-1)^k q_i^{-k} (p_i p_j)^k\right) \\
&= (n-1)\left(\sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} q_i^{-k} (p_i p_j)^k\right) \\
&\le (n-1)\alpha_m \sum_{k=1}^{m} \left(\frac{p_i p_j}{q_i}\right)^k \\
&= \alpha_m (n-1) \frac{\frac{p_i p_j}{q_i} - \left(\frac{p_i p_j}{q_i}\right)^{m+1}}{1 - \frac{p_i p_j}{q_i}} \\
&\le \alpha_m (n-1) \frac{\frac{p_i p_j}{q_i}}{1 - p_i \frac{p_j}{q_i}} \\
&\le \alpha_m (n-1) \frac{\frac{p_i p_j}{q_i}}{1 - p_i} \\
&= \alpha_m (n-1) \frac{p_i p_j}{q_i^2} \\
&\le \alpha_m (n-1) \left(\frac{p}{q}\right)^2 \qquad (3.3)
\end{aligned}$$

where $\alpha_m = \max\{\binom{m}{\frac{m-1}{2}}, \binom{m}{\frac{m}{2}}\}$ and $q = 1 - p$.

Hence, by (2.1), (2.5) and (3.3), we have

$$|P(W \in A) - Poi_\lambda(A)| \le (n-1) C_{\lambda,m,A} \left(\frac{p}{q}\right)^2$$

where $\lambda = \sum_{i=1}^{n} {q_i}^m$ and $C_{\lambda,m,A} = \max\{\binom{m}{\frac{m-1}{2}}, \binom{m}{\frac{m}{2}}\} \min\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\}$. □

Proof of Corollary 1.4. From equation (3.3), let $\max_{1\le i\le n} p_i = \frac{1}{n^\delta}$.

Case 1: Suppose that $\delta \in \left(\frac{1}{2}, 1\right)$, we have

$$\begin{aligned}
\mathrm{E}(|W - W_i|) &\le \alpha_m (n-1)\left(\frac{1}{n^\delta - 1}\right)^2 \\
&\le \alpha_m \frac{n}{n^{2\delta} - 2n^\delta + 1} \\
&\le \alpha_m \frac{1}{n^{2\delta-1} - 2n^{\delta-1}} \\
&= \alpha_m \frac{1}{n^{2\delta-1} - 2\left(\frac{1}{n^{1-\delta}}\right)} \\
&\le \alpha_m \frac{1}{n^{2\delta-1} - 2}.
\end{aligned} \tag{3.4}$$

Hence, by (2.1), (2.5) and (3.4), we have

$$|P(W \in A) - Poi_\lambda(A)| \le \frac{C_{\lambda,m,A}}{n^{2\delta-1} - 2}.$$

Case 2: Suppose that $\delta \ge 1$, we have

$$\begin{aligned}
\mathrm{E}(|W - W_i|) &\le \alpha_m (n-1)\left(\frac{1}{n^\delta - 1}\right)^2 \\
&\le \alpha_m \frac{n-1}{(n-1)(n^\delta - 1)} \\
&= \alpha_m \frac{1}{n^\delta - 1}.
\end{aligned} \tag{3.5}$$

Hence, by (2.1), (2.5) and (3.5), we have

$$|P(W \in A) - Poi_\lambda(A)| \le \frac{C_{\lambda,m,A}}{n^\delta - 1}.$$

□

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