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AN ELEGANT SOLUTION FOR DRAWING A FIXED POINT

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Abstract: This paper presents a newly developed theorem for the construction of fixed points under congruent and similar transformations, which differs from existing theorems based on Euclidean geometry. It also explains the random-dot pattern that facilitated the invention of this new theorem.

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Key Words: fixed point, construction, congruent transformation, similar transformation, random-dot pattern

1. Elegant Solution

It is probably a common aspiration among mathematicians to find at least one theorem during their lifetime. The theorem presented here is for the construction of fixed points, and the only theorem I can be proud of among those I have invented. I like this theorem, which I personally think is elegant. Suppose that two squares are randomly placed as shown in Figure 1. What is the simplest method to find a fixed point to match the one with the other?

The process of congruent transformation is divided into rotational translation (Figure 2(1)) and parallel translation (Figure 2(2)) or also symmetry

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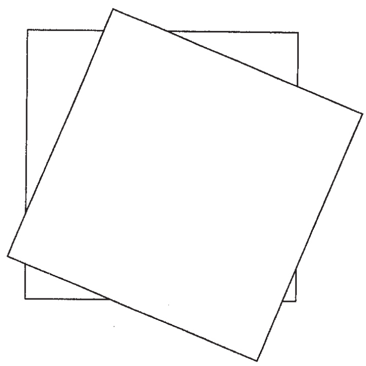


Figure 1

translation although we are not aware of such a division in everyday life. These translations may be made in any order.

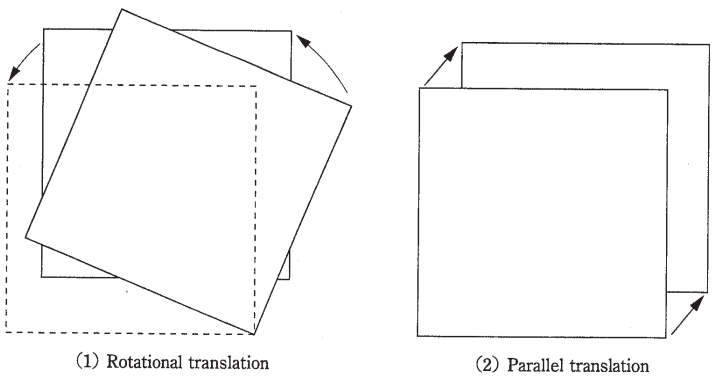


Figure 2: Transformation of figure

The combination of parallel and rotational translations can be replaced by one rotational translation, the center of which is called a fixed point in congruent transformation. A general method to find a fixed point is based on Euclidean geometry, as shown in Figure 3. In order to match Square ABCD and Square A'B'C'D', Side AD must be put on Side A'D'. This means that Vertex A and Vertex D moved to Vertex A' and Vertex D' respectively. Point O where the perpendicular bisector of Segment AA' intersects with that of Segment DD' is the center of rotation, or the fixed point, as indicated by the fact that Triangle

AOD and Triangle A'OD' are congruent.

Is there a more efficient and elegant method to construct a fixed point? In Figure 4, I take two squares, and draw two straight lines on the square above. If the square above is held by a compass at the point at which the two lines intersect and rotated, the one square completely matches the other. When I tried this experimental movement for the first time, I suspected that the two squares were matched by pure chance. However, repeated attempts from different positions on the paper convinced me of the fact that a fixed point could be constructed in that way. By just drawing two straight lines, a fixed point can be constructed from their intersection (Nishiyama, February 1982, [2]).

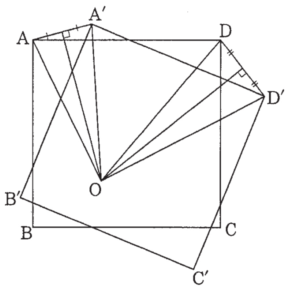


Figure 3: Construction by perpendicular bisectors

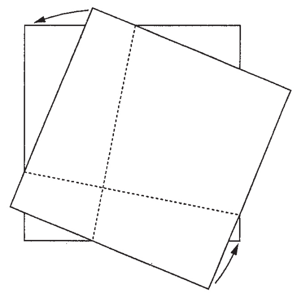


Figure 4: Elegant solution

While the existing construction method shown in Figure 3 requires a compass and a ruler to find a fixed point, one feature of my method shown in Figure 4 is that it does not need a compass. The necessary condition of this new way to construct a fixed point is that opposite sides of a quadrilateral are parallel. My theorem therefore applies not only to squares but also to rectangles and parallelograms. If you do not have square paper at hand, you can conduct the experiment with rectangular writing paper or copy paper.

2. Outline of Proof

Let me mathematically prove that the intersection of the two auxiliary lines drawn on Figure 4 is a fixed point. For the sake of the proof, the two square papers are marked as shown in Figure 5. Points A, B, C and D correspond to each vertex of the square below, and Points A', B', C' and D' to that of the square above. Points P, Q, R and S are the intersections of the sides of the two squares, and Point O is the intersection of Line PR and Line QS.

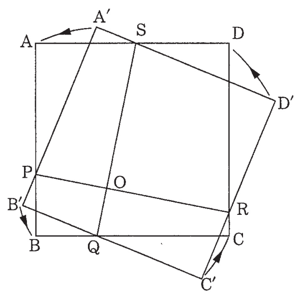


Figure 5

Since it is difficult to give a direct and immediate proof that Point O is a fixed point, I will provide a step-by-step explanation here. As the first step, let me prove "In order to match two lines, a center of rotation must be on the bisector of the intersecting angle." Figure 6 shows Lines l_1 and l_2 intersecting at Point P. Point Q is an arbitrary point on the bisector of the intersecting angle, and Points R and S are foots of perpendicular lines dropped from Point Q to Lines l_1 and l_2. Right-angle Triangles QPR and QPS are congruent since Line QP is common and Angle QPR is equal to Angle QPS. The length of Line QR is therefore equal to that of Line QS. This means that Lines l_1 and l_2 are matched with the rotation centering around Point Q.

The next step is "to determine the conditions required for matching parallel lines." As shown in Figure 7, two parallel Lines l_1 and l_2 with a certain interval intersect with two parallel Lines l_3 and l_4 with the same interval. The intersections are marked as Points E, F, G and H. In order to match Line l_1 with Line l_3, the center of rotation must be on the bisector of an intersecting angle, or Angle E. Similarly, with respect to Lines l_2 and l_4, the center of rotation must be on the bisector of Angle G. Parallelogram EFGH formed by four Lines l_1, l_2, l_3 and l_4 is a lozenge with four sides of the same length, and the bisector of intersecting Angle E exactly corresponds to that of intersecting Angle G. Point Q, an arbitrary point on Line EG, is thus the center of rotation to bring Line l_1 on Line l_3 and Line l_2 on Line l_4 at the same time.

With the above steps in mind, let us look at Figure 5. Sides AB, DC, A'B' and D'C' in Figure 5 can be regarded as Lines l_1, l_2, l_3 and l_4 in the above steps respectively. When Sides AB, DC, A'B' and D'C' are extended, they form a lozenge, and Line PR, a diagonal line of the lozenge, bisects both Angle A'PB

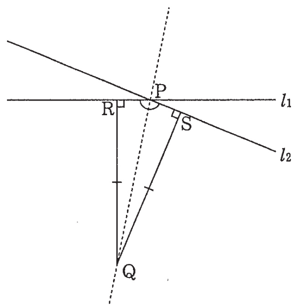


Figure 6: Matching lines

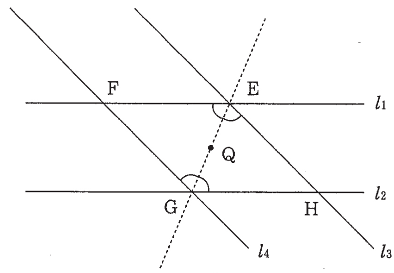


Figure 7: Matching two parallel lines

and Angle DRC'. Any center of rotation on Line PR can therefore match Side A'B' with Side AB and Side D'C' with Side DC. Similarly, any center of rotation on Line QS can match Side B'C' with Side BC and Side A'D' with Side AD. In order to match Square A'B'C'D' with Square ABCD, these two rotational translations must occur simultaneously. Therefore, the Intersection O of Line PR and Line QS is the only center of rotation to match the two squares.

3. Random-Dot Pattern

I would like to describe how the simple method to construct a fixed point of congruent squares shown in Figure 4 was invented. As for the existence of fixed points, the fixed point theorem by E. J. Brouwer is widely known. It says that an arbitrary continuous mapping f of space X into itself has at least one fixed point. Fixed points here are defined as point $x \in X$ that satisfies $f(x) = x$. This theorem demonstrates the existence of fixed points, but it is another issue to construct them.

In 1980, I encountered an interesting article by Jearl Walker in Scientific American (Walker, 1980, [6]).@The article dealt with various experiments using randomly plotted points, and greatly contributed to the invention of my theorem. Being prompted by the article, I made a random-dot pattern (Nishiyama, August 1982, [3]).

The pattern had 2000 points randomly plotted within a 20cm-side square. Computer generated random numbers were used for the (x, y) coordinates of each point, and points were then drawn by a plotter based on the coordinates. Figure 8 shows a random dot pattern plotted in that way.

Further work is required to make this pattern interesting. I printed it on an OHP film, a transparent plastic paper. The film was placed over the original pattern to complete preparation. When I slightly rotated the film above, a concentric circle surfaced from random dots (Figure 9). This illusion would probably surprise and impress anyone who saw it. You would see only one concentric circle, and no other such circle would surface by any chance. When I counter-rotated the film around the center of the concentric circle, the two patterns, the one on the film and the other on the original paper, coincided exactly. The center of the concentric circle is, in other words, a central axis of the rotation or a fixed point.

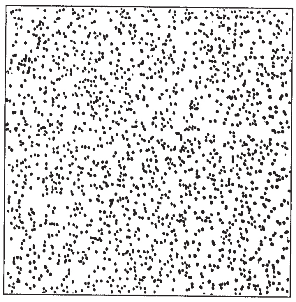


Figure 8: Random-dot pattern

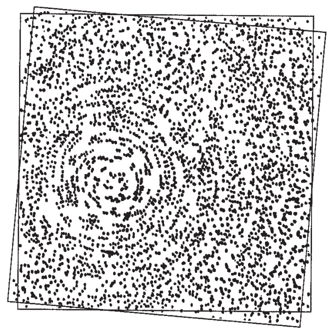


Figure 9: Emerging fixed point

I was fascinated with this enigmatic circle created by random dots, and looked at it all day long. Then, I noticed that the center of the concentric circle corresponded to the intersection of two straight lines connecting the intersection of each side of the two squares. The random assignment of dots is necessary to prevent two or more circles from surfacing, or to make a fixed point visible. In order to explain the reason, let me use a regular-dot pattern with regular placement of dots (Figure 10). The number of dots plotted is 2000 as in the case of a random-dot pattern.

A regular-dot pattern is printed on paper and OHP film, and the two materials are overlaid with the film slightly rotated. A regular-dot pattern produces multiple concentric circles (Figure 11(1)). Wider rotation angle of the film brings smaller and more concentric circles (Figure 11(2)). Among many centers of these concentric circles, only one center shows a fixed point, and the others are dummies. Regularity in dot placement causes this phenomenon. In order to have only one concentric circle, which demonstrates a fixed point, dots must be plotted randomly.

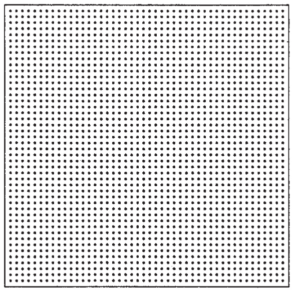


Figure 10: Regular-dot pattern

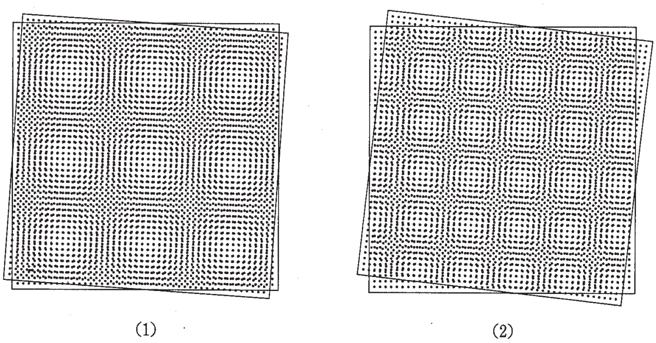


Figure 11: Emerging multiple concentric circles

4. Fixed Points of Circles and Triangles

Since four sides of a square have the same length, there are four types of superimposition alignments of two squares, one of which is shown in Figure 5. Each alignment has a different fixed point. Figures 12(1)-(4) show these four alignments and fixed points. Sides are extended in Figures 12(2) and 12(3) to construct intersections of sides. This expands Nishiyama's theorem so that it can be applied to when corresponding sides of two squares do not cross each other.

In this way, it becomes clear that squares have four fixed points, which are here identified as P_1, P_2, P_3 and P_4 for convenience. What is the positional relationship of them? Figure 13 demonstrates that four fixed points can be put on the same straight line. Although such a relationship seems to be mysterious, circumscribing circles in Figure 13 helps mathematical understanding of it. Rotational translation of squares corresponds to rotational translation of their circumscribing circles.

Squares have four fixed points, and regular octagons have eight fixed points. Two regular n-polygons, each of which has n-sides with the same length, can be placed in n-ways, and the number of fixed points is thus n. The fact that these fixed points form a straight line can be clearly understood by drawing circles to circumscribe regular n-polygons, as in the case of squares. A regular n-polygon with infinite n-sides is namely a circle. Two circles have an infinite number of fixed points, which align on a straight line. In other words, fixed points of congruent circles are on the intersecting line of the two circles, and any point on the line can be a fixed point (Figure 14). Please examine this by yourself (Nishiyama, 1986, [4]).

A fixed point of congruent triangles can be constructed as follows. I would like to construct Point P, a fixed point of Triangle ABC and Triangle A'B'C' (Figure 15(1)). In order to do this, the two triangles are inverted to form Parallelograms ABCD and A'B'C'D', and then a fixed point of them is constructed (Figure 15(2)). A fixed point for parallelograms can be constructed in the same way as that for congruent squares, because a parallelogram also satisfies the aforementioned necessary condition: parallel opposite sides. Point P, a fixed point of parallelograms constructed in this way, is also a fixed point of triangles. In the case of triangles, however, my construction method is more complex and thus less desirable than the method using perpendicular bisectors that is shown in Figure 3.

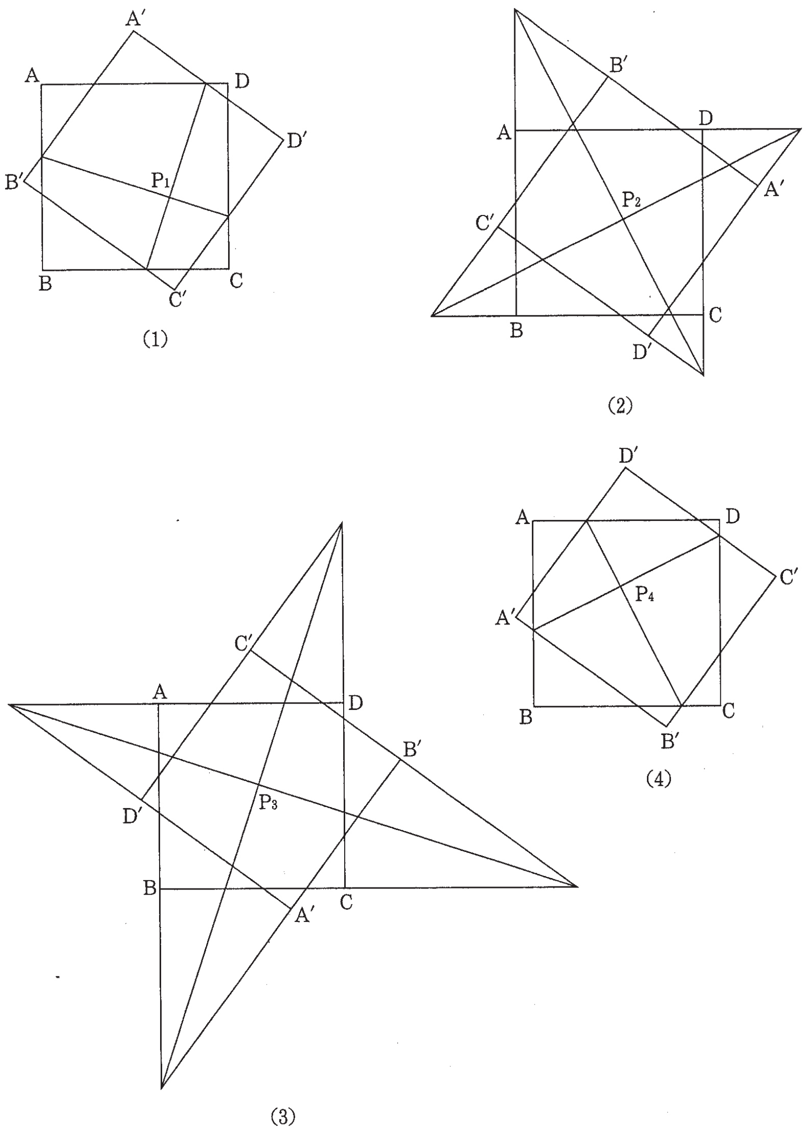


Figure 12: Four fixed points

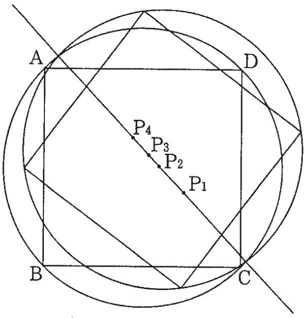


Figure 13: Four fixed points on an intersecting line of two circumscribing circles

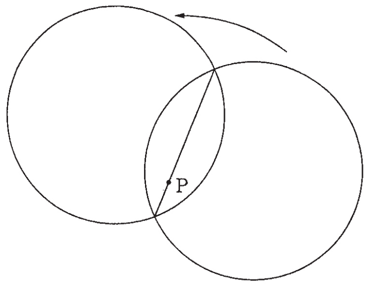


Figure 14: Matcing circles

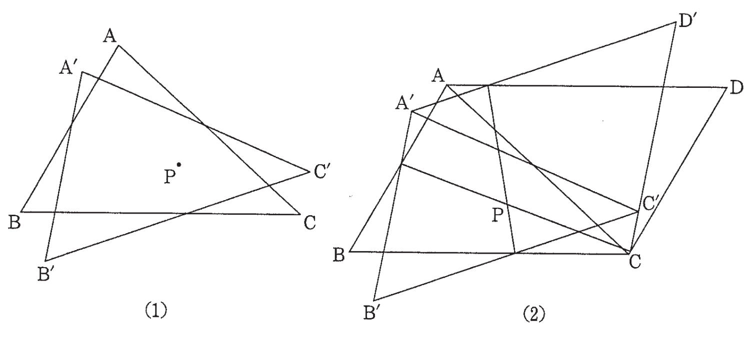


Figure 15: Fixed point of triangles

5. Similar Transformation

Literature by H. S. M. Coxeter provides detailed explanation on similar transformation (Coxeter, 1965, [1]). Coxter takes up a photo enlarger and a pantograph as examples of enlarging transformation and maps with different scales as examples of spiral transformation. Figure 16 illustrates maps piled up in the order of scales. A map with a 1/20,000 scale is placed on a larger map with a 1/10,000 scale. A 1/40,000 scale map is subsequently laid over the 1/20,000 scale map in the same position as the previous two maps. The limit of such a process that is repeated infinitely, according to Coxeter, becomes a fixed point. This is a special case among those in which Brouwer's fixed point theorem, "an arbitrary continuous mapping has at least one fixed point," applies.

Knowing the existence of a fixed point does not necessarily means knowing a method to construct it. As demonstrated by congruent transformation, Nishiyama's theorem is extended to construct a fixed point for similar transformation. Figure 17 shows Rectangle ABCD and Rectangle A'B'C'D' above, a rectangle scaled-down from Rectangle ABCD. Points P, Q, R and S are the intersections of sides of Rectangle ABCD and corresponding sides of Rectangle A'B'C'D'. When sides of the two rectangles do not intersect, such sides are extended to make an intersection. Point O, the intersection of two diagonal lines PR and QS is a fixed point.

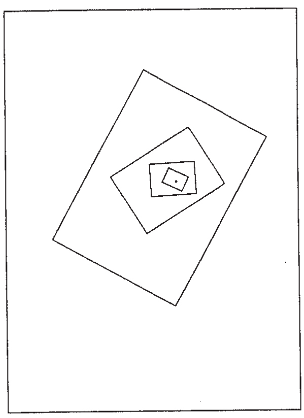


Figure 16: Fixed point of similar transformation

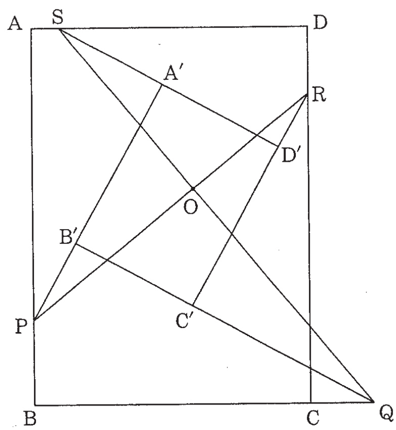


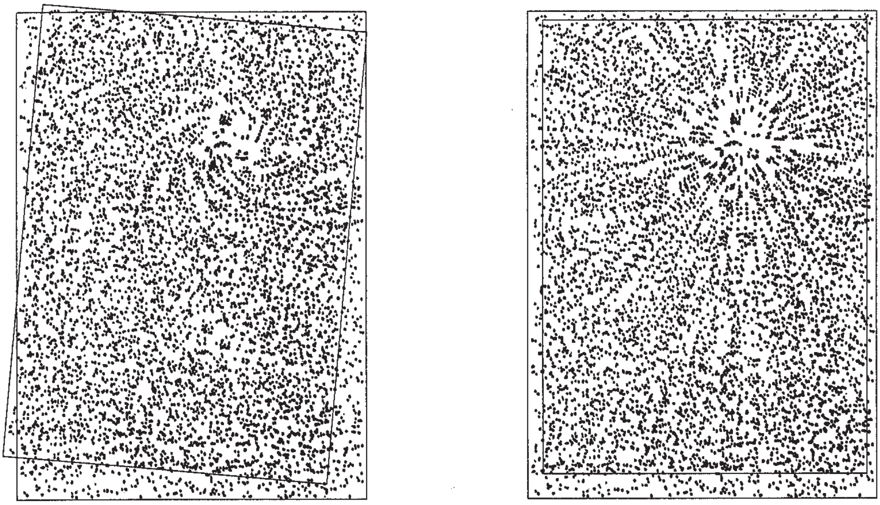
Figure 17: Construction of a fixed point

Norio Morihara provides the following proof that Intersection O is a fixed point (Okabe, 1989, [5]). A horizontal line on Rectangle ABCD below is moved from BC to AD. A corresponding horizontal line on Rectangle A'B'C'D' above is moved from B'C' to A'D'. Line QS can be regarded as the locus of intersections between the horizontal line on Rectangle ABCD and the corresponding horizontal line on Rectangle A'B'C'D' moving at the same time. Similarly, Line PR can be regarded as the locus of intersections between a vertical line on Rectangle ABCD moving from AB to DC and a corresponding vertical line on Rectangle A'B'C'D' moving from A'B' to D'C'. A fixed point must meet conditions both vertically and horizontally. Therefore, a fixed point is Point O, an intersection of Line PR and Line QS.

This mathematical demonstration by Norio Morihara can be also applied to a fixed point of congruent transformation explained in Section 2. above. A fixed point of similar transformation is also confirmed by a random-dot pattern. When a random-dot pattern is microcopied on an OHP film, the reduction ratio should be up to 90%. A reduction ratio of 50% would make the confirmation of a fixed point difficult. As you can see in Figure 18(1), only one fixed point surfaces, but a pattern surrounding it is different from that in congruent transformation. When the film above is rotated slightly, a spiral, not a concentric circle, appears. Right-handed rotation creates a dextrorsal spiral, and left-handed one brings a sinistrorsal spiral. When the film is moved in parallel, a radiological pattern is created around a fixed point as shown in Figure 18(2).

6. Computer Program to Produce a Random-Dot Pattern

Since many people now have access to a computer, let me introduce a computer program using Visual BASIC to produce a random-dot pattern. With such a program, a random-dot pattern is easily produced. A built-in RND function is used to generate random numbers. The RND function automatically calculates uniform random numbers in $(0, 1)$ intervals. With a size of a square fixed, multiplication by random numbers decides (x, y) coordinates. I here assume that each side of a square is 8000, as indicated by $wx = 8000$ and $wy = 8000$. Given larger dots make for easier plotting, "+" is formed by Line command. The number of dots is 2000. Only 14 lines constitute the command.



(1) Spiral pattern created (2) Radiological pattern created

Figure 18: Random-dot pattern for similar transformation

```
Private Sub Command1 _ Click()
wx = 8000
wy = 8000
sx = 100
sy = 100
Line (sx, sy)-(wx + sx, wy + sy), , B
For i = 1 To 2000
X = Rnd * wx + sx
Y = Rnd * wy + sy
d = 30
Line (X - d, Y)-(X + d, Y)
Line (X, Y - d)-(X, Y + d)
Next i
End Sub
```

Table 1. Visual BASIC program

A random-dot pattern appears on a screen when you run the program. The screen is then hardcopied. If the copy is not a square, you should adjust wx or wy. The pattern created is then printed on OHP film. A copy machine can be also used as printing equipment. Please try this process to make a set of materials for a random-dot pattern by yourself.

I would like to introduce another mysterious phenomena caused by a random-dot pattern, which is used to demonstrate a fixed point. As shown in Figure 19, two random-dot patterns are overlaid, and the OHP film above is rotated slightly to create a concentric circle in the center. When the film is moved horizontally, the concentric circle makes a vertical movement. When the film is moved vertically, it makes a horizontal movement. The direction of the film's movements and that of the concentric circle's movements, or the fixed point's movements, have a near-orthogonal relationship. Although an angle formed by these two movements is not an exact right angle, it infinitely approaches 90 degrees as a rotational angle goes toward zero. This relationship is similar to a precession, which can be seen in an orthogonal movement of spinning top's axis when you touch axis of the top with your finger.

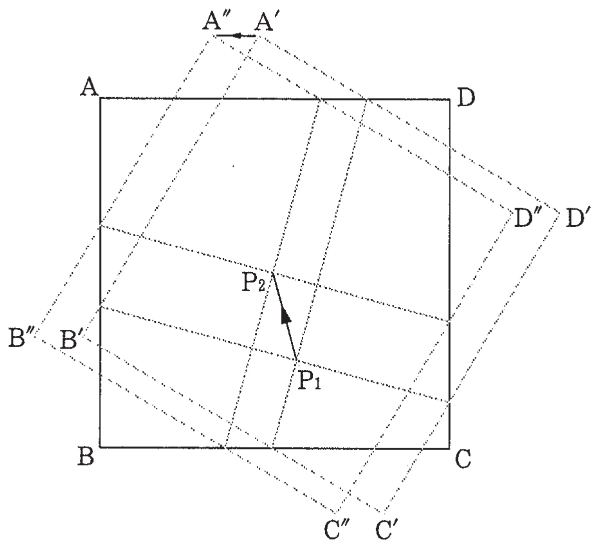


Figure 19: Movements of film and concentric circle

The moving velocity of a concentric circle presents another interesting phenomenon. Only a slight movement of a film brings about rapid movement of the concentric circle. The velocity ratio between the film and circle is as much as several dozen times. The reason for such dynamic movement can be clarified by analyzing Figure 5 for fixed point construction. I encourage you to observe for yourself.

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