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ON REGULAR DUO po-Γ-SEMIGROUPS

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Abstract: In this paper, we give some characterizations of regular duo po-Γ-semigroups. We show that po-Γ-semigroup M is regular duo if and only if M is the semilattice of B-simple ordered sub-Γ-semigroups of M which are analogous to the characterizations of bi-ideals of poe-semigroups considered by Zhu, see [1].

AMS Subject Classification: po-Γ-semigroup, regular duo, bi-ideal, B-simple, semilattice congruence
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1. Introduction and Preliminaries

M.K. Sen (see [2]) introduced Γ-semigroups in 1981. M.K. Sen and N.K. Saha (see [3]) have Γ-semigroups different from the first definition of Γ-semigroups in the sense of Sen (1981). From Sen (see [2]) we recall the following definition of Γ-semigroup.

Let M and Γ be any two nonempty sets. M is called a Γ-semigroup if:

(1) $M\Gamma M \subseteq M$, $\Gamma M\Gamma \subseteq \Gamma$.

(2) $(axb)yc = a(xby)c = ax(byc)$, for all $a, b, c \in M$ and $x, y \in \Gamma$.

In 1996, Y.I. Kwon and S.K. Lee introduced po-Γ-semigroup (partially ordered Γ-semigroups).

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A *po*-Γ-semigroup is an ordered set M at the same time a Γ-semigroup such that:

$$a \leq b \Longrightarrow a\gamma x \leq b\gamma x \text{ and } x\mu a \leq x\mu b,$$

$\forall a, b, x \in M$ and $\forall \gamma, \mu \in \Gamma$. Let K be a nonempty subset of M. K is called a sub-Γ- semigroup of M if $a\gamma b \in K$ for all $a, b \in K, \gamma \in \Gamma$.

Notation. For subsets A, B of M, let

$$A\Gamma B := \{a\gamma b | a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

For $H \subseteq M$, we denote $(H] := \{t \in M | t \leq h$ for some $h \in H\}$. For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$ $(a \in M)$.

Definition 1.1. A *po*-Γ-semigroup M is called regular if for every $a \in M$ there exists $x \in M$ such that $a \leq a\gamma x\mu a$ for some $\gamma, \mu \in \Gamma$.

Equivalent Definition. $a \in (a\Gamma M\Gamma a]$ for every $a \in M$.

Definition 1.2. Let M be a *po*-Γ-semigroup and $\emptyset \neq A \subseteq M$. A is called a left (resp. right)ideal of M if:

(1) $M\Gamma A \subseteq A$ (resp. $A\Gamma M \subseteq A$).

(2) $a \in A, M \ni b \leq a \Rightarrow b \in A$.

A subset A of M is called an ideal of M if it is both a right and left ideal of M. A sub-Γ-semigroup T of a *po*-Γ-semigroup M is called left (resp. right) simple if it has no proper left (resp. right) ideal.

Definition 1.3. Let M be a *po*-Γ-semigroup and $\emptyset \neq B \subseteq M$. B is called a bi-ideal of M if:

(1) $B\Gamma M\Gamma B \subseteq B$.

(2) $a \in B, M \ni b \leq a \Rightarrow b \in M$.

Let M be a *po*-Γ-semigroup, we denote by $L(a), R(a)$ and $B(a)$ the left ideal, right ideal and bi-ideal of M generated by $a \in M$ respectively. One can easily prove that $L(a) = (a \bigcup M\Gamma a], R(a) = (a \bigcup a\Gamma M]$ and $B(a) = (a \bigcup a\Gamma a \bigcup a\Gamma M\Gamma a]$. If M is regular, then it is clear that $L(a) = (M\Gamma a], R(a) = (a\Gamma M], B(a) = (a\Gamma M\Gamma a]$. M is called left (resp. right) duo if the left (resp. right) ideals of M are also right (resp. left) ideals of M. M is called duo if M is left duo and right duo. A *po*-Γ-semigroup M is called B-simple if it has no proper bi-ideals.

Definition 1.4. A sub-Γ-semigroup F of a *po*-Γ-semigroup M is called a filter of M if

(1) $a, b \in M, \gamma \in \Gamma, a\gamma b \in F \Rightarrow a \in F, b \in F$.

(2) $a \in F, M \ni c \geq a \Rightarrow c \in F$.

Definition 1.5. Let M be a po-Γ-semigroup and $\emptyset \neq T \subseteq M$. T is called semiprime if

$$A \subseteq M, A\Gamma A \subseteq T \Rightarrow A \subseteq T.$$

Equivalent Definition. $a \in M, a\gamma a \in T(\gamma \in \Gamma) \Rightarrow a \in T$.

Definition 1.6. Let M be a po-Γ-semigroup and $\emptyset \neq T \subseteq M$. T is called prime if

$$A, B \subseteq M, A\Gamma B \subseteq T \Rightarrow A \subseteq T \text{ or } B \subseteq T.$$

Equivalent Definition. $a, b \in M, a\gamma b \in T(\gamma \in \Gamma) \Rightarrow a \in T$ or $b \in T$.

Definition 1.7 Let M be a po-Γ-semigroup. An equivalence relation σ on M is called congruence if

$$(a, b) \in \sigma \Rightarrow (a\gamma c, b\gamma c) \in \sigma, (c\gamma a, c\gamma b) \in \sigma, \forall a, b, c \in M, \gamma \in \Gamma.$$

A congruence σ on M is called a semilattice congruence if

$$(a\gamma a, a) \in \sigma, (a\gamma b, b\gamma a) \in \sigma, \forall a, b \in M, \gamma \in \Gamma.$$

M is called a semilattice of B-simple sub-Γ-semigroups if there exists a semilattice congruence σ on M such that the σ-class $(x)_\sigma$ of M containing x is a B-simple sub-Γ-semigroups of M for every $x \in M$.

For a po-Γ-semigroup M, if σ a semilattice congruence on M, then the set $M/\sigma := \{(x)_\sigma | x \in M\}$ is a commutative Γ−semigroup under the multiplication defined by $(x)_\sigma \gamma (y)_\sigma = (x\gamma y)_\sigma \forall x, y \in M, \gamma \in \Gamma$[5]. We denote by "$\preceq$" the order on the Γ−semigroup $(M/\sigma, \cdot)$ defined by:

$$(x)_\sigma \preceq (y)_\sigma \Leftrightarrow (x)_\sigma = (x\gamma y)_\sigma \forall x, y \in M, \forall \gamma \in \Gamma$$

Then $(M/\sigma, \cdot, \preceq)$ is a $po - \Gamma-$semigroup.

In the following, when we say that the bi-ideals of M form a chain we suppose them endowed with the inclusion relation "$\subseteq$".

Definition 1.8. Let M be a po-Γ-semigroup. M is called a chain of B-simple ordered semigroups if there exists a semilattice congruence σ on M such that $(x)_\sigma$ is a B-simple ordered subsemigroup of M for every $x \in M$ and $(M/\sigma, \preceq)$ is a chain.

Let M be a po-Γ-semigroup. We define the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{B}$, $\mathcal{N}$ on M as follows:

$$a\mathcal{L}b \text{ if and only if } L(a) = L(b).$$
$$a\mathcal{R}b \text{ if and only if } R(a) = R(b).$$
$$a\mathcal{B}b \text{ if and only if } B(a) = B(b).$$
$$a\mathcal{N}b \text{ if and only if } N(a) = N(b).$$

where $N(x)$ is a filter of M generated by x. One can easily prove that the relation $\mathcal{L}$, $\mathcal{R}$, $\mathcal{B}$, $\mathcal{N}$ are the equivalence relations on M.

Lemma 1.1. (see [4]) *Let M be a po-Γ-semigroup. Then we have:*

(1) $A \subseteq (A], \forall A \subseteq M$.

(2) If $A \subseteq B \subseteq M$, then$(A] \subseteq (B]$.

(3) $(A]\Gamma(B] \subseteq (A\Gamma B], \forall A, B \subseteq M$.

(4) $((A]] = (A], \forall A \subseteq M$.

(5) For every right ideal, left ideal, ideal T of M, we have $(T] = T$.

(6) If A, B are ideals of M, then $(A\Gamma B], A \bigcap B, A \bigcup B$ are also ideals of M.

(7) $(M\Gamma a]$ (resp. $(a\Gamma M]$) is a left (resp. right) ideal of M, $(M\Gamma a\Gamma M]$ is an ideal of M for every $a \in M$.

Lemma 1.2. *For a po-Γ-semigroup M, $\mathcal{N}$ is a semilattice congruence on M.*

Proof. Let $(x, y) \in \mathcal{N}, z \in M$. Since $z\gamma x \in N(z\gamma x)$, we have $z, x \in N(z\gamma x)$. Since $x \in N(z\gamma x), N(x) \subseteq N(z\gamma x)$. Thus $y \in N(z\gamma x)$. Since $z, y \in N(z\gamma x)$, we have $z\gamma y \in N(z\gamma x)$, and $N(z\gamma y) \subseteq N(z\gamma x)$. By symmetry, $N(z\gamma x) \subseteq N(z\gamma y)$. Thus $N(z\gamma x) = N(z\gamma y)$ i.e. $(z\gamma x, z\gamma y) \in \mathcal{N}$. By symmetry, $(x\gamma z, y\gamma z) \in \mathcal{N}$ for every $z \in M$.

Let $x \in M$. Since $x\gamma x \in N(x\gamma x)$, we have $x \in N(x\gamma x)$, and $N(x) \subseteq N(x\gamma x)$. Since $x \in N(x)$, we have $x\gamma x \in N(x)$, and $N(x\gamma x) \subseteq N(x)$. Then $(x\gamma x, x) \in \mathcal{N}$.

Let $x, y \in M$. Since $x\gamma y \in N(x\gamma y)$, we have $x, y \in N(x\gamma y), y\gamma x \in N(x\gamma y), N(y\gamma x) \subseteq N(x\gamma y)$. By symmetry, $N(x\gamma y) \subseteq N(y\gamma x)$. Thus

$$(x\gamma y, y\gamma x) \in \mathcal{N}.$$

Lemma 1.3. *Let M be a po-Γ-semigroup. Then M is left(resp. right) simple if and only if $(M\Gamma a] = M$(resp. $(a\Gamma M] = M$) for every $a \in M$.*

Proof. $\Rightarrow$: Let M be a left simple. For every $a \in M$, by Lemma 1.1, $(M\Gamma a]$ is a left ideal of M. Then $(M\Gamma a] = M$.

$\Leftarrow$: Let L be a left ideal of M, for any $a \in L$. We have $M = (M\Gamma a] \subseteq (M\Gamma L] \subseteq (L] = L$. Then $M = L$. Thus M is left simple.

Lemma 1.4. *Let M be a po-Γ-semigroup. Then the following statements are equivalent:*

(1) M is B-simple.

(2) $(a\Gamma M\Gamma a] = M$ for all $a \in M$.

(3) M is a left and right simple sub-Γ-semigroup of M.

Proof. (1) $\Rightarrow$ (2) Let $a \in M$, it is clear that $(a\Gamma M\Gamma a]$ is a bi-ideal of M. Since M is B-simple, $(a\Gamma M\Gamma a] = M$ for all $a \in M$.

(2) $\Rightarrow$ (3) For every $a \in M$, it is clear that

$$(a\Gamma M\Gamma a] \subseteq (M\Gamma a] \subseteq M \text{ and } (a\Gamma M\Gamma a] \subseteq (a\Gamma M] \subseteq M$$

By our hypothesis, we have $(a\Gamma M\Gamma a] = M$ and so for all $a \in M$, we have

$$(M\Gamma a] = M \text{ and } (a\Gamma M] = M.$$

Thus, by Lemma 1.3, M is both left and right simple.

(3) $\Rightarrow$ (1) Let B be a bi-ideal of M. Then, for $a \in B$, we have $(a\Gamma M] = (M\Gamma a] = M$ since M is a left and a right simple sub-Γ-semigroup. Thus, we have

$$M = (a\Gamma M] = (a\Gamma(M\Gamma a]] \subseteq (a\Gamma M\Gamma a] \subseteq (B\Gamma M\Gamma B] \subseteq (B] = B.$$

This lead to $B = M$ and hence M is B-simple.

Lemma 1.5. *Let M be a regular po-Γ-semigroup. Then the following conditions are equivalent:*

(1) M is left duo.

(2) For every $x \in M, (x\Gamma M\Gamma x] = (x\Gamma M]$.

Proof. (1) $\Rightarrow$ (2) It is clear that $x \in M, (x\Gamma M\Gamma x] \subseteq (x\Gamma M]$. Since M is regular, for any $x \in M$, we have

$$(x\Gamma M] \subseteq ((x\Gamma M\Gamma x]\Gamma M] \subseteq (x\Gamma M\Gamma x\Gamma M] \subseteq (x\Gamma(M\Gamma x]\Gamma M]$$

Since $(M\Gamma x]$ is a left ideal of M, by (1), it follows that $(M\Gamma x]\Gamma M \subseteq (M\Gamma x]$. Thus

$$(x\Gamma M] \subseteq (x\Gamma(M\Gamma x]] = (x\Gamma M\Gamma x].$$

(2) $\Rightarrow$ (1) Let L be a left ideal of M, for any $x \in L\Gamma M$. Then there exists $u \in L, \gamma \in \Gamma$ and $m \in M$ such that $x = u\gamma m \in L\Gamma M \subseteq (L\Gamma M]$. By (2), we have $x = u\gamma m \in (u\Gamma M] = (u\Gamma M\Gamma u] \subseteq (u\Gamma M\Gamma L] \subseteq (M\Gamma M\Gamma L] \subseteq (M\Gamma L] \subseteq L$. It implies that L is a right ideal of M.

Corollary. *Let M be a po-Γ-semigroup. Then M is right duo if and only if for $x \in M, (x\Gamma M\Gamma x] = (M\Gamma x]$. In particular, M is duo if and only if for $x \in M, (x\Gamma M\Gamma x] = (M\Gamma x] = (x\Gamma M]$.*

2. Main Result

Theorem 2.1. *Let M be a po-Γ-semigroup. Then the following conditions are equivalent:*

(1) M is regular duo.

(2) $B(a\gamma b) = B(a) \cap B(b), \forall a, b \in M, \gamma \in \Gamma$.

(3) M is duo and every left and right ideal of M is semiprime.

(4) $\forall x \in M$, $N(x)=\{y \in M | x \in (y\Gamma M\Gamma y]\}$.

(5) $\mathcal{N} = \mathcal{B}$.

(6) For every bi-ideal B of M, $B = \cup\{(x)_{\mathcal{N}} | x \in B\}$.

(7) $(x)_{\mathcal{N}}$ is a B-simple sub-Γ-semigroups of M, $\forall x \in M$.

(8) M is the semilattice of B-simple ordered sub-Γ-semigroups.

Proof. (1) $\Rightarrow$(2) Let M be a regular duo. For every $a \in M$, we can easily prove that $B(a) = (a\Gamma M\Gamma a]$. Let $a, b \in M, \gamma \in \Gamma$, for any $t \in B(a\gamma b)$. Then

$$\begin{aligned} t \in (a\gamma b\Gamma M\Gamma a\gamma b] &\subseteq (M\Gamma M\Gamma M\Gamma M\Gamma b] \subseteq (M\Gamma b] = (b\Gamma M\Gamma b] \\ &\subseteq (B(b)\Gamma M\Gamma B(b)] \subseteq (B(b)] = B(b) \Longrightarrow t \in B(b), \end{aligned}$$

$$\begin{aligned} t \in (a\gamma b\Gamma M\Gamma a\gamma b] &\subseteq (a\Gamma M\Gamma M\Gamma M\Gamma M] \subseteq (a\Gamma M] = (a\Gamma M\Gamma a] \\ &\subseteq (B(a)\Gamma M\Gamma B(a)] \subseteq (B(a)] = B(a) \Longrightarrow t \in B(a). \end{aligned}$$

Then $B(a\gamma b) \subseteq B(a) \cap B(b)$.

On the other hand, if $t \in B(a) \cap B(b)$. Since M is regular duo, then

$$t \in (t\Gamma M\Gamma t] \subseteq ((a\Gamma M\Gamma a]\Gamma M\Gamma(b\Gamma M\Gamma b]] \subseteq (a\Gamma M\Gamma b] \subseteq ((a\Gamma M]\Gamma b] \subseteq ((a\Gamma M\Gamma a]\Gamma b] \subseteq (M\Gamma a\Gamma b].$$

Hence for some $m \in M, \mu, \gamma \in \Gamma$, we have

$$t \leq m\mu a\gamma b \in M\Gamma a\gamma b \subseteq (M\Gamma a\gamma b] = (a\gamma b\Gamma M\Gamma a\gamma b] = B(a\gamma b) \Longrightarrow t \in B(a\gamma b).$$

(2)⇒(3) For any $a \in M, \gamma, \mu \in \Gamma$, by (2), $a \in B(a) = B(a\gamma a) = B(a\gamma a\mu a\gamma a)$. Then we have

$$x \in B(x\gamma x\mu x\gamma x) = (x\gamma x\mu x\gamma x \cup x\gamma x\mu x\gamma x\Gamma x\gamma x\mu x\gamma x \cup x\gamma x\mu x\gamma x\Gamma M\Gamma x\gamma x\mu x\gamma x] \subseteq (x\Gamma M\Gamma x].$$

Thus M is regular.

Let $a \in M$, for any $x \in (a\Gamma M]$. Then there exists $y \in M, \gamma \in \Gamma$ such that $x \leq a\gamma y$. By (2),

$$B(a\gamma y) = B(a) \cap B(y) = B(a) \cap B(y) \cap B(a) = B(a\gamma y\mu a) = (a\gamma y\mu a\Gamma M\Gamma a\gamma y\mu a] \subseteq (a\Gamma M\Gamma a](\mu \in \Gamma).$$

That is, $x \in (a\Gamma M\Gamma a]$. Then we have $(a\Gamma M] \subseteq (a\Gamma M\Gamma a]$. Obviously, $(a\Gamma M\Gamma a] \subseteq (a\Gamma M]$. Then $(a\Gamma M\Gamma a] = (a\Gamma M]$. In similar way, we have $(a\Gamma M\Gamma a] = (M\Gamma a]$. By Corollary, M is duo.

Let L be a left ideal of $M, \forall a \in M$ such that $a\gamma a \in L(\gamma \in \Gamma)$. By M is regular duo, we have

$$a \in (a\Gamma M\Gamma a] \subseteq ((a\Gamma M]\Gamma a] \subseteq ((a\Gamma M\Gamma a]\Gamma a] \subseteq (a\Gamma M\Gamma a\Gamma a] \subseteq (a\Gamma M\Gamma L] \subseteq L.$$

Then L is semiprime. Similarly, we can prove that every right ideal of M is semiprime.

(3) ⇒ (4) Let $x \in M$, $T=\{y \in M | x \in (y\Gamma M\Gamma y]\}$. Then T is a filter of M containing x. In fact: For $x \in M$, since $(x\Gamma M]$ is a left ideal of M, and $x\gamma x \in (x\Gamma M]$. By (3) and Corollary, we have $x \in (x\Gamma M] = (x\Gamma M\Gamma x] \Longrightarrow x \in T$.

Let $a, b \in T$. Then $x \in (a\Gamma M\Gamma a], x \in (b\Gamma M\Gamma b]$. Since $x \in (x\Gamma M\Gamma x]$(the proof is the same as in above), we have

$$x \in (x\Gamma M\Gamma x] \subseteq ((a\Gamma M\Gamma a]\Gamma M\Gamma(b\Gamma M\Gamma b]] \subseteq (a\Gamma M\Gamma a\Gamma M\Gamma b\Gamma M\Gamma b] \subseteq (a\Gamma M\Gamma b] \subseteq (a\Gamma(M\Gamma b]] \subseteq (a\Gamma b\Gamma M].$$

Thus, for some $m \in M, \gamma, \mu \in \Gamma$, we have $x \leq a\gamma b\mu m$. By Corollary, Then

$$x \leq a\gamma b\mu m \in (a\gamma b\Gamma M] = (a\gamma b\Gamma M\Gamma a\gamma b] \Longrightarrow a\gamma b \in T.$$

If $a\gamma b \in T$, then $x \in (a\gamma b\Gamma M a\gamma b] \subseteq (a\Gamma M] = (a\Gamma M\Gamma a]$, and

$$x \in (a\gamma b\Gamma M a\gamma b] \subseteq (M\Gamma b] = (b\Gamma M\Gamma b].$$

Therefore $a \in T, b \in T$. If $a \in T, M \ni b \geq a$, then $x \leq a\gamma m\mu a$, for some $\gamma, \mu \in \Gamma, m \in M$. Thus, $x \leq a\gamma m\mu a \leq b\gamma m\mu b \in (b\Gamma M\mu b]$ i.e. $b \in T$.

Let F be a filter of M such that $x \in F$. For any $a \in T$, $x \leq a\gamma m\mu a$, for some $\gamma, \mu \in \Gamma, m \in M$, then we have $a\gamma m\mu a \in F$. Thus $a \in F$. Therefore T is the least filter of M containing x, that is, $T = N(x)$.

(4) $\Rightarrow$ (5) For any $a, b \in M$, if $(a, b) \in \mathcal{N}$, then $N(a) = N(b)$, by (4), we have $a \in (b\Gamma M\Gamma b] \subseteq B(b)$, and $b \in (a\Gamma M\Gamma a] \subseteq B(a)$. Thus $B(a) = B(b)$, that is $(a, b) \in \mathcal{B}$.

Conversely, if $(a, b) \in \mathcal{B}$, then $a \in B(b) = (b \cup b\Gamma b \cup b\Gamma M\Gamma b]$. By hypothesis, we have $b \in N(a) \Longrightarrow N(b) \subseteq N(a)$. From $b \in B(a)$, by symmetry, we have $N(a) \subseteq N(b)$. Thus $(a, b) \in \mathcal{N}$.

(5) $\Rightarrow$ (6) Let B be a bi-ideal of M. If $y \in B$, then $y \in (y)_{\mathcal{N}} \subseteq \cup\{(x)_{\mathcal{N}} | x \in B\}$. Let $y \in (x)_{\mathcal{N}}, x \in B$. Then $(y, x) \in \mathcal{N} = \mathcal{B}$, thus $B(y) = B(x)$ and $y \in B(x)$. Besides, $x \in B$, implies $B(x) \subseteq B$. Hence $y \in B$.

(6) $\Rightarrow$ (7) By Lemma 1.2, it is clear that $(x)_{\mathcal{N}}$ is an ordered sub-Γ-semigroup of M with respet to the order of M. Let B be a bi-ideal of $(x)_{\mathcal{N}}$. For every $a \in (x)_{\mathcal{N}}$, assume that $b \in B$. Since $(b\Gamma M\Gamma b]$ is a bi-ideal of M. By (6), we have $(b\Gamma M\Gamma b] = \cup\{(y)_{\mathcal{N}} | y \in (b\Gamma M\Gamma b]$. Since $b\gamma b\gamma b \in (b\Gamma M\Gamma b]$, then we have $(b\gamma b\gamma b)_{\mathcal{N}} \subseteq (b\Gamma M\Gamma b]$. Because $\mathcal{N}$ is a semilattice congruence on M. Hence $(b\gamma b\gamma b)_{\mathcal{N}} = (b\gamma b)_{\mathcal{N}} = (b)_{\mathcal{N}} = (x)_{\mathcal{N}} = (a)_{\mathcal{N}}$, it follows that $a \in (a)_{\mathcal{N}} \subseteq (b\Gamma M\Gamma b] \subseteq B$. That is $a \in B$. Therefore $(x)_{\mathcal{N}} \subseteq B$, that is $B = (x)_{\mathcal{N}}$.

(7) $\Rightarrow$ (8) Since $\mathcal{N}$ is a semilattice congruence on M. By (7), M is the semilattice of B-simple ordered sub-Γ-semigroups.

(8) $\Rightarrow$ (1) Let σ be a semilattice congruence on M such that $(x)_\sigma$ is B-simple ordered sub-Γ-semigroups of M for every $x \in M$. By Lemma 4, we have $(x)_\sigma = (x\Gamma(x)_\sigma\Gamma x] \subseteq (x\Gamma M\Gamma x]$. Thus M is regular.

For any $x \in M$, then $B(x) \cap (x)_\sigma$ is a bi-ideal of $(x)_\sigma$. In fact:

$$\emptyset \neq B(x) \cap (x)_\sigma \subseteq (x)_\sigma (x \in B(x) \cap (x)_\sigma),$$

$$\begin{aligned}
&(B(x) \cap (x)_\sigma)\Gamma(x)_\sigma\Gamma(B(x) \cap (x)_\sigma) \\
&\subseteq B(x)\Gamma(x)_\sigma\Gamma B(x) \cap B(x)\Gamma(x)_\sigma\Gamma(x)_\sigma \cap (x)_\sigma\Gamma(x)_\sigma\Gamma B(x) \cap (x)_\sigma\Gamma(x)_\sigma\Gamma(x)_\sigma \\
&\qquad \subseteq B(x) \cap B(x)\Gamma(x)_\sigma \cap (x)_\sigma\Gamma B(x) \cap (x)_\sigma \subseteq B(x) \cap (x)_\sigma.
\end{aligned}$$

Let $y \in B(x) \cap (x)_\sigma, (x)_\sigma \ni z \leq y$, since $z \leq y \in B(x), B(x)$ is a bi-ideal of M, we have $z \in B(x)$. Thus $z \in B(x) \cap (x)_\sigma$. Since $(x)_\sigma$ is B-simple, we have $B(x) \cap (x)_\sigma = (x)_\sigma$, that is $B(x) = (x)_\sigma$. Therefore $B(x)$ is B-simple. By Lemma 4, then $B(x)$ is also left and right simple. It implies that $B(x) = L(x) = R(x)$. Because M is regular, we have $(x\Gamma M\Gamma x] = (x\Gamma M] = (M\Gamma x]$. By Corollary, we have M is duo.

Theorem 2.2. *Let M be a po-Γ-semigroup. Then the following conditions are equivalent:*

(1) M is a chain of B-simple ordered semigroups.

(2) M is duo and every bi-ideal of M is prime.

(3) M is regular duo and the bi-ideals of M is a chain.

Proof. (1) $\Rightarrow$(2) Let σ be a semilattice congruence on M such that $(x)_\sigma$ is a B-simple ordered subsemigroup of M for every $x \in M$ and $(M/\sigma, \preceq)$ is a chain. By Theorem 2.1, M is duo. Let I be a bi-ideal of M, $a, b \in M, \gamma \in \Gamma$ such that $a\gamma b \in I$. Then $I \cap (a\gamma b)_\sigma$ is a bi-ideal of $(a\gamma b)_\sigma$. Since $(a\gamma b)_\sigma$ is B-simple, we have $I \cap (a\gamma b)_\sigma = (a\gamma b)_\sigma$. On the other hand, we have $(a)_\sigma \preceq (b)_\sigma$ or $(b)_\sigma \preceq (a)_\sigma$. If $(a)_\sigma \preceq (b)_\sigma$, then $(a)_\sigma = (a\gamma b)_\sigma$ for $\gamma \in \Gamma$. Thus $I \cap (a)_\sigma = (a)_\sigma$, that is $a \in I$. If $(b)_\sigma \preceq (a)_\sigma$, then $(b)_\sigma = (b\gamma a)_\sigma = (a\gamma b)_\sigma$ for $\gamma \in \Gamma$. Thus $I \cap (b)_\sigma = (b)_\sigma$, that is $b \in I$. Consequently, I is prime.

(2) $\Rightarrow$(3) Since $a\gamma a\mu a \in (a\Gamma M\Gamma a]$ for any $a \in M, \gamma, \mu \in \Gamma$. By (2), we have $a \in (a\Gamma M\Gamma a]$, that is M is regular. Since M is duo, by Corollary, we have $(x\Gamma M\Gamma x] = (M\Gamma x] = (x\Gamma M]$, for any $x \in M$. Let A, B be bi-ideals of M, one can easily prove that A, B be ideals of M. Since $(A\Gamma B]$ is a bi-ideal of M and $A\Gamma B \subseteq (A\Gamma B]$, we have

$$A \subseteq (A\Gamma B] \subseteq (M\Gamma B] \subseteq (B] = B$$

or

$$B \subseteq (A\Gamma B] \subseteq (A\Gamma M] \subseteq (A] = A.$$

Thus the bi-ideals of M is a chain.

(3) $\Rightarrow$(1) By Theorem 2.1, M is a semilattice of B-simple sub-Γ-semigroups. For any $a, b \in M, \gamma \in \Gamma$, we have $B(a\gamma b) = B(a) \cap B(b)$. On the other hand, by (3), $B(a) \subseteq B(b)$ or $B(b) \subseteq B(a)$. Hence $B(a\gamma b) = B(a)$ or $B(a\gamma b) = B(b)$. Since $\mathcal{N} = \mathcal{B}$. We have $N(a\gamma b) = N(a)$ or $N(a\gamma b) = N(b)$. Thus $(M/\mathcal{N}, \preceq)$ is a chain.

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