International Journal of Pure and Applied Mathematics
Volume 78 No. 4 2012, 509-522
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

MULTIPLIER GENERALIZED DOUBLE SEQUENCE SPACES OF FUZZY NUMBERS DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS

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Abstract: In this paper we introduce multiplier generalized double sequence spaces of fuzzy numbers defined by a sequence of Orlicz functions $\mathcal{M} = (M_{k,l})$ and multiplier function $u = (u_{k,l})$. We also make an effort to prove some topological properties and inclusion relation between these spaces.

AMS Subject Classification: 40A05, 40D25
Key Words: fuzzy numbers, Musielak-Orlicz function, de La Vallee Poussin means, statistical convergence, multiplier function

1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [16] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [8] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. In [9] Nanda studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Savas [13]

Received: April 9, 2012

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introduced and discussed double convergent sequences of fuzzy numbers and showed that the space of all double convergent sequences of fuzzy numbers is complete. Recently Basarir and Mursaleen [2] introduced and studied some new sequence spaces of fuzzy numbers generated by nonnegative regular matrix.

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Parashar and Choudhary [11] have introduced and discussed some properties of the sequence spaces defined by using a Orlicz function M which generalized the well-known Orlicz sequence space l_M and strongly summable sequence spaces $[\mathcal{C}, 1, p], [\mathcal{C}, 1, p]_0$ and$[\mathcal{C}, 1, p]_\infty$. Later on, Basarir and Mursaleen [1], Tripathy and Mahanta [15] used the idea of an Orlicz function to construct some spaces of complex sequences.

The concept of statistical convergence was introduced by Fast [6] and also independently by Buck [4] and Schoenberg [14] for real and complex sequences. Further this concept was studied by Fridy [7], Connor [5] and many others. Statistical convergence is closely related to the concept of convergence in probability. The existing literature on statistical convergence appears to have been restricted to real or complex analysis, but at the first time Nurray and Savas [10] extended the idea to apply the sequences of fuzzy numbers.

2. Definitions and Preliminaries

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called the modulus function see, [12]. An orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $X : \mathbb{R}^n \to [0, 1]$ which satisfies the following four conditions:

1. X is normal, i.e., there exist an $x_0 \in \mathbb{R}^n$ such that $X(x_0) = 1$;
2. X is fuzzy convex, i.e., for $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1, X(\lambda x + (1 - \lambda)y) \geq \min[X(x), X(y)]$;
3. X is upper semi-continuous;
4. the closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$, denoted by $[X]^0$, is compact.

Let $C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A \text{ compact and convex }\}$. The spaces $C(\mathbb{R}^n)$ has a linear structure induced by the operations

$$A + B = \{a + b : a \in A, b \in B\}$$

and

$$\lambda A = \{\lambda a : a \in A\}$$

for $A, B \in C(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The Hausdorff distance between A and B of $C(\mathbb{R}^n)$ is defined as

$$\delta_\infty(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|\}$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete (not separable) metric space.

Let $\lambda = (\lambda_{m,n})$ be a non-decreasing sequence of positive real numbers tending to infinity such that

$$\lambda_{m+1,n} \le \lambda_{m,n} + 1, \lambda_{m,n+1} \le \lambda_{m,n} + 1,$$

$$\lambda_{m,n} - \lambda_{m+1,n} \le \lambda_{m,n+1} - \lambda_{m+1,n+1}, \lambda_{1,1} = 1$$

and

$$I_{m,n} = \{(k, l) : m - \lambda_{m,n} + 1 \le k \le m, n - \lambda_{m,n} + 1 \le l \le n\}.$$

The generalized double de la Vallee-Poussin mean is defined by

$$t_{m,n} = t_{m,n}(X_{k,l}) = \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} (X_{k,l}).$$

A Fuzzy double sequence is a double infinite array of fuzzy numbers. We denote a fuzzy double sequence by $(X_{k,l})$, where $X_{k,l}$'s are fuzzy numbers for each $k, l \in \mathbb{N}$. By $s''(F)$ we denote the set of all double sequences of fuzzy numbers.

A double sequence $X = (X_{k,l})$ of fuzzy numbers is said to be convergent in the Pringsheim's sense or P-convergent to a fuzzy number X_0, if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(X_{k,l}, X_0) < \epsilon \text{ for all } k, l > N,$$

where $\mathbb{N}$ is the set of natural numbers, and we denote it also by $P-\lim X = X_0$. The number X_0 is called the Pringsheim limit of $(X_{k,l})$.

More exactly we say that a double sequence $(X_{k,l})$ converges to a finite number X_0 if $X_{k,l}$ tend to X_0 as both k and l tends to ∞ independently of one another.

A double sequence $X = (X_{k,l})$ of fuzzy numbers is said to be λ-statistically convergent to X_0 provided that for each $\epsilon > 0$

$$P - \lim_{m,n} \frac{1}{m,n} |\{(j,k); j \leq m \text{ and } k \leq n : d(X_{k,l}, X_0) \geq \epsilon\}| = 0.$$

We denote the set of all double λ-statistically convergent sequences of fuzzy numbers by $s''(\lambda)^F$.

Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers. In the present paper we define the following classes of sequences:

$$w''(\lambda, \mathcal{M}, u, p)^F$$

$$= \Big\{ X = (X_{k,l}) \in s''(F) : \lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), X_0)}{\rho} \Big) \Big]^{p_{k,l}}$$

$$= 0, \;\text{ uniformly in } m, \;\; n \text{ for some } \;\rho > 0 \Big\},$$

$w''_0(\lambda, \mathcal{M}, u, p)^F =$

$$\Big\{ X = (X_{k,l}) \in s''(F) : \lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), \bar{0})}{\rho} \Big) \Big]^{p_{k,l}}$$

$$= 0, \;\text{ uniformly in } m, \;\; n \text{ for some } \;\rho > 0 \Big\}$$

and

$w''_\infty(\lambda, \mathcal{M}, u, p)^F =$

$$\Big\{ X = (X_{k,l}) \in s''(F) : \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), \bar{0})}{\rho} \Big) \Big]^{p_{k,l}}$$

$$< \infty, \;\text{ uniformly in } m, \;\; n \text{ for some } \;\rho > 0 \Big\}.$$

where

$$\bar{0}(t) = \begin{cases} 1, & t = (0,0,0,\cdots,0) \\ 0, & \text{otherwise.} \end{cases}$$

If $X \in w''(\lambda, \mathcal{M}, u, p)^F$, we say that X is strongly almost λ-convergent with respect to the sequence of Orlicz function. In this case we write $X_{k,l} \to X_0(w''(\lambda, \mathcal{M}, u, p)^F)$. The following sequence spaces are defined by giving particular values to $\mathcal{M}, u, p$ and λ;

(i) For $\lambda_{m,n} = 1$, we have

$$w''(\lambda, \mathcal{M}, u, p)^F = w''(\mathcal{M}, u, p)^F, w_0''(\lambda, \mathcal{M}, u, p)^F = w_0''(\mathcal{M}, u, p)^F$$

and

$$w_\infty''(\lambda, \mathcal{M}, u, p)^F = w_\infty''(\mathcal{M}, u, p)^F,$$

(ii) If $\mathcal{M} = M_{k,l}(x) = x$ for all k, l we get

$$w''(\lambda, \mathcal{M}, u, p)^F = w''(\lambda, u, p)^F, \quad w_0''(\lambda, \mathcal{M}, u, p)^F = w_0''(\lambda, u, p)^F$$

and

$$w_\infty''(\lambda, \mathcal{M}, u, p)^F = w_\infty''(\lambda, u, p)^F,$$

(iii) If $p_{k,l} = 1$ for all $k, l \in \mathbb{N}$, then

$$w''(\lambda, \mathcal{M}, u, p)^F = w''(\lambda, \mathcal{M}, u)^F, \quad w_0''(\lambda, \mathcal{M}, u, p)^F = w_0''(\lambda, \mathcal{M}, u)^F$$

and

$$w_\infty''(\lambda, \mathcal{M}, u, p)^F = w_\infty''(\lambda, \mathcal{M}, u)^F,$$

(iv) If $\mathcal{M} = M_{k,l}(x) = x$ for all k, l and $p_{k,l} = 1$ for all $k, l \in \mathbb{N}$, then

$$w''(\lambda, \mathcal{M}, u, p)^F = w''(\lambda, u)^F, w_0''(\lambda, \mathcal{M}, u, p)^F = w_0''(\lambda, u)^F$$

and

$$w_\infty''(\lambda, \mathcal{M}, u, p)^F = w_\infty''(\lambda, u)^F,$$

(v) If $p_{k,l} = 1$ for all $k, l \in \mathbb{N}$, and $u_{k,l} = 1$ for all k, l then

$$w''(\lambda, \mathcal{M}, u, p)^F = w''(\lambda, \mathcal{M})^F, \quad w_0''(\lambda, \mathcal{M}, u, p)^F = w_0''(\lambda, \mathcal{M})^F$$

and

$$w_\infty''(\lambda, \mathcal{M}, u, p)^F = w_\infty''(\lambda, \mathcal{M})^F,$$

(vi) If $\mathcal{M} = M_{k,l}(x) = x, p_{k,l} = 1$ and $u_{k,l} = 1$ for all k, l then

$$w''(\lambda, \mathcal{M}, u, p)^F = w''(\lambda)^F, w_0''(\lambda, \mathcal{M}, u, p)^F = w_0''(\lambda)^F$$

and

$$w''_\infty(\lambda, \mathcal{M}, u, p)^F = w''_\infty(\lambda)^F.$$

The following inequality will be used throughout the paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \leq \sup_{k,l} p_{k,l} = H$, and let $D = \max\{1, 2^{H-1}\}$. Then, for the factorable sequences a_k and b_k in the complex plane, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq D(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}). \tag{2.1}$$

In the present paper we study some topological properties and inclusion relation between the above defined sequence spaces.

3. Main Results

Theorem 3.1. *Suppose $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of positive real numbers then*

$$w''_0(\lambda, \mathcal{M}, u, p)^F \subset w''(\lambda, \mathcal{M}, u, p)^F \subset w''_\infty(\lambda, \mathcal{M}, u, p)^F.$$

Proof. The inclusion $w''_0(\lambda, \mathcal{M}, u, p)^F \subset w''(\lambda, \mathcal{M}, u, p)^F$ is obvious. Let $X \in w''(\lambda, \mathcal{M}, u, p)^F$. Then we get

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l}\Big(\frac{d(t_{mn}(X), \bar{0})}{2\rho}\Big)\Big]^{p_{k,l}}$$

$$\begin{aligned}
&\leq \frac{D}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \frac{1}{2^{p_{k,l}}} u_{k,l} \Big[M_{k,l}\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big]^{p_{k,l}} \\
&+ \frac{D}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \frac{1}{2^{p_{k,l}}} u_{k,l} \Big[M_{k,l}\Big(\frac{d(X_0, \bar{0})}{\rho}\Big)\Big]^{p_{k,l}} \\
&\leq \frac{D}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l}\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big]^{p_{k,l}} \\
&+ D \max_{k,l \in I_{mn}} \Big\{ \max\Big\{1, \sup u_{k,l} \Big[M_{k,l}\Big(\frac{d(X_0, \bar{0})}{\rho}\Big)\Big]^H\Big\}\Big\}.
\end{aligned}$$

where $\sup_{k,l} p_{k,l} = H$ and $D = \max(1, 2^{H-1})$. Thus we get $X \in w''_\infty(\lambda, \mathcal{M}, u, p)^F$.

Theorem 3.2. *Suppose $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers then $w''(\lambda, \mathcal{M}, u, p)^F$, $w''_0(\lambda, \mathcal{M}, u, p)^F$, and $w''_\infty(\lambda, \mathcal{M}, u, p)^F$ are linear spaces.*

Proof. Let $X = (X_{k,l}), Y = (Y_{k,l}) \in w''_\infty(\lambda, \mathcal{M}, u, p)^F$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1, ρ_2 such that

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), \bar{0})}{\rho_1} \Big) \Big]^{p_{k,l}} < \infty, \text{ uniformly in } m, \ n.$$

and

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), \bar{0})}{\rho_2} \Big) \Big]^{p_{k,l}} < \infty, \text{ uniformly in } m, \ n.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$.

Since $\mathcal{M} = (M_{k,l})$ is non-decreasing and convex, we have

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{\alpha d(t_{mn}(X), \bar{0}) + \beta d(t_{mn}(Y), \bar{0})}{\rho_3} \Big) \Big]^{p_{k,l}}$$

$$\begin{aligned}
&\leq \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{\alpha d(t_{mn}(X), \bar{0})}{\rho_3} + \frac{\beta d(t_{mn}(Y), \bar{0})}{\rho_3} \Big) \Big]^{p_{k,l}} \\
&\leq \frac{1}{2} \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), \bar{0})}{\rho_1} \Big) \Big]^{p_{k,l}} \\
&+ \frac{1}{2} \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), \bar{0})}{\rho_2} \Big) \Big]^{p_{k,l}} \\
&< \infty.
\end{aligned}$$

This proves that $w''_\infty(\lambda, \mathcal{M}, u, p)^F$ is a linear space. Similarly, we can prove others.

Theorem 3.3. *Let $0 < p_{k,l} \leq r_{k,l}$ for all $k, l \in \mathbb{N}$ and $(\frac{r_{k,l}}{p_{k,l}})$ be bounded. Then we have*

$$w''_\infty(\lambda, \mathcal{M}, u, r)^F \subset w''_\infty(\lambda, \mathcal{M}, u, p)^F.$$

Proof. Let $X = (X_{k,l}) \in w''_\infty(\lambda, \mathcal{M}, u, r)^F$. Then

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),\bar{0})}{\rho}\Big)\Big]^{r_{k,l}} < \infty, \text{ uniformly in } m, \ n.$$

Let $s_{k,l} = \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),\bar{0})}{\rho}\Big)\Big]^{r_{k,l}}$ and $\lambda_{k,l} = \frac{p_{k,l}}{r_{k,l}}$. Since $p_{k,l} \le r_{k,l}$, we have $0 \le \lambda_{k,l} \le 1$. Take $0 < \lambda < \lambda_{k,l}$.

Define

$$u_{k,l} = \begin{cases} s_{k,l} & \text{if } s_{k,l} \ge 1 \\ 0 & \text{if } s_{k,l} < 1 \end{cases}$$

and

$$v_{k,l} = \begin{cases} 0 & \text{if } s_{k,l} \ge 1 \\ s_{k,l} & \text{if } s_{k,l} < 1 \end{cases}$$

$s_{k,l} = u_{k,l} + v_{k,l}$, $s_{k,l}^{\lambda_{k,l}} = u_{k,l}^{\lambda_{k,l}} + v_{k,l}^{\lambda_{k,l}}$. It follows that $u_{k,l}^{\lambda_{k,l}} \le u_{k,l} \le s_{k,l}$, $v_{k,l}^{\lambda_{k,l}} \le v_{k,l}^{\lambda}$. since $s_{k,l}^{\lambda_{k,l}} = u_{k,l}^{\lambda_{k,l}} + v_{k,l}^{\lambda_{k,l}}$, then $s_{k,l}^{\lambda_{k,l}} \le s_{k,l} + v_{k,l}^{\lambda}$

$$\begin{aligned}
&\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[\Big(M_{k,l}\Big(\frac{d(t_{mn}(X),\bar{0})}{\rho}\Big)\Big)^{r_{k,l}}\Big]^{\lambda_{k,l}} \\
&\qquad \le \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),\bar{0})}{\rho}\Big)\Big]^{r_{k,l}} \\
&\Longrightarrow \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[\Big(M_{k,l}\Big(\frac{d(t_{mn}(X),\bar{0})}{\rho}\Big)\Big)^{r_{k,l}}\Big]^{p_{k,l}/r_{k,l}} \\
&\qquad \le \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),\bar{0})}{\rho}\Big)\Big]^{r_{k,l}} \\
&\Longrightarrow \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[\Big(M_{k,l}\Big(\frac{d(t_{mn}(X),\bar{0})}{\rho}\Big)\Big)\Big]^{p_{k,l}} \\
&\qquad \le \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),\bar{0})}{\rho}\Big)\Big]^{r_{k,l}}
\end{aligned}$$

But

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),\bar{0})}{\rho}\Big)\Big]^{r_{k,l}} < \infty, \text{ uniformly in } m, \ n.$$

Therefore

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), \bar{0})}{\rho} \Big) \Big]^{p_{k,l}} < \infty, \text{ uniformly in } m, \ n.$$

Hence $X \in w''_\infty(\lambda, \mathcal{M}, u, p)^F$. Thus we get $w''_\infty(\lambda, \mathcal{M}, u, r)^F \subset w''_\infty(\lambda, \mathcal{M}, u, p)^F$.

Theorem 3.4. *Suppose $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers. If $\sup_{k,l} (M_{k,l}(x))^{p_{k,l}} < \infty$ for all fixed $x > 0$, then*

$$w''(\lambda, \mathcal{M}, u, p)^F \subset w''_\infty(\lambda, \mathcal{M}, u, p)^F.$$

Proof. Let $X \in w''(\lambda, \mathcal{M}, u, p)^F$. Then there exists a positive number $\rho_1 > 0$ such that

$$\lim_{m,n \to \infty} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), X_0)}{\rho_1} \Big) \Big]^{p_{k,l}} = 0, \text{ uniformly in } m, \ n.$$

Define $\rho = 2\rho_1$. Since $\mathcal{M} = (M_{k,l})$ is non-decreasing and convex, for each k, l so by using (2.1), we have

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), \bar{0})}{\rho} \Big) \Big]^{p_{k,l}}$$

$$\begin{aligned}
&\leq \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), X_0) + d(X_0, \bar{0})}{\rho} \Big) \Big]^{p_{k,l}} \\
&\leq D \Big\{ \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \frac{1}{2^{p_{k,l}}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), X_0)}{\rho_1} \Big) \Big]^{p_{k,l}} \\
&+ \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \frac{1}{2^{p_{k,l}}} u_{k,l} \Big[M_{k,l} \Big(\frac{d(t_{mn}(X), \bar{0})}{\rho_1} \Big) \Big]^{p_{k,l}} \Big\} \\
&< \infty.
\end{aligned}$$

Thus $X \in w''_\infty(\lambda, \mathcal{M}, u, p)$. This completes the proof of the theorem.

Theorem 3.5. *Let $0 < h = \inf p_{k,l} \leq p_{k,l} \leq \sup p_{k,l} = H < \infty$. Then for a sequence of Orlicz functions $\mathcal{M} = (M_{k,l})$ which satisfies the Δ_2-condition,* we have $w_0''(\lambda, u, p)^F \subset w_0''(\lambda, \mathcal{M}, u, p)^F, w''(\lambda, u, p)^F \subset w''(\lambda, \mathcal{M}, u, p)^F$ and $w_\infty''(\lambda, u, p)^F \subset w_\infty''(\lambda, \mathcal{M}, u, p)^F$.

Proof. Let $X \in w''(\lambda, u, p)^F$. Then we have

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big]^{p_{k,l}} \to 0 \text{ as } m, n \to \infty, \text{uniformly in } m, \ n.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_{k,l}(t) < \epsilon$ for $0 \leq t \leq \delta$. Then

$$\begin{aligned}
&\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big]^{p_{k,l}} \\
&= \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}, d(t_{mn}(X), X_0) \leq \delta} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big]^{p_{k,l}} \\
&+ \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}, d(t_{mn}(X), X_0) > \delta} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big]^{p_{k,l}} \\
&= \sum_1 + \sum_2 .
\end{aligned}$$

where

$$\sum_1 = \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}, d(t_{mn}(X), X_0) \leq \delta} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big]^{p_{k,l}} < \max(\epsilon, \epsilon^H)$$

by using continuity of $(M_{k,l})$. For the second summation, we shall make the following procedure. Thus we have

$$\frac{d(t_{mn}(X), X_0)}{\rho} < 1 + \frac{\frac{d(t_{mn}(X), X_0)}{\rho}}{\delta}.$$

Since $\mathcal{M} = (M_{k,l})$ is non-decreasing and convex, so we have

$$\begin{aligned}
u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big] &< u_{k,l}\Big[M_{k,l}\Big\{1 + \frac{\frac{d(t_{mn}(X), X_0)}{\rho}}{\delta}\Big\}\Big] \\
&\leq \frac{1}{2} u_{k,l}\Big[M_{k,l}(2)\Big] + \frac{1}{2} u_{k,l}\Big[M_{k,l}\Big\{2\frac{\frac{d(t_{mn}(X), X_0)}{\rho}}{\delta}\Big\}\Big].
\end{aligned}$$

Again, since $\mathcal{M} = (M_{k,l})$ satisfies the Δ_2-condition, it follows that

$$u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big] \leq \frac{1}{2} L\Big\{\frac{\frac{d(t_{mn}(X), X_0)}{\rho}}{\delta}\Big\} u_{k,l}\Big[M_{k,l}(2)\Big]$$

$$\begin{aligned} &+ \frac{1}{2} L\Big\{\frac{\frac{d(t_{mn}(X),X_0)}{\rho}}{\delta}\Big\} u_{k,l}\Big[M_{k,l}(2)\Big] \\ &= L\Big\{\frac{\frac{d(t_{mn}(X),X_0)}{\rho}}{\delta}\Big\} u_{k,l}\Big[M_{k,l}(2)\Big] \end{aligned}$$

Thus, it follows that

$$\sum_2 = \max_{k,l \in I_{m,n}} \Big\{1, \Big[\frac{L u_{k,l}[M_{k,l}(2)]}{\delta}\Big]^H\Big\} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big]^{p_{k,l}}.$$

Taking the limit as $\epsilon \to 0$ and $m, n \to \infty$, it follows that $X \in w''(\lambda, \mathcal{M}, u, p)^F$. Similarly, we can prove that $w''_0(\lambda, u, p)^F \subset w''_0(\lambda, \mathcal{M}, u, p)^F$ and $w''_\infty(\lambda, u, p)^F \subset w''_\infty(\lambda, \mathcal{M}, u, p)^F$.

Theorem 3.6. *Suppose $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers. Then:*

(i) If $0 < \inf p_{k,l} \le p_{k,l} \le 1$ for all k, l then $w''(\lambda, \mathcal{M}, u)^F \subseteq w''(\lambda, \mathcal{M}, u, p)^F$

(ii) If $1 \le p_{k,l} \le \sup p_{k,l} = H < \infty$ then $w''(\lambda, \mathcal{M}, u, p)^F \subseteq w''(\lambda, \mathcal{M}, u)^F$.

Proof. (i) Let $X \in w''(\lambda, \mathcal{M}, u)^F$. Then since $0 < \inf p_{k,l} \le p_{k,l} \le 1$, we get

$$\begin{aligned} &\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big]^{p_{k,l}} \\ &\qquad \le \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big] \end{aligned}$$

and hence $X \in w''(\lambda, \mathcal{M}, u, p)^F$.

(ii) Let $X \in w''(\lambda, \mathcal{M}, u, p)^F$ and $1 \le p_{k,l} \le \sup p_{k,l} = H < \infty$. Then for every $0 < \epsilon < 1$, there exists a positive integers m_0, n_0 such that

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X), X_0)}{\rho}\Big)\Big]^{p_{k,l}} \le \epsilon < 1$$

for all $m \ge m_0, n \ge n_0$. It follows that

$$\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),X_0)}{\rho}\Big)\Big]$$
$$\leq \frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),X_0)}{\rho}\Big)\Big]^{p_{k,l}}.$$

Hence $X \in w''(\lambda,\mathcal{M},u)^F$.

Theorem 3.7. *Suppose $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers, $u = (u_{k,l})$ be a sequence of strictly positive real numbers and $0 < h = \inf p_{k,l} \leq p_{k,l} \leq \sup p_{k,l} = H < \infty$. Then $w''(\lambda,\mathcal{M},u,p)^F \subset s''(\lambda)^F$.*

Proof. The proof of the theorem follows from the following inequality:

$$\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),X_0)}{\rho}\Big)\Big]^{p_{k,l}}$$

$$\begin{aligned}
&\geq \frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),X_0)}{\rho}\Big)\Big]^{p_{k,l}}\\
&\geq \frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}, d(t_{mn}(X),X_0)\geq\epsilon} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),X_0)}{\rho}\Big)\Big]^{p_{k,l}}\\
&\geq \frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}, d(t_{mn}(X),X_0)\geq\epsilon} \min\Big\{u_{k,l}[M_{k,l}(\epsilon_1)]^h, u_{k,l}[M_{k,l}(\epsilon_1)]^H\Big\}\\
&\geq \frac{1}{\lambda_{m,n}}\Big|\Big\{(k,l)\in I_{m,n} : d(t_{mn}(X),X_0)\geq\epsilon\Big\}\Big|\\
&\qquad\times \min_{k,l\in I_{m,n}}\Big\{u_{k,l}[M_{k,l}(\epsilon_1)]^h, u_{k,l}[M_{k,l}(\epsilon_1)]^H\Big\},
\end{aligned}$$

where $\epsilon_1 = \frac{\epsilon}{\rho}$.

Theorem 3.8. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $X = (X_{k,l})$ be a double bounded sequence of Fuzzy numbers and $0 < h = \inf p_{k,l} \leq p_{k,l} \leq \sup p_{k,l} = H < \infty$. Then $s''(\lambda)^F \subset w''(\lambda,\mathcal{M},u,p)^F$.*

Proof. Suppose that $X \in l_\infty^F$ and $X_{k,l} \to X_0(s''(\lambda)^F)$. Since $X \in l_\infty^F$, there exists a constant $K > 0$ such that $d(t_{mn}(X),X_0) \leq K$ for all m,n. Given $\epsilon > 0$, we have

$$\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),X_0)}{\rho}\Big)\Big]^{p_{k,l}}$$

$$
\begin{aligned}
&= \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}, d(t_{mn}(X),X_0)\geq\epsilon} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),X_0)}{\rho}\Big)\Big]^{p_{k,l}} \\
&+ \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}, d(t_{mn}(X),X_0)<\epsilon} u_{k,l}\Big[M_{k,l}\Big(\frac{d(t_{mn}(X),X_0)}{\rho}\Big)\Big]^{p_{k,l}} \\
&\leq \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}, d(t_{mn}(X),X_0)\geq\epsilon} \max\Big\{u_{k,l}[M_{k,l}(\frac{K}{\rho})]^h, u_{k,l}[M_{k,l}(\frac{K}{\rho})]^H\Big\} \\
&+ \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}, d(t_{mn}(X),X_0)<\epsilon} u_{k,l}[M_{k,l}(\frac{\epsilon}{\rho})]^{p_{k,l}} \\
&\leq \max_{k,l\in I_{m,n}} \Big\{u_{k,l}[M_{k,l}(T)]^h, u_{k,l}[M_{k,l}(T)]^H\Big\} \\
&\qquad \times\frac{1}{\lambda_{m,n}}\Big|\Big\{(k,l)\in I_{m,n} : d(t_{mn}(X),X_0)\geq\epsilon\Big\}\Big| \\
&+ \max_{k,l\in I_{m,n}} \Big\{u_{k,l}[M_{k,l}(\epsilon_1)]^h, u_{k,l}[M_{k,l}(\epsilon_1)]^H\Big\}
\end{aligned}
$$

where $T = \frac{K}{\rho}, \frac{\epsilon}{\rho} = \epsilon_1$. Hence $X \in w''(\lambda, \mathcal{M}, u, p)^F$.

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