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$\mu-$SEMI COMPACTNESS AND $\mu-$SEMI LINDELÖFNESS IN GENERALIZED TOPOLOGICAL SPACES

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Abstract: The purpose of the present paper is to introduce the concepts of $\mu-$semi compact and $\mu-$semi Lindelöf spaces in generalized topological spaces and study some of their properties and characterizations.

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1. Introduction and Preliminaries

In [2] - [13], Á. Császár introduced the concepts of generalized neighborhood systems and generalized topological spaces. He also introduced the concepts of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In particular, he investigated characterizations for the generalized continuous function by using a closure operator defined on generalized neighborhood systems. In [1], A. Al-Omari and T. Noiri introduced the notions of $contra-(\mu,\lambda)-continuity$, $contra-(\alpha,\lambda)-continuity$, $contra-(\sigma,\lambda)-continuity$, $contra-(\pi,\lambda)-continuity$ and $contra-(\beta,\lambda)-continuity$ on generalized topological spaces.

In this paper, we introduce the concepts of $\mu-$semi compact and $\mu-$semi Lindelöf spaces in generalized topological spaces and study some of their properties and characterizations.

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We recall some basic definitions and notations. Let X be a set and denote $expX$ the power set of X. A subset μ of $expX$ is said to be a generalized topology [4] (briefly GT) on X if $\phi \in \mu$ and the arbitrary union of elements of μ belongs to μ. A set X with a GT μ on it is called a generalized topological space and is denoted by (X, μ). Let μ be a GT on X, the elements of μ are called $\mu - open$ sets and the complements of $\mu - open$ sets are called $\mu - closed$ sets. If $A \subseteq X$, then $i_\mu(A)$ denotes the union of all $\mu - open$ sets contained in A and $c_\mu(A)$ is the intersection of all $\mu - closed$ sets containing A [7]. According to [9], for $A \subseteq X$ and $x \in X$, we have $x \in c_\mu(A)$ if and only if $x \in M \in \mu$ implies $M \cap A \neq \phi$.

Definition 1.1. [7] Let (X, μ) be a generalized topological space and $A \subseteq X$. Then A is said to be $\mu - semi - open$ if $A \subseteq c_\mu(i_\mu(A))$.

The complement of a $\mu - semi - open$ set is called $\mu - semi - closed$.

Definition 1.2. [4] Let (X, μ) and $(Y, \mu^{'})$ be generalized topological spaces. A function $f : (X, \mu) \rightarrow (Y, \mu^{'})$ is said to be $(\mu, \mu^{'}) - continuous$ if $M^{'} \in \mu^{'}$ implies $f^{-1}(M^{'}) \in \mu$.

Definition 1.3. Let (X, μ) and $(Y, \mu^{'})$ be generalized topological spaces. A function $f : (X, \mu) \rightarrow (Y, \mu^{'})$ is said to be $\mu - pre - semi - open$ (resp. $\mu - pre - semi - closed$) if it maps $\mu - semi - open$ (resp. $\mu - semi - closed$) subsets of X onto $\mu - semi - open$ (resp. $\mu - semi - closed$) subsets of Y.

Definition 1.4. [1] Let (X, μ) and $(Y, \mu^{'})$ be generalized topological spaces. A *function* $f : (X, \mu) \rightarrow (Y, \mu^{'})$ is said to be $contra-(\sigma, \mu')-continuous$ if $f^{-1}(V)$ is $\mu - semi - closed$ in X for each $\mu' - open$ set V of Y.

2. $\mu-$ Semi Compact and $\mu-$ Semi Lindelöf Spaces

Definition 2.1. A collection $\{U_\alpha : \alpha \in \Delta\}$ of $\mu - semi - open$ sets in a generalized topological space (X, μ) is called a $\mu - semi - open$ cover of a subset B of X if $B \subseteq \cup\{U_\alpha : \alpha \in \Delta\}$ holds.

Definition 2.2. A generalized topological space (X, μ) is called $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) if every $\mu - semi - open$ cover of X has a finite (resp. countable) subcover.

The proof of the following theorem is straightforward and thus omitted.

Theorem 2.3. *If X is finite (resp. countable) then (X, μ) is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) for any generalized topology μ on X.*

Definition 2.4. A subset B of a generalized topological space (X, μ) is said to be $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to X if, for every collection $\{U_\alpha : \alpha \in \Delta\}$ of $\mu - semi - open$ subsets of X such that $B \subseteq \{U_\alpha : \alpha \in \Delta\}$, there exists a finite subset Δ_0 of Δ such that $B \subseteq \cup\{U_\alpha : \alpha \in \Delta_0\}$.

Notice that if (X, μ) is a generalized topological space and $A \subseteq X$ then $\mu_A = \{U \cap A : U \in \mu\}$ is a generalized topology on A.

(A, μ_A) is called a generalize subspace of (X, μ).

Definition 2.5. A subset B of a generalized topological space (X, μ) is said to be $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) if B is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) as a generalized subspace of X.

The proof of the following theorem is straightforward, and thus omitted.

Theorem 2.6. *The finite (resp. countable) union of subsets of X which are $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to X is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to X.*

Theorem 2.7. *Let A and B be two subsets of a generalized topological space X with $A \subseteq B$. If A is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative X, then A is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to B.*

Proof. We will show the case when A is $\mu-$ semi compact relative to X, the other case is similar. Suppose that $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ is a cover of A by $\mu - semi - open$ sets in B. Then $U_\alpha = S_\alpha \cap B$ for each $\alpha \in \Delta$, where S_α is $\mu - semi - open$ in X for each $\alpha \in \Delta$. Thus $\check{S} = \{S_\alpha : \alpha \in \Delta\}$ is a cover of A by $\mu - semi - open$ sets in X, but A is $\mu-$ semi compact relative X, so there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{S_\alpha : \alpha \in \Delta_0\}$, and thus $A \subseteq \cup\{S_\alpha \cap B : \alpha \in \Delta_0\} = \cup\{U_\alpha : \alpha \in \Delta_0\}$. Hence A is $\mu-$ semi compact relative to B. □

Corollary 2.8. *Let A be a subset of a generalized topological space X. If A is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to X, then A is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf).*

Theorem 2.9. *Let A and B be two subsets of a generalized topological space X with $A \subseteq B$. Then A is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative X if and only if A is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to B.*

Proof. Necessity: Follows from Theorem 2.7.

Sufficiency: We will show the case when A is $\mu-$ semi compact relative to B, the other case is similar. Suppose that $\check{S} = \{S_\alpha : \alpha \in \Delta\}$ is a cover of A by

$\mu - semi - open$ sets in X. Then $\tilde{U} = \{S_\alpha \cap B : \alpha \in \Delta\}$ is a cover of A. Since S_α is $\mu - semi - open$ in X for each $\alpha \in \Delta$, it follows that $S_\alpha \cap B$ is $\mu - semi - open$ in B for each $\alpha \in \Delta$, but A is $\mu-$ semi compact relative B, so there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{S_\alpha \cap B : \alpha \in \Delta_0\} \subseteq \cup\{S_\alpha : \alpha \in \Delta_0\}$. Hence A is $\mu-$ semi compact relative to X. □

Corollary 2.10. *A subset A of a generalized topological space X is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) if and only if A is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to X.*

Theorem 2.11. *If a subset A of X is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to X and B is a $\mu - semi - closed$ subset of X, then $A \cap B$ is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to X. In particular, a $\mu - semi - closed$ subset of a $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) space X is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to X.*

Proof. We will show the case when A is $\mu-$ semi compact relative to X, the other case is similar. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $A \cap B$ by $\mu - semi - open$ subsets of X. Then $\tilde{U}^* = \tilde{U} \cup \{X - B\}$ is a cover of A by $\mu - semi - open$ sets in X, but A is $\mu-$ semi compact relative to X, so there exists a finite subset Δ_0 of Δ such that $A \subseteq (\cup\{U_\alpha : \alpha \in \Delta_0\}) \cup (X - B)$. Thus $A \cap B \subseteq \cup\{U_\alpha \cap B : \alpha \in \Delta_0\} \subseteq \cup\{U_\alpha : \alpha \in \Delta_0\}$. Hence $A \cap B$ is $\mu-$ semi compact relative to X. □

Definition 2.12. A subset F of a space X is called $\mu - semi - F_\sigma$ -set if $F = \cup\{F_i : i = 1, 2, ...\}$ where F_i is a $\mu - semi - closed$ subset of X for each $i = 1, 2, ...$.

Theorem 2.13. *A $\mu - semi - F_\sigma$ -subset F of a $\mu-$ semi Lindelöf space X is $\mu-$ semi Lindelöf relative to X.*

Proof. Let $F = \cup\{F_i : i = 1, 2, ...\}$ where F_i is a $\mu - semi - closed$ subset of X for each $i = 1, 2, ...$ Let $\tilde{U}$ be a cover of F by $\mu - semi - open$ sets in X, then $\tilde{U}$ is a cover of $F_i, i = 1, 2, ...$ by $\mu - semi - open$ subsets of X. Since F_i is $\mu-$ semi Lindelöf relative to X, $\tilde{U}$ has a countable subcover $\tilde{U}_i = \{\tilde{U}_{i_1}, \tilde{U}_{i_2}, ...\}$ for F_i for each $i = 1, 2, ...$ Now $\tilde{U}^* = \cup\{\tilde{U}_i : i = 1, 2, ...\} = \{U_{i_n} : i, n = 1, 2, ...\}$ is a countable subcover of $\tilde{U}$ for F. So F is $\mu-$ semi Lindelöf relative to X. □

Theorem 2.14. *Every generalized subspace of a generalized topological space (X, μ) is $\mu-$ semi Lindelöf relative to X if and only if every $\mu - semi - open$ generalized subspace of X is $\mu-$ semi Lindelöf relative to X.*

Proof. $\Rightarrow$) Is clear.

$\Leftarrow$) Let Y be a generalized subspace of X and let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of Y by $\mu - semi - open$ sets in X. Now, let $V = \cup\tilde{U}$ then V is a $\mu - semi - open$ subset of X, so it is $\mu-$ semi Lindelöf relative to X. But $\tilde{U}$ is a cover of V so $\tilde{U}$ has a countable subcover $\tilde{U}^*$ for V. Then $V \subseteq \cup\tilde{U}^*$ and therefore $Y \subseteq V \subseteq \cup\tilde{U}^*$. So $\tilde{U}^*$ is a countable subcover of $\tilde{U}$ for Y. then Y is $\mu-$ semi Lindelöf relative to X. □

The proofs of the following two theorems are straightforward, and thus omitted.

Theorem 2.15. *A generalized topological space (X, μ) is $\mu-$ semi compact if and only if every $\mu - semi - closed$ family of subsets of X with empty intersection, has a finite subfamily with empty intersection.*

Theorem 2.16. *A generalized topological space (X, μ) is $\mu-$ semi compact if and only if every $\mu - semi - closed$ family of subsets of X having the finite intersection property, has a nonempty intersection.*

Theorem 2.17. *Let $f : (X, \mu) \to (Y, \mu^{'})$ be a $(\mu, \mu^{'}) - continuous$ function. Then, if A is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to X, then $f(A)$ is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to Y.*

Proof. We will show the case when A is $\mu-$ semi compact relative to X, the other case is similar. Suppose that $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ is a cover of $f(A)$ by $\mu - semi - open$ subsets of Y. Then $\tilde{U}^* = \{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is a cover of A by $\mu - semi - open$ subsets of X. Since A is $\mu-$ semi compact relative to X, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup\{f^{-1}(U_\alpha) : \alpha \in \Delta_0\}$. Thus $f(A) \subseteq \cup\{f(f^{-1}(U_\alpha)) : \alpha \in \Delta_0\} \subseteq \cup\{U_\alpha : \alpha \in \Delta_0\}$. Hence $f(A)$ is $\mu-$ semi compact relative to X. □

Theorem 2.18. *Let $f : (X, \mu) \to (Y, \mu^{'})$ be a $\mu - pre - semi - closed$ surjection. If for each $y \in Y$, $f^{-1}(y)$ is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to X, then $f^{-1}(A)$ is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to X whenever A is $\mu-$ semi compact (resp. $\mu-$ semi Lindelöf) relative to Y.*

Proof. We will show the case when A is $\mu-$ semi compact relative to X, the other case is similar. Suppose that $\check{S} = \{S_\alpha : \alpha \in \Delta\}$ is a cover of $f^{-1}(A)$ by $\mu - semi - open$ sets in X. Then it follows by assumption that for each $y \in A$, there exists a finite subset Δ_y of Δ such that $f^{-1}(y) \subseteq \cup\{S_\alpha : \alpha \in \Delta_y\}$. Let $V_y = \cup\{S_\alpha : \alpha \in \Delta_y\}$. Then V_y is $\mu - semi - open$ in X as any union of

$\mu-semi-open$ sets is $\mu-semi-open$. Let $W_y = Y - f(X - V_y)$. Then W_y is $\mu-semi-open$ in Y as f is $\mu-pre-semi-closed$, also $y \in W_y$ for each $y \in A$ as $f^{-1}(y) \subseteq V_y$. Thus, $\hat{W} = \{W_y : y \in A\}$ is a cover of A by $\mu-semi-open$ sets in Y, but A is $\mu-$ semi compact relative to Y, so there exist y_1, y_2, ..., $y_n \in A$ such that $A \subseteq \cup\{W_{y_i} : i = 1, 2, ..., n\}$. Thus, $f^{-1}(A) \subseteq \cup\{f^{-1}(W_{y_i}) : i = 1, 2, ..., n\} \subseteq \cup\{V_{y_i} : i = 1, 2, ..., n\}$. Since $S^i = \{S_\alpha : \alpha \in \Delta_{y_i}\}$ is a finite subcollection of $\check{S}$ for each $i \in \{1, 2, ..., n\}$, it follows that $\cup\{S^i : i = 1, 2, ..., n\}$ is a finite subcollection of $\check{S}$. Hence, $f^{-1}(A)$ is $\mu-$ semi compact relative to X. □

Theorem 2.19. *For a function $f : (X, \mu) \to (Y, \mu^{'})$, the following are equivalent:*

a) f is $contra-(\sigma, \mu')-continuous$.

b) For every $\mu'-$closed subset F of Y, $f^{-1}(F)$ is $\mu-semi-open$ in X.

c) For each $x \in X$ and each $\mu'-$closed subset F of Y with $f(x) \in F$, there exists a $\mu-semi-open$ subset U of X with $x \in U$ such that $f(U) \subseteq F$.

Proof. The implications $(a) \Leftrightarrow (b)$ and $(b) \Rightarrow (c)$ are obvious.

$(c) \Rightarrow (b)$. Let F be any $\mu'-$closed subset of Y. If $x \in f^{-1}(F)$ then $f(x) \in F$ and there exists a $\mu-semi-open$ subset U_x of X with $x \in U_x$ such that $f(U_x) \subseteq F$. Therefore, we obtain $f^{-1}(F) = \cup\{U_x : x \in f^{-1}(F)\}$. Therefore, $f^{-1}(F)$ is $\mu-semi-open$. □

Definition 2.20. A generalized topological space (X, μ) is said to be *strongly* $\mu-closed$ if every $\mu-$closed cover of X has a finite subcover.

Theorem 2.21. *If $f : (X, \mu) \to (Y, \mu^{'})$ is $contra-(\sigma, \mu')-continuous$ and K is $\mu-$ semi compact relative to X, then $f(K)$ is strongly $\mu'-closed$ in Y.*

Proof. Let $\{C_\alpha : \alpha \in \Delta\}$ be any cover of $f(K)$ by $\mu'-$closed subsets of $f(K)$. For each $\alpha \in \Delta$, there exists a $\mu'-$closed set F_α of Y such that $C_\alpha = F_\alpha \cap f(K)$. For each $x \in K$, there exists $\alpha_x \in \Delta$ such that $f(x) \in F_{\alpha_x}$. Now by Theorem , there exists a $\mu-semi-open$ set U_x of X with $x \in U_x$ such that $f(U_x) \subseteq F_{\alpha_x}$. Since the family $\{U_x : x \in K\}$ is a $\mu-semi-open$ cover of K by sets $\mu-semi-open$ in X, there exists a finite subset K_0 of K such that $K \subseteq \cup\{U_x : x \in K_0\}$. Therefore we obtain $f(K) \subseteq \cup\{f(U_x) : x \in K_0\}$ which is a subset of $\cup\{F_{\alpha_x} : x \in K_0\}$. Thus $f(K) \subseteq \cup\{C_{\alpha_x} : x \in K_0\}$ and hence $f(K)$ is *strongly* $\mu-closed$. □

Corollary 2.22. *If $f:(X,\mu)\rightarrow(Y,\mu^{'})$ is contra $-(\sigma,\mu')-$ continuous surjection and X is $\mu-$ semi compact, then Y is strongly $\mu'-$ closed.*

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