**International Journal of Pure and Applied Mathematics**
**Volume 78 No. 4 2012, 577-585**
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

# A SUPPLEMENT TO POINCARÉ'S THEOREM ON DIFFERENCE EQUATIONS

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**Abstract:** The second-order linear difference equation of Poincaré type

$$u(n+2) + (a + \alpha(n))u(n+1) + (b + \beta(n))u(n) = 0, \; n = 0, 1, \dots,$$

with Buslaev's restrictions on coefficients

$$\limsup_{n\to\infty} \sqrt[n]{|\alpha(n)|} \le q < 1, \quad \limsup_{n\to\infty} \sqrt[n]{|\beta(n)|} \le q < 1$$

is considered. It is assumed that the characteristic roots of the equation have the same modulus. The set $\mathcal{A}$ of all accumulation points for the sequence $\{\frac{u(n+1)}{u(n)}\}_{n=0}^{\infty}$ for any nontrivial solutions of the equation is described.

**AMS Subject Classification:** 39A10
**Key Words:** second-order difference equation of Poincaré type, Poincaré's theorem, asymptotic behavior, Padé approximants

## 1. Introduction and Preliminaries

We consider the second-order linear difference equation of Poincaré type

$$u(n+2) + (a + \alpha(n))u(n+1) + (b + \beta(n))u(n) = 0, \; n = 0, 1, \dots, \tag{1}$$

where $a, b \in \mathbb{C}$, $\alpha(n)$, $\beta(n)$ are complex sequences, and $\lim_{n\to\infty} \alpha(n) = 0$, $\lim_{n\to\infty} \beta(n) = 0$. The polynomial $\lambda^2 + a\lambda + b$ is called *the characteristic polynomial* and its roots $\lambda_1, \lambda_2$ are *the characteristic roots* of equation (1).

Received: June 6, 2012

The fundamental result in the asymptotic theory of equation (1) is Poincaré's theorem which states that, if $|\lambda_1| \neq |\lambda_2|$, then for any solution $\{u(n)\}_{n=0}^{\infty}$ of equation (1) either $u(n) = 0$ for all sufficiently large $n$ or there exists $\lim\limits_{n\to\infty} \frac{u(n+1)}{u(n)}$, and this limit is equal to one of the characteristic roots $\lambda_1, \lambda_2$. The theorem is also true for the higher-order equations (see, e.g., [1, Ch.8]). Simple examples show that the statement of the theorem becomes false if $|\lambda_1| = |\lambda_2|$.

The purpose of this paper is to study the asymptotic behavior of the sequence $\{\frac{u(n+1)}{u(n)}\}_{n=0}^{\infty}$ in the case of $|\lambda_1| = |\lambda_2| \neq 0$ and to obtain the description for the set $\mathcal{A}$ of all accumulation points of this sequence under some additional assumptions on $\alpha(n), \beta(n)$.

The solution $\{u(n)\}_{n=0}^{\infty}$ of equation (1) such that $u(n) = 0$ for all sufficiently large $n$ will be called *trivial*. We will further consider only nontrivial solutions. Hence for all $n$ at least one of the numbers $u(n), u(n+1)$ does not vanish. If there exists subsequence $n_k$ such that $u(n_k) = 0$, then we put $0, \infty \in \mathcal{A}$. For example, we have $\mathcal{A} = \{0, \infty\}$ for the solution of the equation $u(n+2) - u(n) = 0$ with initial conditions $u(0) = 0, u(1) = 1$ .

For a nontrivial solution $\{u(n)\}_{n=0}^{\infty}$ let us introduce its generating function $U(z) = \sum_{n=0}^{\infty} u(n)z^n$. It is known that for any nontrivial solution $\limsup\limits_{n\to\infty} \sqrt[n]{|u(n)|}$ is equal to the modulus of one of the characteristic roots [2, 3]. Hence in our case the generating function $U(z)$ for any nontrivial solutions is analytic in the disk $|z| < R_0$, where $R_0 = |\lambda_1|^{-1}$.

We will assume that the following additional restrictions are fulfilled:

$$\limsup_{n\to\infty} \sqrt[n]{|\alpha(n)|} \leq q < 1, \quad \limsup_{n\to\infty} \sqrt[n]{|\beta(n)|} \leq q < 1. \tag{2}$$

Conditions (2) were introduced by V.I. Buslaev and S.F. Buslaeva [4]. It turns out that (2) guarantee that the function $U(z)$ can be meromorphically continued to the disk $|z| < R$, where $R = q^{-1}R_0$, and all poles of $U(z)$ belong to the set $\{\lambda_1^{-1}, \lambda_2^{-1}\}$ [4, Theorem 1, item 2]. Thus, in our case the meromorphic function $U(z)$ have in the disk $|z| < R$ either one pole $\lambda_1^{-1}$ or $\lambda_2^{-1}$, or two poles $\lambda_1^{-1}, \lambda_2^{-1}$.

Statements on the asymptotic behavior of the sequence $\{\frac{u(n+1)}{u(n)}\}_{n=0}^{\infty}$ can be formulated as statements on the asymptotic behavior of poles of the first row of a Padé table for the power series $U(z)$.

Recall a definition of Padé approximants in the Frobenius form.

**Definition 1.** (see Frobenius, [5]) Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$. The rational function $\pi_{n,m}(z) = \frac{P_{n,m}(z)}{Q_{n,m}(z)}$ such that $Q_{n,m}(z) \not\equiv 0, \deg Q_{n,m}(z) \leq m$, $\deg P_{n,m}(z) \leq n$, and

$$f(z)Q_{n,m}(z) - P_{n,m}(z) = R_{n+m+1}z^{n+m+1} + R_{n+m+2}z^{n+m+2} + \cdots, \tag{3}$$

is called the Padé approximant of order $(n,m)$ for a power series $f(z)$.

Let us consider the approximants $\pi_{n,1}(z)$, $n = 0, 1, \dots$ (*the first row of the Padé table*). It is easy to obtain from equation (3) that $Q_{n,1}(z) = f_n - f_{n+1}z$, and $P_{n,1}(z) = f_0 f_n + (f_1 f_n - f_0 f_{n+1})z + \cdots + (f_n f_n - f_{n-1} f_{n+1})z^n$. Thus, a pole $\xi_n$ of $\pi_{n,1}(z)$ coincides with $f_n/f_{n+1}$. If $f_{n+1} = 0$, we put $\xi_n = \infty$. To study the asymptotic behavior of $f_n/f_{n+1}$ we must know the asymptotic behavior of $\xi_n$. A complete description of the set of accumulation points of the sequence $\{\xi_n\}_{n=0}^{\infty}$ for the first row was obtained only for a meromorphic function $f(z)$ having in the disk $|z| < R$ one pole (de Montessus de Ballore's theorem [5], Ch.6) or two poles [6]. For this reason in this paper we consider only a second-order difference equation of Poincaré type.

Our main results are Theorem 1 and Theorem 2. Two illustrative examples are given.

## 2. Main Results

First, we study the case of $\lambda_1 = \lambda_2 \neq 0$.

**Theorem 1.** *Suppose $\lambda_1 = \lambda_2 \neq 0$ and conditions (2) are fulfilled. If $\{u(n)\}_{n=0}^{\infty}$ is any nontrivial solution of equation (1), then there exists*

$$\lim_{n\to\infty} \frac{u(n+1)}{u(n)} = \lambda_1.$$

*Proof.* Let $U(z)$ be the generating function of the solution $\{u(n)\}_{n=0}^{\infty}$. By virtue of (2) $U(z)$ is meromorphic in the disk $|z| < R$.

First, we suppose $U(z)$ has a simple pole (coinciding with $\lambda_1^{-1}$) in this disk. Then for the first row of the Padé table for $U(z)$ de Montessus de Ballore's theorem holds. Hence $u(n+1) \neq 0$ for all sufficiently large $n$ and the monic denominator $\widehat{Q}_{n,1}(z) = z - u(n)/u(n+1)$ of the Padé approximant $\pi_{n,1}(z)$ converge to $z - \lambda_1^{-1}$. This means that exists $\lim\limits_{n\to\infty} \frac{u(n+1)}{u(n)} = \lambda_1$.

Now let $\lambda_1^{-1}$ be the pole of $U(z)$ with multiplicity 2. Thus, the meromorphic function $U(z)$ has in the disk $|z| < R$ two poles and the first row of Padé table for $U(z)$ is the so-called last intermediate row. Therefore we can use the convergence theory for this row developed in [6]. In our case the pole $\lambda_1^{-1}$ is the unique *dominant pole* of $U(z)$ and by Theorem 2.4 [6, p.166, p. 199] the sequence of the normalized denominators $\widehat{Q}_{n,1}(z) = z - u(n)/u(n+1)$ has the limit $z - \lambda_1^{-1}$. □

Note that without assumption (2) the statement of the theorem is false in general. The corresponding example can be found in [7].

The case of $|\lambda_1| = |\lambda_2| \neq 0$, $\lambda_1 \neq \lambda_2$, demonstrates more interesting asymptotic behavior of the sequence $\{\frac{u(n+1)}{u(n)}\}_{n=0}^{\infty}$.

**Theorem 2.** *Suppose $|\lambda_1| = |\lambda_2| \neq 0$, $\lambda_1 \neq \lambda_2$, and conditions (2) are fulfilled. Let $\{u(n)\}_{n=0}^{\infty}$ be any nontrivial solution of equation (1). Denote by $A_k$ the residue of the generating function $U(z)$ at the point $\lambda_k^{-1}$, $k = 1, 2$, and $\rho = |A_2/A_1|$, $0 \leq \rho \leq \infty$. Then the following statements hold:*

1. $\lim\limits_{n\to\infty} \left|\frac{u(n+1)-\lambda_1 u(n)}{u(n+1)-\lambda_2 u(n)}\right|$ *exists and is equal to $\rho$.*

2. *If $\rho = 0$, then* $\lim\limits_{n\to\infty} \frac{u(n+1)}{u(n)} = \lambda_1$.

3. *If $\rho = \infty$, then* $\lim\limits_{n\to\infty} \frac{u(n+1)}{u(n)} = \lambda_2$ .

4. *If $0 < \rho < \infty$, and $\lambda_1/\lambda_2$ is a primitive $\sigma$th root of unity, then*

$$\lim_{n\to\infty} \frac{u(n+1)}{u(n)} = \frac{A_1\lambda_1^{j+2} + A_2\lambda_2^{j+2}}{A_1\lambda_1^{j+1} + A_2\lambda_2^{j+1}}, \quad n \equiv j \pmod{\sigma}.$$

*The set $\mathcal{A}$ consists of $\sigma$ accumulation points $\alpha_j := \frac{A_1\lambda_1^{j+2}+A_2\lambda_2^{j+2}}{A_1\lambda_1^{j+1}+A_2\lambda_2^{j+1}}$, $j = 0, 1, \dots, \sigma-1$, lying on the Apollonian circle $\left|\frac{z-\lambda_1}{z-\lambda_2}\right| = \rho$.*

*The point $\infty$ belongs to $\mathcal{A}$ if the condition $A_1\lambda_1^{j_0+1} + A_2\lambda_2^{j_0+1} = 0$ is fulfilled for some $j_0 \in \{0, \dots, \sigma-1\}$. In this case $\rho = 1$ (the Apollonian circle is a straight line), and $0 \in \mathcal{A}$.*

5. *If $0 < \rho < \infty$, and $\lambda_1/\lambda_2$ is not a root of unity, then $\mathcal{A}$ coincides with the Apollonian circle $\left|\frac{z-\lambda_1}{z-\lambda_2}\right| = \rho$.*

*Proof.* Evidently, $\rho = 0$ if and only if $U(z)$ has one pole $\lambda_1^{-1}$ in the disk $|z| < R$. As in Theorem 1 by virtue of de Montessus de Ballore's theorem we have $\lim\limits_{n\to\infty} \frac{u(n+1)}{u(n)} = \lambda_1$. Then $\lim\limits_{n\to\infty} \left|\frac{u(n+1)-\lambda_1 u(n)}{u(n+1)-\lambda_2 u(n)}\right| = 0$. The case of $\rho = \infty$ can be considered similarly. Thus, for $\rho = 0, \infty$ the theorem is true.

Let now $0 < \rho < \infty$. Then $U(z)$ has in the disk $|z| < R$ two simple poles $z_1 = \lambda_1^{-1}$, $z_1 = \lambda_2^{-1}$, lying on the circle $|z| = R_0$, the first row of Padé table for $U(z)$ is the last intermediate row, and $z_1, z_2$ are the dominant poles. Hence, we can use Theorem 2.2, Theorem 2.5, and Theorem 2.7 from [6].

Let us consider the first case, when $z_2/z_1 = \lambda_1/\lambda_2$ is a primitive $\sigma$th root of unity, i.e. $z_1$, $z_2$ are vertices of a regular $\sigma$-gon with the lowest possible $\sigma$.

Define the numbers $C_k = (z_2 - z_1)^{-2} A_k^{-1}$, $k = 1, 2$, by formula 4.14 from [6]. Suppose that $S_0(j) := C_1 z_1^{j+2} + C_2 z_2^{j+2} \neq 0$ for given $j \in \{0, 1, \dots, \sigma - 1\}$.

Then, by Theorem 2.2 and Theorem 2.7 from [6], we have $u(n+1) \neq 0$ for all sufficiently large $n \equiv j \pmod{\sigma}$ , the denominator $Q_{n,1}(z) = u(n) - u(n+1)z$ can be 1-normalized, and the 1-normalized denominator $\hat{Q}(z) = z - u(n)/u(n+1)$ tends to the polynomial $\hat{\omega}_j(z) = \frac{C_1(z-z_2)z_1^{j+2} + C_2(z-z_1)z_2^{j+2}}{C_1 z_1^{j+2} + C_2 z_2^{j+2}}$ as $n \to \infty$ in the residue class $n \equiv j \pmod{\sigma}$. This means that there exists a finite limit

$$\lim_{n\to\infty} \frac{u(n)}{u(n+1)} = \frac{C_1 z_2 z_1^{j+2} + C_2 z_1 z_2^{j+2}}{C_1 z_1^{j+2} + C_2 z_2^{j+2}}$$
$$= \frac{A_1 \lambda_1^{j+1} + A_2 \lambda_2^{j+1}}{A_1 \lambda_1^{j+2} + A_2 \lambda_2^{j+2}}, \quad n \equiv j \pmod{\sigma}.$$

If $A_1 \lambda_1^{j+1} + A_2 \lambda_2^{j+1} \neq 0$, then there exists a finite limit

$$\lim_{n\to\infty} \frac{u(n+1)}{u(n)} = \frac{A_1 \lambda_1^{j+2} + A_2 \lambda_2^{j+2}}{A_1 \lambda_1^{j+1} + A_2 \lambda_2^{j+1}}.$$

If $A_1 \lambda_1^{j_0+1} + A_2 \lambda_2^{j_0+1} = 0$ for some (unique) $j_0 \in \{0, 1, \dots, \sigma - 1\}$, then $\lim\limits_{n\to\infty} \frac{u(n+1)}{u(n)} = \infty$, $n \equiv j_0 \pmod{\sigma}$, and $\lim\limits_{n\to\infty} \frac{u(n+1)}{u(n)} = 0$, $n \equiv j_0 - 1 \pmod{\sigma}$.

It is not difficult to verify that the finite accumulation points $\alpha_j$ lie on the Apollonian circle. It follows from this that in the first case the statement of item 1 is true also.

The statement of item 5 is an immediate consequence of Theorem 2.5 from [6]. □

Let us illustrate Theorem 2 by the following examples. All calculations were made with the help of Maple.

**Example 1.** Consider the solution of the equation

$$u(n+2) - (2\sqrt{3} + n/3^n)u(n+1) + (4 + 1/2^n)u(n) = 0$$

with the initial values $u(0) = 1$, $u(1) = 1 - i$. Here $\lambda_1 = \sqrt{3} - i, \lambda_2 = \sqrt{3} + i$, $\lambda_1/\lambda_2$ is a primitive $\sigma$th root of unity, $\sigma = 6$. Conditions (2) are fulfilled with $q = 1/2$.

We can use the partial fractions of the Padé approximants $\pi_{100,2}(z)$ for the series $U(z)$ to find approximately the residue $A_k$ of the continuation of $U(z)$ at the point $\lambda_k^{-1}$. In this way we obtain the following value

$$A_1 = -0.7402 - 0.1784i, \ A_2 = -0.3614 - 0.2275i, \ \rho = 0.5609.$$

Then we can calculate the accumulation points according to the formula $\alpha_j = \frac{A_1\lambda_1^{j+2}+A_2\lambda_2^{j+2}}{A_1\lambda_1^{j+1}+A_2\lambda_2^{j+1}}$, $j = 0, 1, \ldots, 5$, from item 5 of Theorem 2:

$$\alpha_0 = 1.0157 - 0.4463i, \quad \alpha_1 = 0.1634 - 1.4503i,$$
$$\alpha_2 = 3.1571 - 2.7233i, \quad \alpha_3 = 2.7376 - 0.6266i,$$
$$\alpha_4 = 2.0757 - 0.3177i, \quad \alpha_5 = 1.5812 - 0.2882i.$$

These values coincide with $u(6k + j + 1)/u(6k + j)$, $j = 0, 1, \ldots, 5$, even for $k = 39$. As shown in Figure 1, all accumulation points $\alpha_j$ lie on the Apollonian circle $\left|\frac{z-\lambda_1}{z-\lambda_2}\right| = 0.5609$.

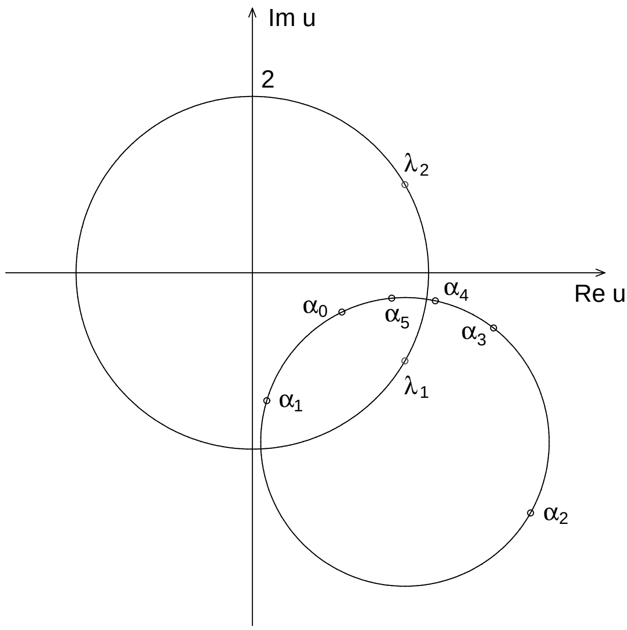


Figure 1: Example 1. Accumulation points of $\{u(n+1)/u(n)\}$ if $\lambda_1/\lambda_2$ is a primitive 6th root of unity.

**Example 2.** For the equation

$$u(n + 2) = (1 + e^{2\pi i\sqrt{2}} + 1/2^n)u(n + 1) + (1/3^n - e^{2\pi i\sqrt{2}})u(n)$$

we have $\lambda_1 = 1$, $\lambda_2 = e^{2\pi i\sqrt{2}}$, and $\lambda_1/\lambda_2$ is not a root of unity. Hence the accumulation points of the sequence $\{\frac{u(n+1)}{u(n)}\}_{n=0}^{\infty}$ for any solution of the equation must fill out the whole corresponding Apollonian circle.

In order to calculate $\rho$ let us use the formula from item 1 of Theorem 2. The calculation show that the sequence $\rho_n := \left|\frac{u(n+1)-\lambda_1 u(n)}{u(n+1)-\lambda_2 u(n)}\right|$ is stabilized for $n \geq 200$. We put $\rho \approx \rho_{200}$ and obtain the following result for some initial values:

$$
\begin{array}{lll}
u(0) = 1, & u(1) = i, & \rho = 0.2298, \\
u(0) = 1 + i, & u(1) = -2, & \rho = 0.5088, \\
u(0) = -2.0 + 1.i, & u(1) = 1.0 - 1.i, & \rho = 0.8808, \\
u(0) = -1.05 + 1.i, & u(1) = 1 - i, & \rho = 1.1468, \\
u(0) = -1.35 + 1.i, & u(1) = 1. - 1.1i, & \rho = 2.0710, \\
u(0) = -1.3 + 1.i, & u(1) = 1. - 1.3i, & \rho = 4.7284.
\end{array}
$$

As shown in Figure 2 the points $\frac{u(n+1)}{u(n)}$, even for $n = 50, \ldots, 200$, lie on the corresponding Apollonian circle and they sufficiently densely fill out this circle.

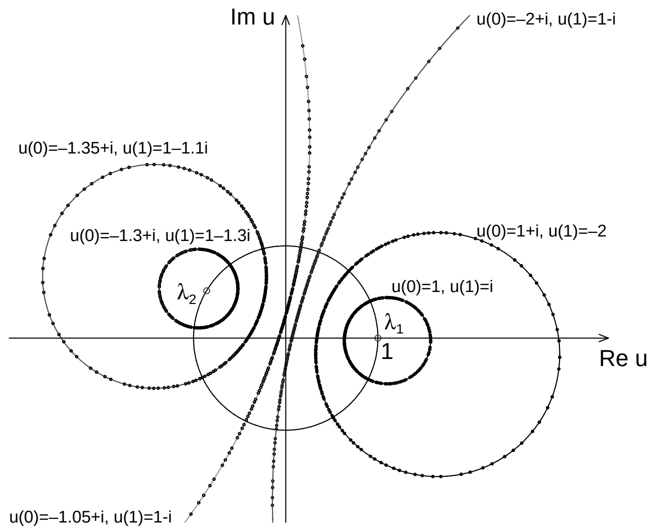


Figure 2: Example 2. The points $\frac{u(n+1)}{u(n)}$, $n = 50, \ldots, 200$, for some initial conditions, when $\lambda_1/\lambda_2$ is not a root of unity.

## 3. Conclusion

Under Buslaev's condition (2) the asymptotic behavior of the sequence

$$\left\{\frac{u(n+1)}{u(n)}\right\}_{n=0}^{\infty}$$

for any nontrivial solution $\{u(n)\}_{n=0}^{\infty}$ of equation (1) is completely determined by the residue $A_1$, $A_2$ of the generating function $U(z)$ at the points $\lambda_1^{-1}$, $\lambda_1^{-1}$, and the arithmetic nature of $\lambda_1/\lambda_2$. All accumulation points of this sequence lie on the Apollonian circle $\left|\frac{z-\lambda_1}{z-\lambda_2}\right| = |A_2/A_1|$. Numerical experiments show that probably the similar asymptotic behavior holds without restrictions (2).

Unfortunately, a dependence of $A_1$, $A_2$ (or $\rho = |A_2/A_1|$) on the initial values $u(0)$, $u(1)$ still remains unknown; we can calculate $A_1$, $A_2$ only approximately. If $b + \beta(n) = 0$ for some $n$, then it is not difficult to prove that $\rho$ equals the unique value for any $u(0)$, $u(1)$.

**Open Problem.** In what cases will the function $\rho(u(0), u(1))$ take all values from $[0, \infty]$ or at least take values 0, $\infty$?

## Acknowledgment

I would like to thank Professor M.M. Kipnis for numerous comments during the preparation of this paper.

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