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OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SECOND ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH "MAXIMA"

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Abstract: The authors establish some new criteria for the oscillation and asymptotic behavior of all solutions of the equation.

$$(a(t)(x(t)+p(t)x(\tau(t)))')' + q(t)\max_{[\sigma(t),t]} x^{\alpha}(s) = 0, \quad t \geq t_0 \geq 0,$$

where $a(t) > 0$, $q(t) \geq 0$, $\tau(t) \leq t$, $\sigma(t) \leq t$, α is the ratio of odd positive integers, and $\int_0^{\infty} \frac{dt}{a(t)} < \infty$. Examples are included to illustrate the results.

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1. Introduction

Consider the second order neutral delay differential equations with "maxima" of the form

$$(a(t)(x(t)+p(t)x(\tau(t)))')' + q(t)\max_{[\sigma(t),t]} x^{\alpha}(s) = 0, t \geq t_0 \geq 0, \tag{1.1}$$

subject to the following conditions:

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(C1) $\tau(t)$ and $\sigma(t)$ are continuous functions with $\tau(t) \leq t$, $\sigma(t) \leq t$ and α is the ratio of odd positive integers;

(C2) $p(t) \in C([t_0, \infty), R)$ with $0 \leq p(t) \leq p < 1$, and $q(t) \in C([t_0, \infty), R_+)$;

(C3) $a(t) \in C([t_0, \infty), (0, \infty))$ and $\int_0^\infty \frac{dt}{a(t)} < \infty$.

By a solution of equation (1.1) we mean a continuous function, x on $[t_0, \infty)$ such that $x(t) + p(t)x(\tau(t))$ is continuously differentiable and x satisfies equations (1.1). A nontrivial , a solution of equation (1.1) is said to be oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory. A solution $x(t)$ of equation (1.1) is said to be almost oscillation if $x(t)$ is either oscillatory or $|x(t)| \to 0$ monotonically as $t \to \infty$.

The qualitative theory of differential equations with "maxima" received very little attention even though such equations often arise in the problem of automatic regulation of various real systems, see for example [1, 7, 9]. The oscillatory behavior of differential equations with maxima are discussed in [1-6, 8, 10], and the references cited therein. In [2],the authors studied the oscillatory behavior of solutions of equation (1.1) when $\alpha = 1$ and $a(t) \equiv 1$. Motivated by this observation, in this paper, we investigate the oscillatory and asymptotic behavior of solutions of equation (1.1) under the condition

$$\int_{t_0}^\infty \frac{dt}{a(t)} < \infty.$$

In Section 2, we establish sufficient conditions for the almost oscillation of all solutions of equation (1.1). In Section 3, we present sufficient conditions for the existence of nonoscillatory solutions for the equation (1.1) using contraction mapping principle. In Section 4, we present some examples to illustrate the main results.

2. Oscillation Results

In this section we present some new sufficient conditions for the almost oscillation of equation (1.1).

Define

$$z(t) = x(t) + p(t)x(\tau(t)),$$

and

$$A(t) = \int_t^\infty \frac{ds}{a(s)}.$$

Lemma 2.1. *Let $x(t)$ be an eventually positive solution of equation* (1.1)*. Then one of the following holds*

(I) $z(t) > 0, \quad z'(t) > 0, \quad and \quad (a(t)z'(t))' \leq 0;$

(II) $z(t) > 0, \quad z'(t) < 0, \quad and \quad (a(t)z'(t))' \leq 0.$

Proof. Let $x(t)$ be an eventually positive solutions of equation (1.1). Then we may assume that $x(\sigma(t)) > 0, x(\tau(t) > 0 \; for \; all \; t \geq T$. Then inview of (C2), we have $z(t) > 0$ for $t \geq T$. From the equation (1.1) we obtain $(a(t)z'(t))' = -q(t) \max_{[\sigma(t),t]} x^{\alpha}(s) \leq 0,$. Hence $a(t)z'(t)$ and $z(t)$ are of eventually of one sign. This completes the proof. □

Lemma 2.2. *Let $x(t)$ be an eventually negative solution of equation* (1.1)*. Then one of the following holds*

(I) $z(t) < 0, \quad z'(t) < 0, \quad and \quad (a(t)z'(t))' \geq 0;$

(II) $z(t) < 0, \quad z'(t) > 0, \quad and \quad (a(t)z'(t))' \geq 0.$

The proof of Lemma 2.2 is analogous to that of Lemma 2.1.

Lemma 2.3. *The function $x(t)$ is a negative solution of* (1.1) *if and only if $-x(t)$ is a positive solution of the equation*

$$(a(t)(y(t) + p(t)y(\tau(t)))')' + q(t) \min_{[\sigma(t),t]} x^{\alpha}(s) = 0.$$

The assertion of Lemma 2.3 is verified easily.

Lemma 2.4. *Let $x(t)$ be a positive solution of equation* (1.1) *and let the corresponding $z(t)$ satisfy Lemma 2.1 (II). If*

$$\int_{t_0}^{\infty} q(t)A(t)dt = \infty \tag{2.1}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0.$

Proof. Let $x(t)$ be a positive solution of equation (1.1). Then by Lemma 2.1 (II) we have

$$z(t) > 0 \text{ and } z'(t) < 0 \text{ for all } t \geq T.$$

Therefore $z(t) \to L \geq 0$ as $t \to \infty$. If $L > 0$, then for $\epsilon = \frac{L(1-p)}{2p} > 0$; there exists $T \geq t_0$ such that $L < z(t) < L + \epsilon$ for $t \geq T$. Then for $t \geq T$, we have

$$x(t) = z(t) - p(t)x(\tau(t)) > L - p(t)z(t) > L - p(t)(L+\epsilon) \geq L - p(L+\epsilon) = L_1.$$

From equation (1.1), we have

$$(a(t)z'(t))' \leq -q(t) \max_{[\sigma(t),t]} x^{\alpha}(s).$$

Integrating from T to ∞ and using the fact that $a(t)z'(t)$ is positive and decreasing we obtain

$$a(t)z'(t) \geq \int_t^{\infty} q(s) \max_{[\sigma(s),s]} x^{\alpha}(s)ds \geq L_1^{\alpha} \int_t^{\infty} q(s)ds.$$

Divide the last inequality by $a(t)$ and then integrating the resulting inequality we obtain

$$z(t) - z(T) \geq L_1^{\alpha} \int_T^t \frac{1}{a(s)} \Big(\int_s^{\infty} q(u)du \Big) ds$$

or

$$\infty > z(t) \geq L_1^{\alpha} \int_T^{\infty} q(s)A(t)ds$$

which is a contradiction to (2.1) and shows that $L = 0$, that is, $z(t) \to 0$. Since $z(t) > x(t) > 0$ we have $x(t) \to 0$ as $t \to \infty$. □

Theorem 2.5. *Let $\alpha = 1$ in equation (1.1).If (2.1) and*

$$\int_{t_0}^{\infty} q(t) \max_{[\sigma(t),t]} (1 - p(s+\tau))dt = \infty \tag{2.2}$$

hold then every solution of equation (1.1) is almost oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Then either $x(t) > 0$ eventually or $x(t) < 0$ eventually. We shall consider the case when $x(t) > 0$, since the other case can be investigated analogously.

Let $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$ for all $t \geq T$, where T is chosen so large enough that the conclusions of Lemma 2.1 hold for all $t \geq T$.

First we assume Lemma 2.1 (I) holds. Then $z(t) = x(t) + p(t)x(\tau(t))$ implies that

$$x(t) \geq (1 - p(t))z(t) \tag{2.3}$$

and

$$\max_{[\sigma(t),t]} x(s) \geq \max_{[\sigma(t),t]} (1 - p(s))z(s) = z(t) \max_{[\sigma(t),t]} (1 - p(s)). \tag{2.4}$$

From equation (1.1) and (2.4) we have

$$(a(t)z'(t))' + q(t)z(t)\max_{[\sigma(t),t]}(1-p(s)) \le 0, \quad t \ge T. \tag{2.5}$$

Define

$$w(t) = \frac{a(t)z'(t)}{z(t)}, \ t \ge T. \tag{2.6}$$

Then from (2.5) and (2.6) we have

$$w'(t) \le -q(t)\max_{[\sigma(t),t]}(1-p(s)) - \frac{a(t)(z'(t))^2}{z^2(t)} \le -q(t)\max_{[\sigma(t),t]}(1-p(s)).$$

Integrating the last inequality, we obtain

$$\int_T^t q(s)\max_{[\sigma(s),s]}(1-p(v)) \le w(T) - w(t) \le w(T). \tag{2.7}$$

Letting $t \to \infty$ in (2.7) we obtain a contradiction to (2.2).

Next assume that Lemma 2.1 (II) holds. Then by Lemma 2.4 we obtain that $\lim\limits_{t-\infty} x(t) = 0$. This completes the proof. □

Theorem 2.6. *Assume that* $\alpha > 1$. *If* (2.1) *and*

$$\int_{t_0}^{\infty} q(t)A^{\alpha}(t)\max_{[\sigma(t),t]}(1-p(s))^{\alpha}dt = \infty \tag{2.8}$$

hold then every solution of equation (1.1) *is almost oscillatory.*

Proof. Assume that there exists a nonoscillatory solution $x(t)$ of equation (1.1) such that $x(\sigma(t)) > 0$, $x(\tau(t))$ $for\ t \le T\ where\ T$ is chosen large enough that the conclusion of Lemma 2.1 hold for all $t \ge T$.

Integrating (1.1) from $T\ to\ t$ yields

$$a(t)z'(t) - a(T)z'(T) + \int_T^t q(s)\max_{[\sigma(s),s]} x^{\alpha}(u)ds = 0. \tag{2.9}$$

Letting $t \to \infty$, we have

$$\int_T^{\infty} q(s)\max_{[\sigma(s),s]} x^{\alpha}(u)ds < \infty. \tag{2.10}$$

In this case, $z(t)$ is increasing, so there exists a positive number c such that $z(t) > c$ for $t \geq T$. This, together with (2.3) yields, $x^\alpha(t) \geq c^\alpha(1 - p(t))^\alpha$ for $t \geq T$. Now $A^\alpha(t) \to 0$ as $t \to \infty$ so there exists $T_1 \geq T$ such that

$$x^\alpha(t) \geq c^\alpha(1 - p(t))^\alpha A^\alpha(t)$$

for $t \geq T_1$. Then

$$\begin{aligned}\max_{[\sigma(t),t]} x^\alpha(s) &\geq c^\alpha \max_{[\sigma(t),t]}(1 - p(s))^\alpha \max_{[\sigma(t),t]} A^\alpha(s) \\ &\geq c^\alpha \max_{[\sigma(t),t]}(1 - p(s))^\alpha A^\alpha(t) \quad (2.11)\end{aligned}$$

for $t \geq T_1$. Combining (2.10) and (2.11) we have

$$c^\alpha \int_T^\infty q(s)A^\alpha(s) \max_{[\sigma(s),s]}(1 - p(u))^\alpha ds < \infty \quad (2.12)$$

which contradicts (2.10).

Next assume that Lemma 2.1 (II) holds. Then by Lemma 2.4 we obtain that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Theorem 2.7. *Assume that* $0 < \alpha < 1$. *If* (2.1) *and*

$$\int_{t_0}^\infty q(t)A(\sigma(t))dt = \infty \quad (2.13)$$

then every solution of equation (1.1) *is almost oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.2, we have that Lemma 2.1 holds. For case (I), we have (2.10) and (2.11). For large t, we have $A(t) \leq 1$ and $A^\alpha(t) \geq A(t)$, so (2.12) implies

$$\int_T^\infty \max_{[\sigma(t),t]}(1 - p(s))^\alpha A(s)ds < \infty.$$

This contradicts (2.11). Next assume that Lemma 2.1 (II) holds. Then by Lemma 2.4 we obtain that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

3. Existence of Nonoscillatory Solutions

In this section, we provide sufficient conditions for the existence of nonoscillatory solutions of equation (1.1) in case $\alpha > 1$ or $0 < \alpha < 1$. Note that in this section we do not require $p(t) \equiv p$.

Theorem 3.1. *Assume that* $\alpha > 1$. *If*

$$\int_{t_0}^{\infty} q(t)A^{\alpha}(t-\sigma)dt < \infty \tag{3.1}$$

and

$$\lim_{t\to\infty} \frac{A(t-\tau)}{A(t)} = 1 \tag{3.2}$$

then equation (1.1) *has a bounded nonoscillatory solution.*

Proof. Choose $T \geq t_0$ sufficiently large so that

$$(p(t)\frac{A(t-\tau)}{A(t)}) \leq \frac{3p+1}{4}, \tag{3.3}$$

and

$$\int_{T}^{\infty} q(t) \max_{[t-\sigma,t]} A^{\alpha}(s)dt \leq \frac{1-p}{8\alpha} \tag{3.4}$$

for $t \geq T$. Let χ be the set of all bounded continuous function on $[t_0, \infty)$ with norm

$$||x|| = \max_{t\geq t_0}\{\frac{\chi(t)}{A(t)}\},$$

and let

$$S = \{x \in \chi : \frac{3(1-p)}{8} \leq x(t) \leq 1, t \geq t_0\}.$$

Define a mapping $T : S \to \chi$ by

$$(Tx)(t) = \begin{cases} \frac{3p+5}{8}A(t) - p(t)x(t-\tau) + A(t)\int_{T}^{t} q(s) \max_{[s-\sigma,s]} x^{\alpha}(v)ds \\ \qquad + \int_{t}^{\infty} q(s)A(s) \max_{[s-\sigma,s]} x^{\alpha}(v)ds, \qquad t \geq T, \\ (Tx)(T), \qquad\qquad t_0 \leq t < T. \end{cases}$$

Clearly, T is continuous. Now for every $x \in S$ and $t \geq T$, (3.3) implies

$$Tx(t) \geq A(t)\frac{3p+5}{8} - p(t)x(t-\tau) \geq A(t)\Big(\frac{3p+5}{8} - p(t)\frac{A(t-\tau)}{A(t)}\Big)$$
$$\geq \frac{3(1-p)}{8}A(t).$$

Also, (3.4), we have

$$(Tx)(t) \leq A(t)\frac{3p+5}{8} + A(t)\int_T^\infty q(s)\max_{[s-\sigma,s]} x^\alpha(u)ds$$
$$+ \int_t^\infty q(s)A(s)\max_{[s-\sigma,s]} x^\alpha(u)ds$$
$$\leq A(t)\Big(\frac{3p+5}{8} + \int_T^\infty q(s)\max_{[s-\sigma,s]} A^\alpha(u)ds\Big)$$
$$\leq A(t)\Big(\frac{3p+5}{8} + \frac{(1-p)}{8\alpha}\Big) < A(t).$$

Thus, we have that $TS \subset S$. Since S is bounded, closed and convex subset of χ, we only need to show that T is contraction mapping on S in order to apply the contraction mapping principle. For $x, y \in S$ *and* $t \geq T$, we have

$$\tfrac{1}{A(t)}|(Tx)(t) - (Ty)(t)|$$
$$\leq \tfrac{p(t)}{A(t)}\max_{[s-\sigma,s]}|x(t-\tau+\sigma) - y(t-\tau+\sigma)|$$
$$+ \textstyle\int_T^t q(s)\max_{[s-\sigma,s]}|x^\alpha(u) - y^\alpha(u)|ds$$
$$+ \tfrac{1}{A(t)}\textstyle\int_t^\infty q(s)A(s)\max_{[s-\sigma,s]}|x^\alpha(u) - y^\alpha(u)|ds$$
$$\leq \tfrac{p(t)A(t-\tau)}{A(t)}\max_{[t-\sigma,t]}\left|\tfrac{x(s-\tau)}{A(s-\tau)} - \tfrac{y(s-\tau)}{A(s-\tau)}\right|$$
$$+ \textstyle\int_T^\infty q(s)A^\alpha(s-\sigma)\max_{[v-\sigma,\sigma]}\left|\Big(\tfrac{x(v-\sigma)}{A(v-\sigma)}\Big)^\alpha - \Big(\tfrac{y(v-\sigma)}{A(v-\sigma)}\Big)^\alpha\right|ds.$$

By the Mean Value Theorem for derivatives applied to the function $f(u) = u^\alpha, \alpha > 1$, we see that for any $x, y \in S$, we have $|x^\alpha - y^\alpha| \leq 2\alpha|x-y|$ for all $t \geq T$. Hence,

$$||Tx - Ty|| \leq \frac{3p+1}{4}||x-y|| + 2\alpha\int_T^\infty q(s)A^\alpha(s-\sigma)||x-y||ds$$

$$\leq \Big(\frac{3p+1}{4} + \frac{1-p}{4}\Big)||x-y|| < ||x-y||.$$

Thus, T is a contraction mapping, so T has a unique fixed point x, that is clearly a positive solution of equation (1.1). This completes the proof of the theorem. □

Theorem 3.2. *Assume that* $0 < \alpha < 1$. *If*

$$\int_0^\infty q(t)A(t)dt < \infty. \tag{3.5}$$

Then equation (1.1) *has a bounded nonoscillatory solution.*

Proof. Choose $T \geq t_0$ sufficiently large so that

$$\int_T^\infty q(t)A(t)dt \leq \frac{(1-p)^2}{8}.$$

Let χ be the set of all bounded continuous function on $[t_0, \infty)$ with norm

$$||x|| = \max_{t > t_0}\{x(t)\},$$

and let

$$S = \{x \in \chi : \frac{3(1-p)}{8} \leq x(t) \leq 1, t \geq t_0\}.$$

Define the operator $T : S \to \chi$ by

$$(Tx)(t) = \begin{cases} \frac{5p+3}{8} - p(t)x(t-\tau) + A(t)\int_T^t q(s)\max_{[s-\sigma,s]} x^\alpha(v)ds \\ \qquad + \int_t^\infty q(s)A(s)\max_{[s-\sigma,s]} x^\alpha(v)ds, & t \geq T, \\ (Tx)(T), & t_0 \leq t < T. \end{cases}$$

It is easy to see that T is continuous, $TS \subset S$, and for $x, y \in S$ and $t \geq T$, we have

$$|(Tx)(t) - (Ty)(t)| \leq p(t)\max_{[t-\sigma,t]}|x(s-\tau) - y(s-\tau)|$$
$$+ \int_T^\infty q(s)A(s)\max_{[s-\sigma,s]}|x^\alpha(v) - y^\alpha(v)|ds.$$

By Mean Value Theorem for derivatives applied to the function $v(u) = u^\alpha, 0 < \alpha < 1$, we see that for any $x, y \in S$, we have $|x^\alpha - y^\alpha| \leq \frac{8\alpha}{3(1-p)}|x-y|$.Thus,

$$||Tx - Ty|| \leq ||x-y||\Big(p + \frac{8\alpha}{3(1-p)}\frac{(1-p)^2}{8}\Big) < ||x-y||,$$

and we see that T is a contraction on S. Hence, T has a unique fixed point which is clearly a positive solution of equation (1.1). This completes the proof of the theorem. □

4. Examples

In this section we present some examples to illustrate the main results.

Example 4.1. Consider the differential equation

$$(e^{2t}(x(t)+\frac{1}{2}x(\tau-1))')'+\frac{e^{2t}(2+e)}{2e}\max_{[t-1,t]}x(s)=0. \tag{4.1}$$

One can easily verify that all conditions of Theorem 2.5 are satisfies and hence every solution of equation (1.1) is almost oscillatory. Infact $x(t)=e^{-t}$ is one such solution of equation (4.1).

Example 4.2. Consider the differential equations

$$(t^2(x(t)+\frac{1}{3}x(t-1))')'+t^3\max_{[t-2,t]}x^3(s)=0, \quad t\geq 1. \tag{4.2}$$

We can easily check that all condition of Theorem 2.6 are satisfying and hence every solution of equation (1.1) is almost oscillatory.

Example 4.3. Consider the differential equations

$$(t^2(x(t)+\frac{1}{3}x(t-2))')'+t^2\max_{[t-3,t]}x^{\frac{1}{3}}(s)=0, \quad t\geq 1. \tag{4.3}$$

It is easy to prove that all condition of Theorem 2.7 are satisfies and hence every solution of equation (1.1) is almost oscillatory.

Example 4.4. Consider the differential equations

$$(t^2(x(t)+\frac{1}{2}x(t-1))')'+t^4\max_{[t-1,t]}x^3(s)=0. \tag{4.4}$$

It is easily verified that all the conditions of Theorem 3.1 are satisfies and hence equation (4.4) has a bounded nonoscillatory solution.

Example 4.5. Consider the differential equations

$$(t^2(x(t)+\frac{1}{3}x(t-1))')'+\frac{1}{t}\max_{[t-1,t]}x^{\frac{1}{3}}(s)=0. \tag{4.5}$$

It is easy verify that all conditions of Theorem 3.2 are satisfies and hence equation (1.1) has a bounded nonoscillatory solution.

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