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OPTIMIZING THE ENERGY OF COLLECTIVE INTELLIGENCE USING NONLINEAR APPROACH

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Abstract: For a fixed initial discrete positive measure, we define the corresponding global energy, which we call it collective intelligence energy. Since this energy map is not a continuous function of measure, we introduce an intensity function to extend it to a continuous mass operator. Moreover, by using nonlinear programming methods, we present some interesting results on maximizing such self-organizing behavior of the collective. We also show that the support structure of the measure -collection of individuals- influences the energy of collective intelligence. Therefore, characterizing the distribution of this agents based model in form of active autonomous individuals is worth to be analyzed.

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Key Words: energy of collective intelligence, self-organizing agents, emergence, measure metric space, nonlinear optimization

1. Introduction

The mathematics of emergence is a recent topic of research in the modeling of population dynamics [6, 7]. Many scientists take this challenge and analyzed these phenomena in many contexts such as flocking, schooling, swarming... see for instance [1, 2, 3, 4, 8]. They have described the collective moves in groups

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by many continuous and discrete models in form of non linear discrete systems or differential equations. The model presented here gives a new idea of rules of self-organizing groups based on the local and global energies of autonomous population agents.

Ants live in large populations. These population show a complicated and strict division of labor for an individual ant, which on the one hand is not determined by the genetic structure of the single ant, and on the other hand makes the whole population react effectively to all kinds of events as if steered by some clever and experienced brain, which however does not exist. The division of labor, which makes an ant become a soldier, and another one a messenger is called emergent. It is very strictly and very stable, but one does not detect it as a program in the individual. How is this to be understood? There is an emergent pattern with local collaborated works. These works are in general benefits for the whole individuals. For more details in the same context, we refer to [9, 11, 8].

In this paper, we try to analyze another context of the model proposed in [16], where the author presents a minimizing conditions for some cases in metric spaces (X, d). Namely, we purpose a nonlinear optimization approach to maximize the energy of the collective intelligence of a population.

Let $M_+(I\!R^n)$ be the set of nonnegative Randon measures on $X = I\!R^n$. We shall however for simplicity, consider in this paper only the case of discrete measures. A measure function is given as $m := \sum_{x\in S(m)} m(x)\delta_x$, where $S(m)$ denotes the discrete support of m and δ_x the Kronecker symbol. For such a function, define the energy map E as

$$E \ : \ M_+(I\!R^n) \to I\!R^+; \quad E(m) = \sum_{d(x,y)\leq\varepsilon} m(x)m(y)d^2(x,y), \tag{1}$$

where $d(\cdot,\cdot)$ is any differentiable metric on $I\!R^n$. It is important to note that E is in general a not continuous function of m.

Example 1. Set for example $X = I\!R$, $S(m) = \{1, 2\}$, $\varepsilon = 1$ and given m with $m = \sum_{x\in\{1,2\}} m(x)\delta_x = \delta_1 + \delta_2$, it follows that $E(m) = 2$. Now let m_j be a sequence of positive measures defined as $m_j = \delta_1 + \delta_{2+\frac{1}{j}}$, it follows $\lim_j m_j = m$, $E(m_j) = 0$, and $E(\lim_j m_j) = 2 \neq \lim_j E(m_j) = 0$. Hence, the map E with definition (1.1) is not continuous in m.

In order to obtain an energy function which depends continuously on m, we extend the definition (1) into the following:

$$E(m) = \int_X \int_X \varphi(x,y) d^2(x,y) m(dx) m(dy) \tag{2}$$
$$= \sum_{(x,y)\in S^2(m)} m(x)m(y)\varphi(x,y)d^2(x,y), \tag{3}$$

where $\varphi : X \times X \longrightarrow [0,1]$ is a continuous function, which satisfies :

$$\varphi(x,y) := \begin{cases} 1, & \text{if} \quad d(x,y) \leq \varepsilon; \\ \phi(x,y), & \text{if} \quad \varepsilon \leq d(x,y) \leq \varepsilon + \theta; \\ 0, & \text{if} \quad d(x,y) \geq \varepsilon + \theta. \end{cases}$$

for $\varepsilon > 0$ and $\theta > 0$ and a continuous function $\phi : X \times X \longrightarrow [0,1]$. The parameters $\varepsilon > 0$ and $\theta > 0$ will be fixed throughout this paper and the function φ will be called **intensity function**.
The figures in 1 refer to a geometrical illustration of an intensity function, which could be easily extended to a two dimensional space.

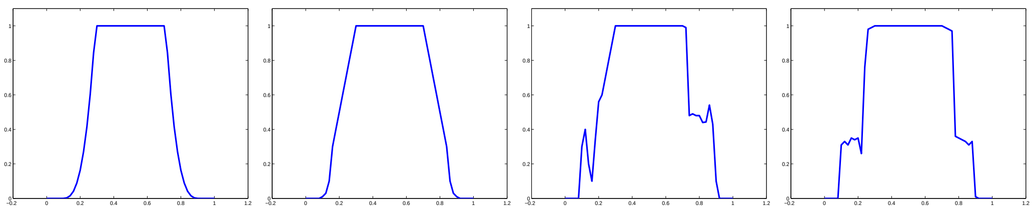

Figure 1: Examples of one dimensional Intensity functions.

Example 2. Let (X,d) a metric space with metric d, The following map

$$\varphi(x,y) := \begin{cases} 1, & \text{if} \quad | x - y | \leq \varepsilon; \\ \left(\frac{\varepsilon+\theta-|x-y|)}{\theta}\right) & \text{if } \varepsilon \leq | x - y | \leq \varepsilon + \theta; \\ 0, & \text{if} \quad | x - y | \geq \varepsilon + \theta. \end{cases}$$

is an intensity function.

2. Problem Statement

This Energy will called Energy of Collective Intelligence $x \in S(m)$. We define the local energy map $e(x,m)$ of the point $x \in X$ (or an Auto-Reacting

Individual) with respect to m as

$$\begin{aligned} e \quad &: \quad X \times M_+(I\!\!R^n) \longrightarrow \mathbb{R}^+, \\ e(x,m) \quad &= \quad \int_X \varphi(x,y) d^2(x,y) m(dy) \\ &= \quad \sum_{y \in S(m)} \varphi(x,y) m(y) d^2(x,y). \end{aligned}$$

Where d is any metric on $I\!\!R^n$. Note that per definition, the map E, which represents the global energy, can be expressed as a function of local energies:

$$E(m) \quad = \quad \int_X e(x,m) m(dx). \tag{4}$$

The main mathematical task of this paper is to analytically solve the following nonlinear optimization problem:

$$\begin{aligned} \max_{m \in M_+^m(I\!\!R^n)} \qquad & E(m) = \int_X e(x,m) m(dx) \\ s.t \qquad & e(x,m) > 0, \quad \forall x \in S(m). \end{aligned}$$

Where $M_+^m(I\!\!R^n)$ is a subset of $M_+(I\!\!R^n)$ such that

$$m(I\!\!R^n) = f(I\!\!R^n), \quad \forall f \in M_+^m(I\!\!R^n).$$

In other words, the problem is to characterize the structure of $S(m^*)$, which satisfies the optimality condition given as:

$$\begin{aligned} E(m^*) \quad &\geq \quad E(f), \quad \forall f \in M_+^m(I\!\!R^n), \quad \text{and} \\ e(x,m^*) \quad &> \quad 0, \quad \forall x \in S(m^*). \end{aligned}$$

Example 3. Consider $X = \mathbb{R}$ and $S(m) = \{x_1, x_2\}$. Suppose that $\theta << 1$, we have to distinct the following cases

- If $x_1 = x_2$, then $e(x_1,m) = e(x_2,m) = 0 \Rightarrow E(m) = 0$.
- If $\varphi(x_1,x_2) = 0$, then $e(x_1,m) = e(x_2,m) = 0 \Rightarrow E(m) = 0$.
- If $\varphi(x_1,x_2) = 1$, then $e(x_1,m), e(x_2,m) > 0 \Rightarrow 0 < E(m) \leq M < \infty$.

3. Optimizing Global Energy by Grouping Individuals

The approach presented in this section aims to develop a method for optimizing the global energy of a group of individuals. Several configurations of the interaction between individuals in terms of reacting energy will be treated and some lemmas giving some related results are exposed.

Definition 1. A ε-connected mass $m \in M_+(I\!R)$ of individuals $i = 1, \ldots, n$ is a subset of $I\!N$ with masses $x_1, \ldots, x_n \in I\!R^+$ such that for all $i = 1, \ldots, n$ the local energy $e(x_i, m) > 0$.

Suppose that there are three individuals which are neighbors and their masses are x, y and z respectively. Different configurations will be treated but we assume, for the sake of simplicity, that the distances between masses equal to the unit. See for examples the figure 2, where each move influences the local and global energy.

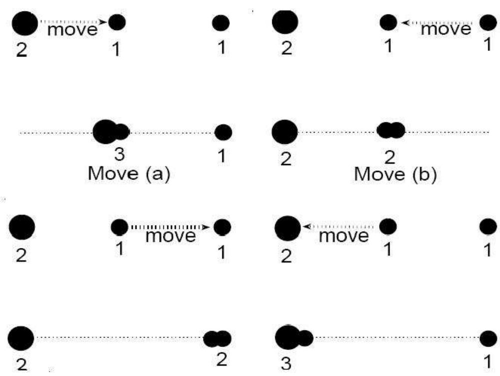

Figure 2: Examples of points reparations and moves.

In this case, the global energy is given by the following lemma.

Lemma 1. *Consider three ordered masses x, y and z placed on a line representing three neighboring individuals such that $x + y + z = M$, where M is constant, then the global energy $E(m)$ of this group of individuals is optimal if one of the following equations holds.*

$$y = x + z; \qquad \text{or} \quad z = x + y \qquad \text{or} \quad x = y + z \tag{5}$$

Proof. Since the proofs for the different cases are similar, we present only the proof for the first case involved in equations (5). The global energy is described by the function

$$E(m) = F(x, y, z) = 2xy + 2zy = 2(x + z)y \tag{6}$$

Noting that $y = M - (x + z)$, the above function can be reduced to a function of two variables

$$F(x, z) = 2(x + z)(M - (x + z)) = 2(x + z)M - 2(x + z)^2$$

Then the partial derivatives are $F_x = 2M - 4(x + z)$ and $F_z = 2M - 4(x + z)$. A necessary condition for (x_0, z_0) to be an extreme point of $F(x, z)$ is that its gradient at (x_0, z_0) must be zero:

$$\nabla F = \begin{pmatrix} F_{x_0} \\ F_{z_0} \end{pmatrix} = 0, \quad \text{or} \quad \begin{cases} 2M - 4(x_0 + z_0) = 0 \\ 2M - 4(x_0 + z_0) = 0 \end{cases}$$

The solution of these simultaneous equations is $M = 2(x_0 + z_0)$. Substituting M by $2(x_0 + z_0)$ in the expression of y shown above, gives $y_0 = x_0 + z_0$. Therefore the optimal value of the global energy is

$$E(m) = F(x, y, z) = 2(x + z)y = 2(x + z)^2 = 2y^2$$

4. Matrix form of the Masses System

Note that the system consisting of the equations given in the lemma can be written in matrix form as follows

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{7}$$

It is obvious that the matrix $\mathcal{A}$ is symmetric. Let us show that the solution given in the above lemma is optimal by finding the Hessian matrix. From equation above, we have

$$\mathcal{H} = \begin{pmatrix} F_{xx} & F_{xz} \\ F_{zx} & F_{zz} \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix}. \tag{8}$$

To state that this matrix is semi-definite negative, their eigenvalues must be negative or zero.

$$\mathcal{H} - \lambda I = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}. \tag{9}$$

From solving $det(\mathcal{H} - \lambda I) = 0$, we get $\lambda = -8$ or$\lambda = 0$, hence the Hessian matrix $\mathcal{H}$ is semi-definite negative and $y = x + z$ is a maximum point for the global energy $E(m)$.

Remark 1. It is possible to prove the previous lemma for n (even) points or $n+1$ (odd) points by grouping individuals in convenient fashion so that these cases could be transformed into the case given in the above lemma.

Lemma 2. *Consider three linked masses x, y and z in the plan representing three neighboring individuals such that $x + y + z = M$ where M is constant and assume that the distance between them is the unit, then the global energy $E(m)$ of this group of individuals is optimal if the following condition holds $x = y = z$.*

Proof. The global energy is expressed by the function

$$\begin{aligned} E(m) = F(x, y, z) &= xz + xy + zx + zy + yz + yx = 2(xz + xy + yz) \text{ and} \\ F(x, y, z) &= 2(x + z)y + 2xz = 2(x + y)z + 2xy \\ &= 2(y + z)x + 2yz. \end{aligned} \tag{10}$$

Noting that $y = M - (x + z)$, the above function can be reduced to a function of two variables

$$F(x, z) = 2(x + z)(M - (x + z)) = 2(x + z)M - 2(x + z)^2 + 2xz$$

Then the partial derivatives of this function are:

$$F_x = 2M - 4(x + z) + 2z \quad \text{and} \quad F_z = 2M - 4(x + z) + 2x. \tag{11}$$

A necessary condition for (x_0, z_0) to be an extreme point of $F(x, z)$ is that its gradient must be zero:

$$\nabla F = \begin{pmatrix} F_{x_0} \\ F_{z_0} \end{pmatrix} = 0 \quad \text{or} \quad \begin{cases} 2M - 4(x + z) + 2z = 0 \\ 2M - 4(x + z) + 2x = 0 \end{cases}$$

The solution of these simultaneous equations leads to $x = z$. In the same way, but taking another expression of the energy

$$E(m) = 2(x + y)M - 2(x + y)^2 + 2xy. \tag{12}$$

Then the partial derivatives of (12) are $F_x = 2M - 4(x + z) + 2y$ and $F_y = 2M - 4(x + z) + 2x$ that yields $x = y$. Therefore $x = y = z$.

Lemma 3. *Given four linked masses x, y, z and t in the plan without link between the masses lying on the diagonals, representing four neighboring individuals such that:*

$$x + y + z + t = M$$

where M is constant and assume that the distance between them is the unit, then the global energy $E(m)$ of this group of individuals cannot be optimal because the masses must satisfy the conditions $x = -z$ and $y = -t$, which impossible given that $x, y, z, t > 0$.

Proof. The global energy is expressed by the function

$$E(m) = F(x, y, z, t) = (xy + xt) + (yx + yz) + (zy + zt) + (tx + tz). \quad (13)$$

Then the partial derivatives of this function are:

$$F_x = 2(y + t) = F_z \quad \text{and} \quad F_y = 2(x + z) = F_t. \quad (14)$$

A necessary condition for (x_0, z_0) to be an extreme point of $F(x, z)$ is that its gradient must be zero:

$$\begin{cases} F_x = 2(y + t) = 0, \\ F_y = 2(x + z) = 0. \end{cases}$$

The solution of these simultaneous equations leads to $y = -t$ and $x = -z$, which is impossible because all masses are positive.

Lemma 4. *Consider four linked masses x, y, z and t in the plan representing four neighboring individuals such that:*

$$x + y + z + t = M,$$

where M is constant and assume that the distance between them is the unit, then the global energy $E(m)$ of this group of individuals is optimal if the following conditions hold.

$$x + z = \frac{M}{2}, \qquad y + t = \frac{M}{2} \quad (15)$$

Proof. The global energy is given by the function

$$E(m) = F(x, y, z, t) = x(y+z+t)+y(x+z+t)+z(x+y+t)+t(x+y+z). \quad (16)$$

Then the partial derivatives of this function are:

$$F_x = (y+z+t) = F_y = (x+z+t) = F_z = (x+y+t) = F_t = (x+y+z) = 0. \quad (17)$$

From (17) we find that

$$x = y \quad \text{and} \quad t = z. \tag{18}$$

Given that $x + y + z + t = M$, we obtain

$$x + z = \frac{M}{2} \quad \text{and} \quad y + t = \frac{M}{2}. \tag{19}$$

5. Mass Displacement

Lemma 5. *Given three aligned masses x, y and z in the plan representing three neighboring individuals such that:*

$$x + y + z = M,$$

where M is constant and assume that the distance between them is the unit. If any mass moves and comes to another mass so that both form one mass then the optimal value of the global energy $E(m)$ of this group of individuals is obtained when the following condition holds $x = y + z = \frac{M}{2}$.

Proof. The global energy is given as

$$E(m) = F(x, y, z) = 2x(y + z) = 2x(y + M - (x + y)) = 2x(M - x). \tag{20}$$

Making the gradient of this function zero, results in the system below:

$$\begin{cases} F_x = 2M - 4x = 0, \\ F_y = F_z = 0, \end{cases}$$

hence $x = \frac{M}{2} = y + z$.

6. Optimality Conditions

This section presents some necessary conditions for optimality which can be interpreted as the connection between the masses and their positions in two different cases. In the first case the masses are aligned but not all connected, while in the second case all masses are connected. The figure 3 illustrate the spacial position of three distances and connected points in a metric spaces (X, d).

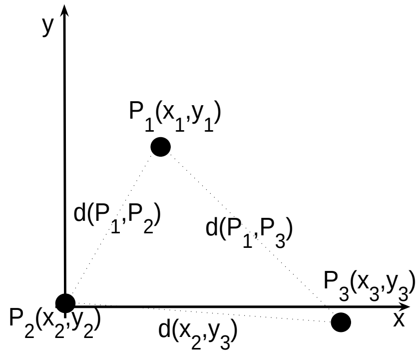

Figure 3: Examples of distanced points.

Theorem 1. *Consider three aligned masses $m(x), m(y)$ and $m(z)$ in positions x, y and z respectively, in the plan representing three neighboring individuals such that:*

$$m(x) + m(y) + m(z) = M,$$

where M is constant and assume the distances between them are $d(x, y)$ and $d(y, z)$ but x, z are not connected. Then the optimal value of the global energy $E(m)$ of this group of individuals is obtained if the following conditions hold.

$$\frac{\partial(m_x d(x,y))}{\partial x} = \frac{\partial(m_z d(y,z))}{\partial z}, \tag{21}$$

$$\frac{\partial m_y}{\partial y} \neq 0, \tag{22}$$

$$\frac{\partial d(x,y)}{\partial y} d(y,z) = \frac{\partial d(y,z)}{\partial y} d(x,y). \tag{23}$$

Proof. The global energy is described by the following function

$$E(m) = F(x,y,z) = 2(m_x m_y d(x,y) + m_y m_z d(y,z)), \tag{24}$$

where $m_x = m(x)$, $m_y = m(y)$ and $m_z = m(z)$.

To find the optimal value of the energy function its gradient must be zero. Then the partial derivatives of this function are equal to zero

$$\frac{\partial m_x}{\partial x} d(x,y) + m_x \frac{\partial d(x,y)}{\partial x} = 0, \tag{25}$$

$$m_x(\frac{\partial m_y}{\partial y}d(x,y)+m_y\frac{\partial d(x,y)}{\partial y})+m_z(\frac{\partial m_y}{\partial y}d(y,z)+m_y\frac{\partial d(y,z)}{\partial y})=0, \quad (26)$$

$$\frac{\partial m_z}{\partial z}d(y,z)+m_z\frac{\partial d(y,z)}{\partial z}=0. \quad (27)$$

Then equations (25) and (8) can be written

$$\frac{\partial(m_x d(x,y))}{\partial x}=0 \quad \text{and} \quad \frac{\partial(m_z d(y,z))}{\partial z}=0. \quad (28)$$

It follows that

$$\frac{\partial(m_x d(x,y))}{\partial x}=\frac{\partial(m_z d(y,z))}{\partial z} \quad (29)$$

Since m_x and m_z are positive, equation (7) gives

$$\frac{\partial m_y}{\partial y}d(x,y)+m_y\frac{\partial d(x,y)}{\partial y}=0, \quad \text{and} \quad \frac{\partial m_y}{\partial y}d(y,z)+m_y\frac{\partial d(y,z)}{\partial y}=0. \quad (30)$$

Noting that m_y is involved in both equations and rearranging them leads to

$$\frac{\partial m_y}{\partial y}(\frac{\partial d(y,z)}{\partial y}d(x,y)-\frac{\partial d(x,y)}{\partial y}d(y,z))=0. \quad (31)$$

If $\frac{\partial m_y}{\partial y}\neq 0$ then

$$\frac{\partial d(y,z)}{\partial y}d(x,y)-\frac{\partial d(x,y)}{\partial y}d(y,z)=0. \quad (32)$$

In the same manner, one can note that others partial derivatives of the global energy function provide the same result.

Theorem 2. *Consider three connected masses $m(x), m(y)$ and $m(z)$ in positions x, y and z respectively, in the plan representing three neighboring individuals such that:*

$$m(x)+m(y)+m(z)=M,$$

where M is constant and assume the distances between them are $d(x,y)$, $d(y,z)$ and $d(x,z)$. Then the optimal value of the global energy $E(m)$ of this group of individuals is obtained when the following conditions hold.

$$\begin{aligned} m_x d(x,z) &= m_y d(y,z), & (33)\\ m_x d(x,y) &= m_z d(y,z), & (34)\\ m_y d(x,y) &= m_z d(x,z). & (35) \end{aligned}$$

Proof. The global energy function is given by the following expression:

$$E(m) = 2(m_x m_y d(x,y) + m_y m_z d(y,z) + m_x m_z d(x,z)). \tag{36}$$

To find the optimal value of the energy function make its gradient equal to zero. That is, the partial derivatives of this function are equal to zero

$$\frac{\partial E(m)}{\partial x} = \frac{\partial m_x}{\partial x}(m_y d(x,y) + m_z d(x,z)) + m_x(m_y \frac{\partial d(x,y)}{\partial x} + m_z \frac{\partial d(x,z)}{\partial x}) = 0. \tag{37}$$

$$\frac{\partial E(m)}{\partial y} = \frac{\partial m_y}{\partial y}(m_x d(x,y) + m_z d(y,z)) + m_y(m_x \frac{\partial d(x,y)}{\partial y} + m_z \frac{\partial d(y,z)}{\partial y}) = 0. \tag{38}$$

$$\frac{\partial E(m)}{\partial z} = \frac{\partial m_z}{\partial z}(m_x d(x,z) + m_y d(y,z)) + m_z(m_x \frac{\partial d(x,z)}{\partial z} + m_y \frac{\partial d(y,z)}{\partial z}) = 0. \tag{39}$$

Let E_1, E_2, E_3 be defined as follows

$$E_1 = m_y d(x,y) + m_z d(x,z)), \tag{40}$$
$$E_2 = m_x d(x,y) + m_z d(y,z)), \tag{41}$$
$$E_3 = m_x d(x,z) + m_y d(x,z)). \tag{42}$$

Note that

$$E_1 = 0 \Rightarrow \frac{\partial E(m)}{\partial x} = 0,$$
$$E_2 = 0 \Rightarrow \frac{\partial E(m)}{\partial y} = 0,$$
$$E_3 = 0 \Rightarrow \frac{\partial E(m)}{\partial z} = 0,$$

equivalently

$$m_y d(x,y) + m_z d(x,z) = m_x d(x,y) + m_z d(y,z) = m_x d(x,z) + m_y d(x,z) = 0 \tag{43}$$

Solving the system consisting of equations (43) results in

$$m_x d(x,z) = m_y d(y,z) \tag{44}$$

In the same way, solving the remaining systems, that is, the first one composed of equations (43), we find these expressions

$$m_x d(x,y) = m_z d(y,z) \quad \text{and} \quad m_y d(x,y) = m_z d(x,z). \tag{45}$$

Remark 2. Theorem 1. and Theorem 2. can be generalized for more than three masses.

Remark 3. Note that in all above lemmas, the mass of a point is independent of its position, while in both theorem 1 and theorem 2, the mass of a point is function of its position.

Example 4. Consider three connected points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ in the plane. We suppose that the distance d is the Euclidian distance

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and

$$m_{P_1} + m_{P_2} + m_{P_3} = M$$

we consider that $P_2(x_2, y_2) = O(0,0)$, then

$$d(O, P_1) = \sqrt{x_1^2 + y_1^2}$$

and

$$d(O, P_3) = \sqrt{x_3^2 + y_3^2}$$

and

$$d(P_1, P_3) = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$$

The global energy is given by :

$$E = 2[m_{P_1} m_O d(O, P_1) + m_{P_3} m_O d(O, P_3) + m_{P_1} m_{P_3} d(P_1, P_3)]$$

$$E = 2[m_{P_1} m_O \sqrt{x_1^2 + y_1^2} + m_{P_3} m_O \sqrt{x_3^2 + y_3^2} + m_{P_1} m_{P_3} \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}]. \tag{46}$$

Applying theorem.2 yields

$$m_O \sqrt{x_3^2 + y_3^2} = m_{P_1} \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$$

and

$$m_{P_1}m_O\sqrt{x_1^2+y_1^2}=m_{P_3}m_O\sqrt{x_3^2+y_3^2}$$

therefore

$$E=2[m_{P_1}m_O\sqrt{x_1^2+y_1^2}+m_{P_3}m_O\sqrt{x_3^2+y_3^2}+m_Om_{P_3}\sqrt{x_3^2+y_3^2}]$$

or

$$E=2[m_Om_{P_3}\sqrt{x_3^2+y_3^2}+m_{P_3}m_O\sqrt{x_3^2+y_3^2}+m_Om_{P_3}\sqrt{x_3^2+y_3^2}]$$

hence

$$E=6[m_Om_{P_3}\sqrt{x_3^2+y_3^2}] \tag{47}$$

7. Concluding Remarks

In this work we have presented another application of the collective intelligence energy. Namely, we have studied how the energy depends on the organizing of autonomous agents, where each member of the collective should adapt and update its position to the groups dynamics. This collective dynamics provide an optimized energy as our theorems demonstrate. It is important to note that the optimal states depends not only from the initial state but also of the moving succession or the reaction order, where the agents form an ε-connected subgroups in different distribution. Moreover, the present study could introduce another example to study consensus and emergence phenomena.

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