International Journal of Pure and Applied Mathematics
Volume 78 No. 7 2012, 1063-1075
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

A NON-UNIFORM BOUND ON POISSON APPROXIMATION BY w-FUNCTIONS

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Abstract: The Stein-Chen method and w-functions are used to give a result in the Poisson approximation to the distribution of a non-negative integer-valued random variable X, in terms of the difference of cumulative distribution function of X and Poisson cumulative distribution function together with its non-uniform bound. For applications, the result obtained in the present study is applied to approximate binomial, negative binomial, hypergeometric and negative hypergeometric cumulative distribution functions.

AMS Subject Classification: 62E17, 60F05
Key Words: cumulative distribution function, non-uniform bound, Poisson approximation, Stein's method; w-functions

1. Introduction

There have been many works on the topics related to the Poisson approximation. These works have yielded useful results in applied probability and statistics, and the most useful results are concerned with the Poisson approximation for

Received: June 24, 2012

sums of Bernoulli random variables and for non-negative integer-valued random variables. In particular, many accurate results in this way are created by the well-known Stein-Chen method, which was introduced by Chen [4]. Because the Stein-Chen method is one of most developed and effective approaches for the Poisson approximation in the last 30 years, a complete review of related works seems to be rather difficult, however, some collections of most results obtained by this method are contained in Stein [9], Barour et al. [2] and Barbour and Chen [1].

In this study, we focus on approximating the distribution of a non-negative integer-valued random variable by Poisson distribution using the Stein-Chen method, and the most useful results are as the following. Let X be a non-negative integer-valued random variable with the probability function $p_X(x) = \mathbb{P}(X = x) > 0$ for every x in the support of X, $\mathcal{S}(x)$, and have mean μ and finite variance σ^2 $(0 < \sigma^2 < \infty)$. It is well known that the distribution of X can be approximated by a Poisson distribution provided that certain conditions on their parameters are satisfied, and if we expect the distribution of X to be closer to the Poisson distribution than other distributions, then it is reasonable to approximate the distribution of X by Poisson distribution. In this direction, bounds on the error of this approximation has already been studied by Papatanasiou and Utev [7], Papadatos and Papatanasiou [6] and Majsnerowska [5], among them, the most general form of bound is given by the last author. He used Stein-Chen method and w-functions to give a uniform bound for the difference of the distribution of X and the Poisson distribution with mean λ as follows:

$$\left|\mathbb{P}(X \in A) - \sum_{x\in A} \frac{e^{-\lambda}\lambda^x}{x!}\right| \leq (1 - e^{-\lambda})\mathbb{E}\left|1 - \frac{\sigma^2 w(X)}{\lambda}\right| + \min\left\{1, \lambda^{-1/2}\right\}|\lambda - \mu|, \quad (1.1)$$

where A is a subset of $\mathbb{N} \cup \{0\}$, and he also applied this result to approximate binomial, negative binomial and hypergeometric distributions. For other applications, Teerapabolarn [12, 13] gave uniform error bounds on Poisson approximation to the beta-negative binomial and negative hypergeometric distributions, respectively. In view of the result in (1.1), if $A = \{0, ..., x_0\}$ as $x_0 \in \mathcal{S}(x)$, then (1.1) becomes

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq (1 - e^{-\lambda})\mathbb{E}\left|1 - \frac{\sigma^2 w(X)}{\lambda}\right| + \min\left\{1, \lambda^{-1/2}\right\}|\lambda - \mu|, \quad (1.2)$$

where $\mathcal{F}_X(x_0) = \sum_{x=0}^{x_0} p_X(x)$ is cumulative distribution function of X at x_0 and $\mathcal{P}_\lambda(x_0) = \sum_{x=0}^{x_0} \frac{e^{-\lambda}\lambda^x}{x!}$ is Poisson cumulative distribution function at x_0. It is observed that the bound in (1.2) does not depend on x_0, or it is a uniform bound with respect to x_0. Hence, the uniform bound in (1.2) may not be sufficiently good for measuring the accuracy of the approximation. In this case, a non-uniform bound with respect to x_0 is required. In this paper, we use the Stein-Chen method and w-functions to give a result in the Poisson approximation to the distribution of X, in terms of $|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)|$ together with its non-uniform bound with respect to $x_0 \in \mathcal{S}(x)$.

The tools for giving the result consist of the Stein-Chen method and w-functions, which are in Section 2. In Section 3, a non-uniform bound for the difference of $\mathcal{F}_X(x_0)$ and $\mathcal{P}_\lambda(x_0)$ with respect to $x_0 \in \mathcal{S}(x)$ is derived. For applications, the obtainable result is directly applied to approximate binomial, negative binomial, hypergeometric and negative hypergeometric cumulative distribution functions, which are in the last section.

2. Method

In order to give our result, we use the same methodology as in Majsnerowska [5], which consists of the Stein-Chen method and w-functions as follows.

For w-functions, Cacoullos and Papathanasiou [3] defined a function w associated with non-negative integer-valued random variable X in the relation

$$w(x)p_X(x) = \frac{1}{\sigma^2}\sum_{i=0}^{x}(\mu - i)p_X(i), \; x \in \mathcal{S}(x) \tag{2.1}$$

and afterwards, Majsnerowska [5] expressed the relation (2.1) in the form of

$$w(0) = \frac{\mu}{\sigma^2}, \quad w(x) = \frac{1}{\sigma^2}\left\{\mu + \frac{\sigma^2 w(x-1)p_X(x-1)}{p_X(x)} - x\right\}, \quad x \in \mathcal{S}(x)\setminus\{0\} \tag{2.2}$$

and

$$w(x) \geq 0, \quad x \in \mathcal{S}(x), \tag{2.3}$$

where $p_X(x) > 0$ for every $x \in \mathcal{S}(x)$. The following relation is an important property for proving the result, which was stated by Cacoullos and Papathanasiou [3].

If a non-negative integer-valued random variable X is defined as in Section 1, then

$$\mathbb{E}[(X-\mu)f(X)] = \sigma^2\mathbb{E}[w(X)\Delta f(X)], \tag{2.4}$$

for any function $f : \mathbb{N}\cup\{0\} \to \mathbb{R}$ for which $\mathbb{E}|w(X)\Delta f(X)| < \infty$, where $\Delta f(x) = f(x+1) - f(x)$. We note that $\mathbb{E}[w(X)] = 1$, which by taking $f(x) = x$.

For Stein-Chen method, Stein [8] introduced a powerful and general method for bounding the error in the normal approximation. This method was first developed and applied in the setting of Poisson approximation by Chen [4], which is refer to as the Stein-Chen method. Stein's equation for Poisson distribution with mean $\lambda > 0$ is, for given h, of the form

$$h(x) - \wp_\lambda(h) = \lambda f(x+1) - xf(x), \tag{2.5}$$

where $\wp_\lambda(h) = e^{-\lambda}\sum_{l=0}^{\infty} h(l)\frac{\lambda^l}{l!}$ and f and h are bounded real valued functions defined on $\mathbb{N}\cup\{0\}$.

For $A \subseteq \mathbb{N}\cup\{0\}$, let function $h_A : \mathbb{N}\cup\{0\} \to \mathbb{R}$ be defined by

$$h_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Following Barbour et al. [2] and writing $C_x = \{0, ..., x\}$, the solution f_A of (2.5) is of the form

$$f_A(x) = \begin{cases} (x-1)!\lambda^{-x}e^{\lambda}[\wp_\lambda(h_{A\cap C_{x-1}}) - \wp_\lambda(h_A)\wp_\lambda(h_{C_{x-1}})] & \text{if } x \geq 1, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.6}$$

Similarly, for $A = C_{x_0}$ where $x_0 \in \mathbb{N}\cup\{0\}$, $f_{C_{x_0}}$ can be expressed as

$$f_{C_{x_0}}(x) = \begin{cases} (x-1)!\lambda^{-x}e^{\lambda}[\wp_\lambda(h_{C_{x-1}})\wp_\lambda(1-h_{C_{x_0}})] & \text{if } x \leq x_0, \\ (x-1)!\lambda^{-x}e^{\lambda}[\wp_\lambda(h_{C_{x_0}})\wp_\lambda(1-h_{C_{x-1}})] & \text{if } x > x_0. \\ 0 & \text{if } x = 0, \end{cases} \tag{2.7}$$

and it is observed that $f_{C_{x_0}}(x) > 0$ for every $x \in \mathbb{N}$.

The following lemma gives non-uniform bounds for $f_{C_{w_0}}$ and $\Delta f_{C_{w_0}}$ that use to determine the main result.

Lemma 2.1. *For $x_0 \in \mathbb{N}\cup\{0\}$ and $x \in \mathbb{N}$, let $\Delta f_{C_{x_0}}(x) = f_{C_{x_0}}(x+1) - f_{C_{x_0}}(x)$, then following inequalities hold:*

$$f_{C_{x_0}}(x) \leq \begin{cases} \lambda^{-1}(1-e^{-\lambda}) & \text{if } x_0 = 0, \\ \min\left\{1, \sqrt{\frac{2}{\lambda e}}, \max\left\{\frac{1}{x_0}, \frac{1}{\lambda}\right\}\frac{2\lambda^{-1}(e^{\lambda}-\lambda-1)}{x_0+1}\right\} & \text{if } x_0 > 0 \end{cases} \tag{2.8}$$

and

$$|\Delta f_{C_{x_0}}(x)| \leq \begin{cases} \lambda^{-2}(e^{-\lambda}+\lambda-1) & \text{if } x_0 = 0, \\ \min\left\{\lambda^{-1}(1-e^{-\lambda}), \dfrac{1}{x_0}\right\} & \text{if } x_0 > 0. \end{cases} \tag{2.9}$$

Proof. First, we have to show that (2.8) holds. Using the properties of $f_{C_{x_0}}$ in Teerapabolarn [11], it follows that $f_{C_{x_0}}$ is decreasing function for $x \in \{x_0+1, x_0+2, ...\}$ and is increasing function for $x \in \{1, ..., x_0\}$. Thus, for $x_0 = 0$, we have $f_{C_0}(x) \leq f_{C_0}(1) = \lambda^{-1}(1-e^{-\lambda})$. For $x_0 > 0$, it follows from Barbour et al. [2] that $f_{C_{x_0}}(x) \leq \min\left\{1, \sqrt{\frac{2}{\lambda e}}\right\}$, and it can be seen that for $x \leq x_0$,

$$\begin{aligned} f_{C_{x_0}}(x) &\leq f_{C_{x_0}}(x_0) \\ &\leq (x_0-1)!\lambda^{-x_0}e^{\lambda}\wp_\lambda(1-h_{C_{x_0}}) \\ &= (x_0-1)!\sum_{k=x_0+1}^{\infty}\frac{\lambda^{k-x_0}}{k!} \\ &= \frac{\lambda}{x_0(x_0+1)} + \frac{\lambda^2}{x_0(x_0+1)(x_0+2)} + \frac{\lambda^3}{x_0(x_0+1)(x_0+2)(x_0+3)} + \cdots \\ &\leq \frac{\lambda^{-1}}{x_0(x_0+1)}\left\{\lambda^2 + \frac{\lambda^3}{3} + \frac{\lambda^4}{12} + \cdots\right\} \\ &= \frac{2\lambda^{-1}(e^{\lambda}-\lambda-1)}{x_0(x_0+1)} \end{aligned}$$

and for $x > x_0$,

$$\begin{aligned} f_{C_{x_0}}(x) &\leq f_{C_{x_0}}(x_0+1) \\ &\leq x_0!\lambda^{-(x_0+1)}e^{\lambda}\wp_\lambda(1-h_{C_{x_0}}) \\ &= x_0!\sum_{k=x_0+1}^{\infty}\frac{\lambda^{k-(x_0+1)}}{k!} \\ &= \frac{1}{x_0+1} + \frac{\lambda}{(x_0+1)(x_0+2)} + \frac{\lambda^2}{(x_0+1)(x_0+2)(x_0+3)} + \cdots \\ &\leq \frac{\lambda^{-2}}{x_0+1}\left\{\lambda^2 + \frac{\lambda^3}{3} + \frac{\lambda^4}{12} + \cdots\right\} \\ &= \frac{2\lambda^{-2}(e^{\lambda}-\lambda-1)}{x_0+1}, \end{aligned}$$

this gives $f_{C_{x_0}}(x) \le \max\left\{\frac{1}{x_0}, \frac{1}{\lambda}\right\} \frac{2\lambda^{-1}(e^{\lambda}-\lambda-1)}{x_0+1}$, and we also have that $f_{C_{x_0}}(x) \le \min\left\{1, \sqrt{\frac{2}{\lambda e}}, \max\left\{\frac{1}{x_0}, \frac{1}{\lambda}\right\} \frac{2\lambda^{-1}(e^{\lambda}-\lambda-1)}{x_0+1}\right\}$. Hence, (2.8) holds.

In the next step, we shall show that (2.9) holds. For $x_0 = 0$, it is obtained from Lemma 2.1 (3) in Teerapabolarn and Neammanee [10]. For $x_0, x \in \mathbb{N}$, let $\Delta f_{\{x_0\}}(x) = f_{\{x_0\}}(x+1) - f_{\{x_0\}}(x)$. Following Stein [9], we obtain

$$\Delta f_{\{x_0\}}(x) \begin{cases} < 0 & \text{if } x \neq x_0, \\ > 0 & \text{if } x = x_0. \end{cases} \tag{2.10}$$

While Barbour et al. [2] showed that

$$\Delta f_{\{x\}}(x) \le \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{x}\right\}. \tag{2.11}$$

By Lemma 2.1 in Teerapabolarn [11], it follows that $\Delta f_{C_{x_0}}(x) \le \Delta f_{C_{x_0}}(x_0)$ for $x \le x_0$. Therefore, by the fact that $\Delta f_{C_{x_0}}(x) = \sum_{k=0}^{x_0} \Delta f_{\{k\}}(x)$ and using inequality (2.10) and (2.11), we have

$$0 < \Delta f_{C_{x_0}}(x) \le \Delta f_{C_{x_0}}(x_0) \le \Delta f_{\{x_0\}}(x_0) \le \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{x_0}\right\}$$

for $x \le x_0$, and for $x > x_0$,

$$0 > \Delta f_{C_{x_0}}(x) \ge \Delta f_{\{x\}^c}(x) = -\Delta f_{\{x\}}(x) \ge -\min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{x_0}\right\},$$

which yields $\left|\Delta f_{C_{x_0}}(x)\right| \le \min\left\{\lambda^{-1}(1-e^{-\lambda}), \frac{1}{x_0}\right\}$ for $x_0 > 0$. Hence, the inequality (2.9) holds. □

3. Result

We use the Stein-Chen method and w-functions to give a result of Poisson approximation to the distribution of a non-negative integer-valued random variable X, in terms of the difference of two cumulative distribution functions together with its non-uniform bound, in the following theorem.

Theorem 3.1. *Let a non-negative integer-valued random variable X together with corresponding w-function $w(X)$ be defined as above. Then we have the following:*

1. For $x_0 = 0$,

$$|\mathcal{F}_X(0) - \mathcal{P}_\lambda(0)| \leq \frac{e^{-\lambda}+\lambda-1}{\lambda}\mathbb{E}\left|1-\frac{\sigma^2 w(X)}{\lambda}\right| + (1-e^{-\lambda})\left|1-\frac{\mu}{\lambda}\right| \tag{3.1}$$

and, if $\lambda = \mu$, *then*

$$|\mathcal{F}_X(0) - \mathcal{P}_\lambda(0)| \leq \frac{e^{-\lambda}+\lambda-1}{\lambda}\mathbb{E}\left|1-\frac{\sigma^2 w(X)}{\lambda}\right|. \tag{3.2}$$

2. For $x_0 \in \mathcal{S}(x) \setminus \{0\}$,

$$\begin{aligned}|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq \min\left\{1-e^{-\lambda}, \frac{\lambda}{x_0}\right\}\mathbb{E}\left|1-\frac{\sigma^2 w(X)}{\lambda}\right| + (1-p_X(0))\\ \times \min\left\{1, \sqrt{\frac{2}{\lambda e}}, \max\left\{\frac{1}{x_0}, \frac{1}{\lambda}\right\}\frac{2(e^{\lambda}-\lambda-1)}{(x_0+1)\lambda}\right\}|\lambda-\mu|\end{aligned} \tag{3.3}$$

and, if $\lambda = \mu$, *then*

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq \min\left\{1-e^{-\lambda}, \frac{\lambda}{x_0}\right\}\mathbb{E}\left|1-\frac{\sigma^2 w(X)}{\lambda}\right|. \tag{3.4}$$

Proof. Substituting h by $h_{C_{x_0}}$, x by X and taking expectation in (2.5), it yields

$$\begin{aligned}\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0) &= \mathbb{E}[\lambda f(X+1) - Xf(X)]\\ &= \lambda\mathbb{E}[f(X+1)] - \mathbb{E}[(X-\mu)f(X)] - \mu\mathbb{E}[f(X)]\\ &= \lambda\mathbb{E}[\Delta f(X)] - \mathbb{E}[(X-\mu)f(X)] + (\lambda-\mu)\mathbb{E}[f(X)],\end{aligned}$$

where $f = f_{C_{x_0}}$ is defined as in (2.7). Since $\mathbb{E}[w(X)] = 1$ and $\mathbb{E}|w(X)\Delta f(X)| = \mathbb{E}[w(X)|\Delta f(X)|] < \infty$. Thus, by (2.4), we have

$$\begin{aligned}|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| &\leq |\lambda\mathbb{E}[\Delta f(X)] - \sigma^2\mathbb{E}[w(X)\Delta f(X)]| + |\lambda-\mu|\mathbb{E}|f(X)|\\ &\leq \mathbb{E}\{|\lambda - \sigma^2 w(X)||\Delta f(X)|\} + |\lambda-\mu|\mathbb{E}|f(X)|\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}|f(X)| &= \sum_{x\in S(x)} |f(x)|p_X(x)\\ &= (1-p_X(0))\sup_{x\geq 1}|f(x)|.\end{aligned}$$

Hence, by Lemma 2.1, the theorem is proved. □

Immediately from the Theorem 3.1, the following corollary is also obtained.

Corollary 3.1. *If $\lambda - \sigma^2 w(x) \geq / < 0$ for every $x \in \mathcal{S}(x)$, then*

1. For $x_0 = 0$,

$$|\mathcal{F}_X(0) - \mathcal{P}_\lambda(0)| \leq \frac{e^{-\lambda} + \lambda - 1}{\lambda}\left|1 - \frac{\sigma^2}{\lambda}\right| + (1 - e^{-\lambda})\left|1 - \frac{\mu}{\lambda}\right| \tag{3.5}$$

and if $\lambda = \mu$, then

$$|\mathcal{F}_X(0) - \mathcal{P}_\lambda(0)| \leq \frac{e^{-\lambda} + \lambda - 1}{\lambda}\left|1 - \frac{\sigma^2}{\lambda}\right|. \tag{3.6}$$

2. For $x_0 \in \mathcal{S}(x) \setminus \{0\}$,

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq \min\left\{1 - e^{-\lambda}, \frac{\lambda}{x_0}\right\}\left|1 - \frac{\sigma^2}{\lambda}\right| + (1 - p_X(0)) \times \min\left\{1, \sqrt{\frac{2}{\lambda e}}, \max\left\{\frac{1}{x_0}, \frac{1}{\lambda}\right\}\frac{2(e^\lambda - \lambda - 1)}{(x_0+1)\lambda}\right\}|\lambda - \mu| \tag{3.7}$$

and if $\lambda = \mu$, then

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq \min\left\{1 - e^{-\lambda}, \frac{\lambda}{x_0}\right\}\left|1 - \frac{\sigma^2}{\lambda}\right|. \tag{3.8}$$

Remark. By comparing the result in (1.2) (uniform bound) and the the result in Theorem 3.1 (non-uniform bound), it can be seen that the bound in Theorem 3.1 is sharper than the bound in (1.2).

4. Applications

For applications of this study, we apply the result in Theorem 3.1 to approximate binomial, negative binomial, hypergeometric and negative hypergeometric cumulative distribution functions.

Example 4.1. Let X be a discrete random variable with probability function as follows:

$$p_X(x) = \binom{n}{x} p^x q^{n-x},\ x = 0, ..., n.$$

Then X has a binomial distribution with parameters $n \in \mathbb{N}$ and $0 < p < 1$, where $q = 1 - p$ and the mean and variance of X are $\mu = np$ and $\sigma^2 = npq$, respectively.

By relation (2.2), we have $w(x) = \frac{(n-x)p}{\sigma^2}$. Setting $\lambda = \mu = np$ in Theorem 3.1, it follows that $\lambda - \sigma^2 w(x) = xp \geq 0$ for all $0 \leq x \leq n$. Thus, by Corollary 3.1, we have the following result.

Corollary 4.1. *If $\lambda = np$, then*

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq \begin{cases} \frac{e^{-\lambda}+\lambda-1}{n} & \text{if } x_0 = 0, \\ \min\left\{1 - e^{-\lambda}, \frac{\lambda}{x_0}\right\} p & \text{if } 1 \leq x_0 \leq n. \end{cases}$$

This result gives a good approximation if p is sufficiently small. For a numerical result, if $n = 100$ and $p = 0.05$, then we have $\lambda = 5$ and a non-uniform bound on the error for this approximation is of the form

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq \begin{cases} 0.04006738 & \text{if } x_0 = 0, \\ 0.04966310 & \text{if } 1 \leq x_0 \leq 5, \\ \frac{0.25}{x_0} & \text{if } 6 \leq x_0 \leq 100, \end{cases}$$

which is sharper than numerical result obtained from (1.2),

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq 0.04966310, \; 0 \leq x_0 \leq 100.$$

Example 4.2. Let X be a discrete random variable with probability function as follows:

$$p_X(x) = \frac{\Gamma(n+x)q^x p^n}{x!\Gamma(n)}, \; x = 0, 1, \ldots.$$

Then X has a negative binomial distribution with parameters $n > 0$ and $0 < p < 1$, where Γ is gamma function and the mean and variance of X are $\mu = \frac{nq}{p}$ and $\sigma^2 = \frac{nq}{p^2}$, respectively.

Using relation (2.2), we have $w(x) = \frac{(n+x)q}{p\sigma^2}$. Setting $\lambda = \mu = \frac{nq}{p}$ in Theorem 3.1, it follows that $\lambda - \sigma^2 w(x) = -\frac{xq}{p} \leq 0$ for all $x \geq 0$. Thus, by Corollary 3.1, the following result can be obtained.

Corollary 4.2. *If $\lambda = \frac{nq}{p}$, then*

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq \begin{cases} \frac{e^{-\lambda}+\lambda-1}{n} & \text{if } x_0 = 0, \\ \min\left\{1 - e^{-\lambda}, \frac{\lambda}{x_0}\right\} \frac{q}{p} & \text{if } x_0 \geq 1. \end{cases}$$

Similarly, this result also gives a good approximation if q is sufficiently small. For a numerical result, if $n = 300$ and $p = 0.99$, then we have $\lambda = \frac{100}{33}$ and a non-uniform bound on the error for this approximation is of the form

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq \begin{cases} 0.00692868 & \text{if } x_0 = 0, \\ 0.00961312 & \text{if } x_0 = 1, 2, 3, \\ \frac{0.03060912}{x_0} & \text{if } x_0 \geq 4, \end{cases}$$

which is sharper than numerical result obtained from (1.2),

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq 0.00961312, \ x_0 \geq 0.$$

Example 4.3. Let X be a discrete random variable with probability function as follows:

$$p_X(x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}, \ x = 0, ..., n.$$

Then X has a hypergeometric distribution with parameters N, m and n, where $N > m > n$ and the mean and variance of X are

$$\mu = \frac{nm}{N} \text{ and } \sigma^2 = \frac{nm(N-n)(N-m)}{N^2(N-1)},$$

respectively.

Following relation (2.2), we obtain

$$w(x) = \frac{(n-x)(m-x)}{N\sigma^2} = \frac{nm - x(n+m-x)}{N\sigma^2}.$$

Setting $\lambda = \mu = \frac{nm}{N}$ in Theorem 3.1, it follows that

$$\lambda - \sigma^2 w(x) = \frac{x(n+m-x)}{N} \geq 0$$

for all $0 \leq x \leq n$. Thus, by Corollary 3.1, we have the following result.

Corollary 4.3. *If* $\lambda = \frac{nm}{N}$, *then*

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq \begin{cases} \frac{e^{-\lambda}+\lambda-1}{\lambda}\left(\frac{n+m-1}{N-1}\right) & \text{if } x_0 = 0, \\ \min\left\{1 - e^{-\lambda}, \frac{\lambda}{x_0}\right\}\left(\frac{n+m-1}{N-1}\right) & \text{if } 1 \leq x_0 \leq n. \end{cases}$$

This result yields a good approximation if $\frac{m}{N}$ and $\frac{n}{N}$ are sufficiently small. For a numerical result, if $N = 1000,\ m = 70$ and $n = 30$, then we have $\lambda = 2.1$ and a non-uniform bound on the error for this approximation is of the form

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq \begin{cases} 0.05768778 & \text{if } x_0 = 0, \\ 0.08696378 & \text{if } x_0 = 1, 2, \\ \frac{0.20810811}{x_0} & \text{if } 3 \leq x_0 \leq 30, \end{cases}$$

which is sharper than numerical result obtained from (1.2),

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq 0.08696378,\ 0 \leq x_0 \leq 300.$$

Example 4.4. Let X be a discrete random variable with probability function as follows:

$$p_X(x) = \frac{\binom{n+x-1}{x}\binom{N-n-x}{m-x}}{\binom{N}{m}},\ x = 0, 1, ..., m.$$

Then X has a negative hypergeometric distribution with parameters N, n and m, where $n \in \{1, ..., N-m\}$ and the mean and variance of X are $\mu = \frac{nm}{N-m+1}$ and $\sigma^2 = \frac{nm(N-m-n+1)(N+1)}{(N-m+1)^2(N-m+2)}$, respectively.

It follows from the relation (2.2) that $w(x) = \frac{(n+x)(m-x)}{(N-m+1)\sigma^2}$. Setting $\lambda = \mu = \frac{nm}{N-m+1}$ in Theorem 3.1, we have $\lambda - \sigma^2 w(x) = \frac{(n-m+x)x}{N-m+1} \geq 0$ for all $0 \leq x \leq m$ when $n \geq m-1$. Thus, by Corollary 3.1, the following result can be obtained.

Corollary 4.4. *If* $\lambda = \frac{nm}{N-m+1}$ *and* $n \geq m-1$, *then*

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq \begin{cases} \frac{e^{-\lambda}+\lambda-1}{\lambda}\frac{(N-m+1)(n+1)-m(N-m-n+1)}{(N-m+1)(N-m+2)} & \textit{if } x_0 = 0, \\ \min\left\{1 - e^{-\lambda}, \frac{\lambda}{x_0}\right\}\frac{(N-m+1)(n+1)-m(N-m-n+1)}{(N-m+1)(N-m+2)} & \textit{if } x_0 \geq 1. \end{cases}$$

This result yields a good approximation if $\frac{n}{N-m}$ is sufficiently small. For a numerical result, if $N = 1000,\ m = 30$ and $n = 50$, then we have $\lambda = 1.54479918$ and a non-uniform bound on the error for this approximation is of the form

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq \begin{cases} 0.01138323 & \text{if } x_0 = 0, \\ 0.01824564 & \text{if } x_0 = 1, 2, \\ \frac{0.03582303}{x_0} & \text{if } 3 \leq x_0 \leq 30, \end{cases}$$

which is sharper than numerical result obtained from (1.2),

$$|\mathcal{F}_X(x_0) - \mathcal{P}_\lambda(x_0)| \leq 0.01824564,\ \ 0 \leq x_0 \leq 30.$$

Acknowledgments

This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

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