International Journal of Pure and Applied Mathematics
Volume 78 No. 7 2012, 981-991
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

SOME RESULTS ON TOPOLOGICAL SPACES WITH ELEMENTS OF FINITE ORDER

Kiran Shrivastava[1 §], Vijeta Iyer[2]
[1,2]Department of mathematics
S.N.G.G.P.G. College
Bhopal, INDIA

Abstract: In this paper various results on order of elements in a set, relative to a topology whose members are necessarily generated by a finite collection of sets are obtained. Various properties of order are explored under different restrictions on generators of such a topology.

AMS Subject Classification: 54A99
Key Words: order of an element, order of a set

*

Definition 1. Let (X, τ) be a topological space and $a \in X$. Then *order of a relative to topology* τ, $O_\tau(a)$ is the number of $\tau-$ open sets containing a.

Definition 2. Let (X, τ) be a topological space and $A \subset X$. Then *order of A relative to topology* τ, $O_\tau(A)$is the number of $\tau-$ open sets containing A.

The following results are obvious consequences of definition of order of an element in a topological space.

1. In an infinite T_1- topological space, order of each element is infinite.
2. Let τ_1 and τ_2 be two finite collection of sets forming two topologies on a set X such that $\tau_1 \cup \tau_2$ is also a topology on X. Then
$$O_{\tau_1 \cup \tau_2}(a) = O_{\tau_1}(a) + O_{\tau_2}(a) - O_{\tau_1 \cap \tau_2}(a).$$
for every $a \in X$.

Received: February 20, 2012

§Correspondence author

The following theorems are proved for topological spaces whose topologies are finite collections of sets.

The proofs of the theorems 1 and 2 directly follow from definition and so are omitted.

Theorem 1. *Let τ be a topology on set X such that all non-trivial members of τ are generated by k non-empty mutually disjoint subsets $A_1, A_2, \ldots, A_k$ of X whose union is not X. Then τ is connected topology on X.*

Theorem 2. *Let τ be a topology on set X such that all non-trivial members of τ are generated by a family $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of non-empty subsets of X such that (i) union of $A_1, A_2, \ldots, A_k$ is not X. (ii) Sets $A_1, A_2, \ldots, A_r$ of X of family $\mathcal{A}$ mutually intersect in same elements and are disjoint from each of the remaining $(k - r)$ mutually disjoint members of family $\mathcal{A}$. Then (X, τ) is connected.*

Theorem 3. *Let τ and $\tau\prime$ be two topologies on X such that $\tau \subset \tau\prime$ and orders of elements of X with respect to these topologies are finite and equal. Then $\tau = \tau\prime$.*

Proof. Suppose $\tau \neq \tau\prime$ then there is an $A \in \tau\prime$ with $A \notin \tau$ and for any $x \in A$, $O_\tau(x) = O_{\tau\prime}(x) = k_x$ say. Let $A_1, A_2, \ldots, A_{k_x}$ are k_x distinct members of τ containing x and hence of $\tau\prime$.

So, $A = A_i$ for some $i, 1 \leq i \leq k$ or $A \in \tau$.

This is a contradiction.

Hence $\tau = \tau\prime$.

Theorem 4. *Let X contain n elements and let A_1 and A_2 be two disjoint subsets of X containing n_1 and n_2 elements respectively.Let τ_1 and τ_2 be two topologies consisting of $\emptyset$ and all supersets of A_1 and A_2 respectively. Then for any $x \in A_1$*

1. *$O_{\tau_1}(x) < O_{\tau_2}(x)$ if $n_1 > n_2 + 1$,*

2. *$O_{\tau_1}(x) = O_{\tau_2}(x)$ if $n_1 = n_2 + 1$,*

3. *$O_{\tau_1}(x) > O_{\tau_2}(x)$ if $n_1 \leq n_2$.*

Proof. The result follows from the fact that $O_{\tau_1}(x) = 2^{n-n_1}$ and $O_{\tau_2}(x) = 2^{n-n_2-1}$ for all $x \in A_1$.

Theorem 5. *Let τ be a topology on set X such that all non-trivial members of τ are generated by k non-empty mutually disjoint subsets $A_1, A_2, \ldots, A_k$ of X whose union is not X.Then τ has $2^k + 1$ members such that $O_\tau(a) = 2^{k-1} + 1$ if $a \in A_i$ for some i; $1 \leq i \leq k$ and $O_\tau(a) = 1$ otherwise.*

Proof. Total members of τ is ${}^kC_0 +^k C_1 + \ldots + {}^kC_k + 1 = 2^k + 1$.
Let $a \in X$. If $a \in A_i$ for some i, $1 \leq i \leq k$ then

$$O_\tau(a) = O_\tau(A_i) = {}^{k-1}C_0 +^{k-1} C_1 + \cdots +^{k-1} C_{k-1} + 1 = 2^{k-1} + 1.$$

If $a \notin A_i$, for any i; $1 \leq i \leq k$ then a is not in any proper open set or $O_\tau(a) = 1$.

Theorem 6. *Let τ be a topology on set X such that all non-trivial members of τ are generated by a family $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of non-empty subsets of X such that (i) union of $A_1, A_2, \ldots, A_k$ is not X. (ii) Two sets A_1 and A_2 of $\mathcal{A}$ neither intersect in each other and are disjoint from each of the remaining mutually disjoint members of $\mathcal{A}$. Then:*

$$O_\tau(a) = \begin{cases} 2^{k-1} + 1 & \text{if } a \in A_1 \cup A_2 - A_1 \cap A_2, \\ 2^k + 1 & \text{if } a \in A_1 \cap A_2, \\ 5.2^{k-3} + 1 & \text{if } a \in A_i, 3 \leq i \leq k, \\ 1 & \text{if } a \notin A_i, 1 \leq i \leq k. \end{cases}$$

Proof, Let $A_1 \cup A_2 - A_1 \cap A_2$, then

$$O_\tau(a) =^{k-1} C_0 +^{k-1} C_1 + \cdots +^{k-1} C_{k-1} + 1 = 2^{k-1} + 1.$$

If $a \in A_1 \cap A_2$ then

$$O_\tau(a) = ({}^{k-1}C_0 +^{k-1} C_1 + \cdots +^{k-1} C_{k-1}) \\ + 2({}^{k-2}C_0 +^{k-2} C_1 + \cdots +^{k-2} C_{k-2}) + 1 = 2^k + 1.$$

If $a \in A_i$, for some i, $3 \leq i \leq k$ then

$$O_\tau(a) = ({}^{k-1}C_0 + {}^{k-1}C_1 + \cdots +^{k-1} C_{k-1}) \\ + ({}^{k-3}C_0 +^{k-3} C_1 + \cdots +^{k-3} C_{k-3}) + 1 = 5.2^{k-3} + 1.$$

If $a \notin A_i$ for any $1 \leq i \leq k$, then X is the only open set containing a.
Consequently $O\ (a) = 1$.

Theorem 7. *Let τ be a topology on set X such that all non-trivial members of τ are generated by a family $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of non-empty subsets of X such that (i) union of $A_1, A_2, \ldots, A_k$ is not X. (ii) Two sets A_1 and A_2 of*

$\mathcal{A}$ intersect with $A_1 \subset A_2$ and are disjoint from each of the remaining mutually disjoint members of $\mathcal{A}$. Then

$$O_\tau(a) = \begin{cases} 2^{k-2}+1 & \text{if } a \in A_2 - A_1, \\ 2^{k-1}+1 & \text{if } a \in A_1, \\ 3.2^{k-3}+1 & \text{if } a \in A_i,\ 3 \le i \le k, \\ 1 & \text{if } a \notin A_i,\ 1 \le i \le k. \end{cases}$$

Proof. Let $a \in A_2 - A_1$, then

$$O_\tau(a) = {}^{k-2}C_0 +^{k-2} C_1 + \dots +^{k-2} C_{k-2} + 1 = 2^{k-2} + 1.$$

If $a \in A_1$ then

$$O_\tau(a) = 2(^{k-2}C_0 +^{k-2} C_1 + \dots +^{k-2} C_{k-2}) + 1 = 2^{k-1} + 1.$$

If $a \in A_i$, for some i; $3 \le i \le k$ then

$$O_\tau(a) = (^{k-2}C_0 +^{k-2} C_1 + \dots +^{k-2} C_{k-2}) \\ + (^{k-3}C_0 +^{k-3} C_1 + \dots +^{k-3} C_{k-3}) + 1 = 32^{k-3} + 1.$$

If $a \notin A_i$ for any i $1 \le i \le k$, $O_\tau(a) = 1$.

With a restriction on generators of non-trivial members of a topology,the next two theorems provides a partial generalization of theorems 6 and 7 respectively.

Theorem 8. *Let τ be a topology on set X such that all non-trivial members of τ are generated by a family $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ of non-empty subsets of X such that (i) union of $A_1, A_2, \dots, A_k$ is not X. (ii) Sets $A_1, A_2, \dots, A_r$ of family $\mathcal{A}$ mutually intersect in same elements but is not equal to any A_i, $1 \le i \le r$ and are disjoint from each of the remaining $(k-r)$ mutually disjoint members of family $\mathcal{A}$. Then*

$$O_\tau(x) = \begin{cases} 2^{k-1}+1 & \text{if } x \in A_1 \cup A_2 \cup \dots \cup A_r - A_1 \cap A_2 \cap \dots A_r, \\ 2^{k}+1 & \text{if } x \in A_1 \cap A_2 \cap \dots A_r, \\ 2^{k-1}+2^{k-r-1}+1 & \text{if } x \in A_i,\ r+1 \le i \le k, \\ 1 & \text{if } x \notin A_i,\ 1 \le i \le k. \end{cases}$$

Proof. Let $x \in A_1 \cup A_2 \cup \dots \cup A_r - A_1 \cap A_2 \cap \dots \cap A_r$. Then

$$O_\tau(x) = {}^{k-1}C_0 +^{k-1} C_1 + \dots +^{k-1} C_{k-1} + 1 = 2^{k-1} + 1.$$

Let $x \in A_1 \cap A_2 \cap \cdots \cap A_r$. Then

$$O_\tau(x) = [({}^{k-1}C_0 + {}^{k-1}C_1 + \cdots + {}^{k-1}C_{k-1}) + ({}^{k-2}C_0 + {}^{k-2}C_1 + \cdots + {}^{k-2}C_{k-2}) + \cdots + ({}^{k-r}C_0 + {}^{k-r}C_1 + \cdots + {}^{k-r}C_{k-r})] + ({}^{k-r}C_0 + {}^{k-r}C_1 + {}^{k-r}C_{k-r}) + 1 = 2^k + 1.$$

If $x \in A_{r+1} \cup A_{r+2} \cup \cdots \cup A_k$, then

$$O_\tau(x) = ({}^{k-1}C_0 + {}^{k-1}C_1 + \cdots + {}^{k-1}C_{k-1}) + ({}^{k-r-1}C_0 + \cdots + {}^{k-r-1}C_{k-r-1}) + 1 = 2^{k-1} + 2^{k-r-1} + 1.$$

If $x \notin A_1 \cup A_2 \cup \cdots \cup A_k$, then $O_\tau(x) = 1$.
Hence the result.

Theorem 9. *Let τ be a topology on set X such that all non-trivial members of τ are generated by a family $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of non-empty subsets of X such that (i) union of $A_1, A_2, \ldots, A_k$ is not X. (ii) Sets $A_1, A_2, \ldots, A_r$ of X of family $\mathcal{A}$ mutually intersect in A_1 and are disjoint from each of the remaining $(k-r)$ mutually disjoint members of family $\mathcal{A}$. Then*

$$O_\tau(x) = \begin{cases} 2^{k-2} + 1 & \text{if } x \in A_1 \cup A_2 \cup \cdots \cup A_r - A_1, \\ 2^{k-1} + 1 & \text{if } x \in A_1 \cap A_2 \cap \ldots A_r, \\ 2^{k-2} + 2^{k-r-1} + 1 & \text{if } x \in A_i,\ r+1 \le i \le k, \\ 1 & \text{if } x \notin A_i,\ 1 \le i \le k. \end{cases}$$

Proof. Let $x \in A_1 \cup A_2 \cup \cdots \cup A_r - A_1$. Then

$$O_\tau(x) = {}^{k-2}C_0 + {}^{k-2}C_1 + \cdots + {}^{k-2}C_{k-2} + 1 = 2^{k-2} + 1.$$

Let $x \in A_1$. Then

$$\emptyset_\tau(x) = [({}^{k-2}C_0 + {}^{k-2}C_1 + \cdots + {}^{k-2}C_{k-2}) + ({}^{k-3}C_0 + {}^{k-3}C_1 + \cdots + {}^{k-3}C_{k-3}) + \cdots + ({}^{k-r}C_0 + {}^{k-r}C_1 + \cdots + {}^{k-r}C_{k-r})] + ({}^{k-r}C_0 + {}^{k-r}C_1 + {}^{k-r}C_{k-r}) + 1 = 2^{k-1} + 1.$$

If $x \in A_{r+1} \cup A_{r+2} \cup \cdots \cup A_k$, then

$$O_\tau(x) = ({}^{k-2}C_0 + {}^{k-2}C_1 + \cdots + {}^{k-2}C_{k-2}) + ({}^{k-r-1}C_0 + \cdots + {}^{k-r-1}C_{k-r-1}) + 1 = 2^{k-2} + 2^{k-r-1} + 1.$$

If $x \notin A_1 \cup A_2 \cup \cdots \cup A_k$, then $O_\tau(x) = 1$.
Hence the result.

Theorem 10. *Let τ be a topology on set X generated by k non-empty mutually disjoint subsets $A_1, A_2, \ldots, A_k$ of X and whose union is not X. Then*

$$O_\tau(A_i) \neq O_\tau(\overline{A_i}) + 2 \quad \text{for all } i,\ 1 \leq i \leq k.$$

Proof. Now $(X, \tau)$is connected and $\overline{A_i} = X - A_j$ for some $j \neq i$, $1 \leq j \leq k$. Therefore $\overline{X - \overline{A_i}} = \overline{A_j}$. Hence A_i and $X - \overline{A_i}$ are not dense in X.
So, $O_\tau(A_i) \neq O_\tau\left(\overline{A_i}\right) + 2$ by theorem 16 [5].

Theorem 11. *Let (X, τ) be a topological space where τ is generated by k non-empty mutually disjoint subsets $A_1, A_2, \ldots, A_k$ of X and whose union is not X. Let A be any open subset of X. Then*

1. $O_\tau\left(\overline{A}\right) = 1$
2. *If* $O_\tau(A_i) = O_\tau(\overline{A_i})$ *then* $A = X$.

Proof. (1) Now $A = \bigcup_{i=1}^{r} A_i, 1 \leq r \leq k$ or $\overline{A} = \bigcup_{j=1}^{r} A_j$.
Hence the only open set containing $\overline{A}$ is X.
So, $O_\tau\left(\overline{A}\right) = 1$.
(2) Since A is open, the result follows from (i).

Theorem 12. *Let τ be a partition topology on set X generated by partition $A_1, A_2, \ldots, A_k$ of X. Then order of each element of X related to τ is same and number of members in τ is twice the order of any element of X.*

Proof. Number of members of τ is ${}^kC_0 +^k C_1 + \cdots +^k C_k = 2^k$.
Let $x \in X$ then $x \in A_i$, for some i, $1 \leq i \leq k$

$$O_\tau(x) = O_\tau(A_i) = {}^{k-1}C_0 + {}^{k-1}C_1 + \cdots + {}^{k-1}C_{k-1} = 2^{k-1}$$

Hence the theorem.

Theorem 13. *Let τ be a topology on a finite set X whose non-empty members are all subsets of X containing x. Then there exists a topology $\tau\prime$ on X finer than τ such that $O_\tau(x) = O_{\tau\prime}(x)$.*

Proof. Let X contain n elements then τ has 2^{n-1} non-empty sets each containing x, or $O_\tau(x) = 2^{n-1}$.
Let Y and Z be non-empty disjoint subsets of X such that $X = Y \bigcup Z$.
Let $x \in Y$ and let Y have n_1 elements and Z have n_2 elements where $n = n_1 + n_2$.

Then $O_{\tau_Y}(x) = ^{n_1-1}C_0 + ^{n_1-1}C_1 + \cdots + ^{n_1-1}C_{n_1-1} = 2^{n_1-1}$.

Now Z intersects every member of τ of the form $\{x, z\}$, $z \in Z$ in singleton $\{z\}$.

Hence the relative topology τ_Z on Z is discrete.

Let $\tau\prime$ be a topology on X generated by non empty members of $\tau_Y \cup \tau_Z$. Then

$$\begin{aligned} O_{\tau\prime}(x) =& [^{n_2}C_0 + ^{n_2}C_1 + \cdots + ^{n_2}C_{n_2}] + [^{n_2}C_0 + ^{n_2}C_1 + \cdots + ^{n_2}C_{n_2}] + \ldots 2^{n_1-1} \\ =& 2^{n_2}.\ 2^{n_1-1} \\ =& 2^{n-1} \end{aligned}$$

or $O_{\tau\prime}(x) = O_\tau(x)$.

As number of non-empty members of τ is $O_\tau(x)$, $\tau \subset \tau\prime$.

Theorem 14. *Let τ be a topology on a finite set X whose non-empty members are all subsets of X containing x. Then $O\tau(x) = O_{\tau\prime}(x)$ where $\tau\prime$ is discrete topology on X.*

Proof. Obvious.

Theorem 15. *Let τ be a topology on a finite set X whose non-empty members of τ are all subsets of X containing x. Then for each element $a \in X$, $a \neq x$, there exist a topology $\tau\prime$ on X such that $\tau \subset \tau\prime$ and $O_{\tau\prime}(a) = O_\tau(a) + 1$.*

Proof. Let X contain n elements and let $a \in X$; $a \neq x$.Then

$$O_\tau(a) = ^{n-2}C_0 + ^{n-2}C_1 + \cdots + ^{n-2}C_{n-2} = 2^{n-2}.$$

Let $X = Y \bigcup Z$ where $Z = \{a\}$ and $Y \bigcap Z = \emptyset$.

Consider the topology $\tau\prime$ on X generated by $\tau_Y \cup \tau_Z$.

As τ_Y contains all subsets of Y containing x and $\tau_Z = \{\emptyset, \{a\}\}$.

$O_{\tau\prime}(a) = 2^{n-1} + 1$ or $O_{\tau\prime}(a) = O_\tau(a) + 1$ Obviously $\tau \subset \tau\prime$.

The above results lead to a very interesting situation where the intersection of all the topologies on a finite set X derived from all the possible partitions of X consisting of two members happens to be a topology on X whose non empty members are generated by a single fixed but arbitrary element of X. The next theorem proves our contention.

Theorem 16. *Let τ be a topology on a finite set X consisting of n elements and whose non-empty members are all subsets of X containing x. Then there exist $2^{n-1} - 1$ number of distinct topologies $\tau_i\prime s$; $1 \leq i \leq 2^{n-1} - 1$ such that for each i, $O_{\tau_i}(x) = O_\tau(x)$ and $\tau \subset \tau_i$. Moreover $\tau = \bigcap_i \tau_i$, $1 \leq i \leq 2^{n-1} - 1$.*

Proof. Now number of partitions of X consisting of two members is

$$^{n-1}C_0 +^{n-1} C_1 +^{n-1} C_{n-2} = 2^{n-1} - 1.$$

By Theorem 5, each such partition gives rise to a distinct topology τ_i where $\tau \subset \tau_i$ and $O_{\tau_i}(x) = O_\tau(x)$, $1 \leq i \leq 2^{n-1} - 1$.

Now $\tau \subset \bigcap_i \tau_i$, $1 \leq i \leq 2^{n-1} - 1$.

Let $\{Y_i, Z_i\}$ be the partition corresponding to topology τ_i on X and $x \in Y_i$, $1 \leq i \leq 2^{n-1} - 1$.

Suppose there is some $A \in \bigcap_i \tau_i$, but $A \notin \tau$ or $A \not\subset Y_i$, $1 \leq i \leq 2^{n-1} - 1$.

Hence

$$A \subset Z_i \text{ for } 1 \leq i \leq 2^{n-1} - 1. \quad (1)$$

By Theorem 7, there exists $\tau_i\prime$s; for which $Z_i\prime$s are singletons. For such $Z_i\prime$s, (1) is not true.

Therefore $\tau = \bigcap_i \tau_i$, $1 \leq i \leq 2^{n-1} - 1$.

1. Some Examples

Example 1. Let $X = \{a, b, c, d, e, f, g, h, i, j, k\}$ and let $A_1 = \{a, b, c\}$, $A_2 = \{c, d, e\}$, $A_3 = \{c, f, g\}$, $A_4 = \{h, i\}$ be generators of non-trivial members of a topology τ on X, where

$$\tau = \{\emptyset, \{a,b,c\}, \{a,b,c,d,e\}, \{a,b,c,f,g\}, \{a,b,c,h,i\}, \{a,b,c,d,e,f,g\},$$
$$\{a,b,c,d,e,h,i\}, \{a,b,c,f,g,h,i\}, \{a,b,c,d,e,f,g,h,i\}, \{c,d,e\}, \{c,d,e,f,g\},$$
$$\{c,d,e,h,i\}, \{c,d,e,f,g,h,i\}, \{c,f,g\}, \{c,f,g,h,i\}, \{h,i\}, \{c\}, \{c,h,i\}, X\}.$$

Here $O_\tau(x) = 2^{4-1} + 1 = 9$ for $x \in A_1 \cup A_2 \cup A_3 - A_1 \cap A_2 \cap A_3$,

$$O_\tau(x) = 2^4 + 1 = 17 \text{ for } x \in A_1 \cap A_2 \cap A_3,$$

$$O_\tau(x) = 2^{4-1} + 2^{4-3-1} + 1 = 10 \text{ for } x \in A_4,$$

and $O_\tau(x) = 1$ for $x \notin A_1 \cup A_2 \cup A_3 \cup A_4$.

Example 2. Let$X = \{a, b, c, d, e\}$ and let τ be a topology on X whose non-empty members are all subsets of X containing a. Then $O_\tau(a) = 2^{5-1} = 16$.
Let $Y = \{a, b, c\}$ and $Z = \{d, e\}$. Then
$\tau_Y = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, Y\}$ and $\tau_Z = \{\emptyset, \{d\}, \{e\}, Z\}$.
Let $\tau\prime$ be generated by non-empty members of $\tau_Y \cup \tau_Z$. Then

$$\tau\prime = \{\emptyset, \{a\}, \{d\}, \{e\}, \{a,b\}, \{a,c\}, \{d,e\}, \{a,b,c\}, \{a,d\}, \{a,e\}, \{a,d,e\}, \{a,b,d\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}, \{a,c,e\}, \{a,b,c,e\}, \{a,b,d,e\}, \{a,c,d,e\}, X\}.$$

Here $O_{\tau\prime}(a) = 16$ and $\tau \subset \tau\prime$.

Example 3. Let $X = \{a, b, c\}\}$. Let $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ be a topology on X where non-empty members of τ are all subsets of X containing a.

The two member partitions of X are $\{\{a\}, \{b, c\}\}, \{\{a, b\}, \{c\}\}, \{\{a, c\}, \{b\}\}$ and the corresponding topologies generated by the unions of relative topologies on members of same partition are:

$$\tau_1 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\},$$
$$\tau_2 = \{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, X\},$$
$$\tau_3 = \{\emptyset, \{a\}, \{b\}, \{a,c\}, \{a,b\}, X\},$$

and

$$\tau_1 \cap \tau_2 \cap \tau_3 = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, X\} = \tau.$$

Also $O_{\tau_1}(a) = O_{\tau_2}(a) = O_{\tau_3}(a) = O_\tau(a) = 4$, and $O_{\tau_2}(c) = 3 = O_\tau(c) + 1$; $O_{\tau_3}(b) = 3 = O_\tau(b)+1$, showing that Theorem 15 holds in case of the topologies derived from partitions{{a,b},{c}} and {{a,c},{b}} where one member of the partition contains a singleton set alone other than {a}.

2. Lattice and Order in a Topological Space

Let τ be a topology on a finite set X.

Define for each $n \in N$, $Y_n = \{a \in X \mid O_\tau(a) = n\}$.

Then $X = \bigcup_{n\in N} Y_n$ and $Y_n \bigcap Y_m = \emptyset$ for $n \neq m$. Let $\mathcal{Y} = \{Y_n\}_{n\in N}$.

Define a relation '$\leq$' on Y as follows $Y_m \leq Y_n$ iff n is a multiple of m. Then '$\leq$' is a partial order relation on $\mathcal{Y}$.

Further define two binary operations $+$ and on Y as follows

$$Y_i + Y_j = \{a \in X \mid O_\tau(a) = LCM(O_\tau(y_i), O_\tau(y_j))\text{where } y_i \in Y_i, y_j \in Y_j\}$$

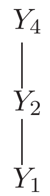
and

$$Y_i.Y_j = \{a \in X \mid O_\tau(a) = HCF(O_\tau(y_i), O_\tau(y_j))\text{where } y_i \in Y_i, y_j \in Y_j\}.$$

Let LCM and HCF of orders of all the elements of X relative to τ be respectively l and h. Suppose there exist elements in X whose orders relative to τ are l and h. Denote Y_h and Y_l by 0 and 1 respectively. Then $< \mathcal{Y}, +, . >$ is a distributive lattice and 0 and 1 are least and greatest elements of the lattice.

The following examples present illustrations of above discussion.

Example 4. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ be a topology on X. Then $\mathcal{Y} = \{Y_1, Y_2, Y_4\}$ where $Y_1 = \{d\}$, $Y_2 = \{c\}$, $Y_4 = \{a, b\}$. Then $< \mathcal{Y}, +, . >$ is a distributive lattice.



Example 5. Let $X = \{a, b, c, d, e\}$ and

$$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, e\}, \{a, b, c, e\}, X\}$$

be a topology on X. Here $O_\tau(a) = O_\tau(b) = 6$, $O_\tau(c) = O_\tau(e) = 3$ and $O_\tau(d) = 1$.

Let $\mathcal{Y} = \{Y_1, Y_3, Y_6\}$. Then $< \mathcal{Y}, +, . >$ is a distributive lattice where Y_1 and Y_6 are least and greatest elements of $\mathcal{Y}$.

Example 6. Let $X = \{a, b, c, d, e, f\}$ and

$$\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, \{a, b, c, d\}, X\}.$$

Here orders ofelements of X relative to τ are 1, 2 and 6.Then $\mathcal{Y} = \{Y_1, Y_2, Y_6\}$ and $< \mathcal{Y}, +, . >$ is a distributive lattice where Y_1 and Y_6 are least and greatest elements of $\mathcal{Y}$.

Example 7. Let (X, τ) be a discrete topological space where order of each element of X relative to τ is k. Then $\mathcal{Y} = \{Y_k\}$ and $< \mathcal{Y}, +, ., \prime, 0, 1 >$ forms a trivial boolean algebra.

Example 8. Partition topology on a set also gives rise to a trivial boolean algebra.

Example 9. Let τ be a topology on a finite set X whose non trivial members are generated by k mutually disjoint proper subsets of x such that union of these sets is not X.

Then by theorem 1; order of any element of X relative to τ is $2^{k-1} + 1$ or 1.

Hence $\mathcal{Y} = \{Y_1, Y_{2^{k-1}+1}\}$ and $0 = Y_1$, $1 = Y_{2^{k-1}+1}$. Thus $< \mathcal{Y}, +, ., \prime, 0, 1 >$ is a two element boolean algebra.

References

[1] J.L. Kelly, *General Topology*, Van Nostrand Reinhold Company, New York (1969).

[2] S. Lipschutz, *Schaum's Outline of Theory and Problems of General Topology* (1965).

[3] James R. Munkers, *Topology, A First Course*, PHI (1987).

[4] K. Shrivastava, S. Duraphe, Order of elements in a topological space, *Ultra Science*, **18**, No. 2 (2006), 191.

[5] K. Shrivastava, S. Duraphe, Order of sets in a topological space, *News Bull. Cal. Math. Soc.*, **30**, No-s: 7-9 (2007), 10-14.

[6] Tremblay, Manohar, *Discrete Mathematical Structure with Applications to Computer Science*, Tata McGraw Hills.