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STABILITY AND HOPF BIFURCATION FOR A DELAYED MODEL WITH CLASSICAL SAVING

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Abstract: In this paper, we study the stability and bifurcation of the Solow-Swan model with classical saving and time delay. We find that there are stability switches and Hopf bifurcations occur when the delay passes through a sequence of critical values.

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Key Words: solow-swan model, classical saving, delay, Hopf bifurcation

1. Introduction

The study of delayed differential equations that arise in economic growth models has received much attention in the last decade. In particular, delayed differential equations have proven to be useful in understanding the dynamics of capital accumulation. Asea and Zak [1] considered the Solow-Swan model with a time lag introduced in production and showed that a delay causes cycles in the economy. In this paper, we propose to modify Zak's model by relaxing the standard assumption of constant saving rate and assuming the saving rate to be a function of the profit rate. Within this framework, the stability of the model is investigated by analyzing the associated characteristic transcendental equation. Conditions on the asymptotic stability of the solution are established. Moreover, it is proved that a sequence of Hopf bifurcations occur at the positive equilibrium as the delay increases. A final comment. In recent years, some

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authors (see, e.g., Ferrara and Guerrini [3-5], Guerrini [6-13]) have investigated the implications of studying economic growth models under a variable population growth law hypothesis. For future research, we aim to investigate the consequence of introducing a time lag in these models.

2. The Model with Classical Saving

We consider the basic Solow-Swan model [15-16] with no population growth and relax the standard assumption of constant saving rate. The saving rate s is now a function of the profit rate ρ, i.e. $s = s(\rho)$, with $s' > 0$. Only capitalists save due to the fact that workers do not have sufficient income to be able to maintain anything but a negligible saving rate. For a competitive economy, ρ will be equal to the marginal product of capital, i.e. $\rho = f'(k_t)$, where k_t denotes the individual's capital stock. Furthermore, $\partial s[f'(k_t)]/\partial k_t < 0$ yields the saving rate to be falling with rising per capita capital. The production function is assumed to be of Cobb-Douglas type, $f(k_t) = k_t^\alpha$, $\alpha \in (0,1)$. Within this framework, the model happens to be described by

$$\dot{k} = s\left(k_t^{\alpha-1}\right) k_t^\alpha - \delta k_t, \tag{1}$$

where $\delta \in (0,1)$ represents the capital depreciation rate. For simplicity, let us assume $s\left(k_t^{\alpha-1}\right) = k_t^{\alpha-1}$. Then Eq. (1) becomes

$$\dot{k_t} = k_t^{2\alpha-1} - \delta k_t. \tag{2}$$

A steady state is a path for k_t such that $\dot{k_t} = 0$. Setting the left hand side of Eq. (2) equal to zero, we derive that there is a unique positive equilibrium $k_* = \delta^{1/(2\alpha-2)}$. Taking a first order Taylor expansion of Eq. (2) around k_* yields $\dot{k_t} = -(2-2\alpha)\delta(k_t - k_*)$, whose solution is $k_t = k_* + (k_0 - k_*)e^{-(2-2\alpha)\delta t}$. Hence, for any k_t in the neighborhood of k_*, the economic system converges to k_*, i.e. the model is asymptotically stable.

3. The Model with Classical Saving and Time Delay

We modify our model by assuming the rate of change in capital stock at moment $T \geq 0$ to be a function of the productive capital stock at $t - T$. As a result, Eq. (2) is replaced by a differential equation with a delay parameter:

$$\dot{k_t} = k_{t-T}^{2\alpha-1} - \delta k_t, \tag{3}$$

for some initial function $k_t = \phi_t$, $t \in [-T, 0]$. Instead of an initial point value for an ordinary differential equation, the initial function ϕ_t is required, which is defined over the range of time delimited by the delay. Of course, steady states of Eq. (3) coincide with the corresponding points for zero delay, $T = 0$. To examine the stability of the positive critical point k_*, we linearize the right hand-side of Eq. (3) at k_* and then make the change of variables $x_t = k_t - k_*$. In this way, k_* is moved to $x_* = 0$. Hence, the resulting equation is

$$\dot{x}_t = (2\alpha - 1)\delta x_{t-T} - \delta x_t. \tag{4}$$

By substituting candidate solutions of the form $e^{-\lambda T}$ into Eq. (4), we get

$$\lambda + \delta - (2\alpha - 1)\delta e^{-\lambda T} = 0. \tag{5}$$

This is the characteristic equation of Eq. (4) at k_*. It is a quasi-polynomial, which exhibits an infinite number of (complex) roots. Obviously, when $T = 0$, $x_* = 0$ is asymptotically stable.

Proposition 1. *Let $|2\alpha - 1| \geq 1/(1 + \delta)$. Then $x_* = 0$ is asymptotically stable for all delay $T \geq 0$.*

Proof. First we show that all roots of Eq. (5) have negative real parts. Let $i\omega_0$ ($\omega_0 > 0$) be a root of Eq. (5). Then we have

$$i\omega_0 + \delta - (2\alpha - 1)\delta e^{-i\omega_0 T} = 0.$$

Separating the real and imaginary parts gives

$$1 - (2\alpha - 1)\cos\omega_0 T = 0, \quad \omega_0 + (2\alpha - 1)\delta \sin\omega_0 T = 0,$$

so that

$$\omega_0^2 = 1 - (2\alpha - 1)^2(1 + \delta)^2.$$

Thus, under inequality $1 - (2\alpha - 1)^2(1 + \delta)^2 \leq 0$, i.e. $|2\alpha - 1| \geq 1/(1 + \delta)$, Eq. (5) has no imaginary roots. Since $\lambda = 0$ is not a root of Eq. (5) and zero is asymptotically stable when $T = 0$, applying the Corollary 2.4 in Ruan and Wei [14], we obtain that all roots of Eq. (5) have negative real parts. Then the zero equilibrium is asymptotically stable for all delay $T \geq 0$, completing the proof. □

Remark 1. $|2\alpha - 1| \geq 1/(1 + \delta)$ yields the inequalities $\alpha \leq \delta/2(1 + \delta)$ and $(2 + \delta)/2(1 + \delta) \leq \alpha < 1$.

Proposition 2. *Let* $|2\alpha - 1| < 1/(1+\delta)$. *Define*

$$\omega_0 = \sqrt{1 - (2\alpha - 1)^2(1+\delta)^2},$$

$$T_j = \frac{1}{\omega_0}\left[arctan\left(-\frac{\omega_0}{\delta}\right) + 2j\pi\right], \quad j = 0, 1, 2, ...$$

If $T \in [0, T_0)$, *all roots of Eq.* (5) *have negative real parts. If* $T = T_0$, *all roots of Eq.* (5), *except* $\pm\, i\omega_0$, *have negative real parts. If* $T \in (T_j, T_{j+1})$ *for* $j = 0, 1, 2, ...$, *Eq.* (5) *has* $2(j+1)$ *roots with positive real parts.*

Proof. From the previous discussion, we know that if $|2\alpha - 1| < 1/(1+\delta)$, Eq. (5) has a pair of purely imaginary roots $\pm i\omega_0$ at the sequence of critical values T_j. Suppose $\lambda_j(T) = \nu_j(T) + i\omega_j(T)$ denotes the root of Eq. (5) near $T = T_j$ satisfying $\nu_j(T_j) = 0$ and $\omega_j(T_j) = \omega_0$, $j = 0, 1, 2, ...$ Then differentiating the characteristic equation (5) with respect to T gives

$$\frac{d\lambda}{dT} + (2\alpha - 1)\delta e^{-\lambda T}\left(T\frac{d\lambda}{dT} + \lambda\right) = 0,$$

that is

$$\left(\frac{d\lambda}{dT}\right)^{-1} = -\frac{1}{\lambda(\lambda+\delta)} - \frac{T}{\lambda}.$$

Thus

$$sign\left[\left.\frac{d\nu_k(T)}{dT}\right|_{i\omega_0}\right] = sign\left[Re\left.\left(\frac{d\lambda}{dT}\right)^{-1}\right|_{i\omega_0}\right]$$
$$= sign\left[Re\left.\left(-\frac{1}{\lambda(\lambda+\delta)} - \frac{T}{\lambda}\right)\right|_{i\omega_0}\right] = sign\left(\frac{1}{\omega_0^2 + \delta^2}\right) > 0.$$

So, by Rouche's Theorem [2], the root $\lambda_j(T)$ near T_j crosses the imaginary axis from the left to the right as T continuously varies from a number less than T_j to one greater than T_j. The statement is now immediate. □

Remark 2. $|2\alpha - 1| < 1/(1+\delta)$. yields $\delta/2(1+\delta) < \alpha < (2+\delta)/2(1+\delta)$.

Spectral properties of Eq. (5) lead immediately to the properties of the positive equilibrium k_* for Eq. (3). Therefore, summarizing the above results, we obtain the following Theorem.

Theorem 1.

1. *If* $|2\alpha - 1| \geq 1/(1+\delta)$ *is satisfied, then* k_* *is asymptotically stable for all* $T \geq 0$.

2. *If* $|2\alpha - 1| < 1/(1+\delta)$ *is satisfied, then* k_* *is asymptotically stable for* $T \in [0, T_0)$ *and unstable for* $T > T_0$*, and Eq.* (3) *undergoes a Hopf bifurcation at* k_* *when* $T = T_j$*, for* $j = 0, 1, 2, ...$

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