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TIME DELAY IN A NEOCLASSICAL GROWTH MODEL WITH DIFFERENTIAL SAVING

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Abstract: In this paper, we undertake the analysis of an extension of the one sector model of Solow-Swan with delay assuming that capital accumulation is generated by the savings behavior of two income groups with different saving propensities. The resulting model happens to have an Hopf bifurcation when the delay passes a critical value.

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Key Words: solow-swan model, Kaldor-Pasinetti, delay, Hopf bifurcation

1. Introduction

In [16] Kalecki considered the issue that a delay in production could cause cycles in the economy. His approach was revived in the work of Asea and Zak [1], who showed that the time delay in production of investment capital induces cyclic behaviour in the model. In this paper, we wish to modify their work by introducing different but constant saving propensities attached to factor shares. The resulting dynamic system is proved to generate a Hopf bifurcation at the positive equilibrium as the delay increases. As well, the length of time delay preserving the stability of the positive equilibrium is estimated. Finally, we recall that recently some authors have considered neoclassical models under the assumption of variable population growth laws (see, e.g., Ferrara and Guerrini [2-4], Guerrini [6-13]). In the future we propose to formulate these models with

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a delay parameter and examine if cycles may arise.

2. The Model with Kaldor-Pasinetti Saving

We consider the extension by Kaldor [15] and Pasinetti [17] of Solow-Swan neoclassical growth model [18-19] to an economy with both workers and capitalists, where the economy's underlying production function takes the Cobb-Douglas form $f(k_t) = k_t^{\alpha}$, $\alpha \in (0,1)$. Here k_t denotes capital per worker. For simplicity, we neglect capital depreciation. Capitalists receive capital income $f'(k_t)k_t$ and have a propensity to save s_c, while workers are paid wages $f(k_t) - f'(k_t)k_t$ and have a lower propensity to save $s_w < s_c$. Under these assumptions, the law of motion of capital is given by the following differential equation

$$\dot{k}_t = [s_w + \alpha(s_c - s_w)]k_t^{\alpha} - nk_t, \tag{1}$$

where $n > 0$ represents the constant labor force growth rate. Eq. (1) is mathematically identical to the fundamental equation of the Solow-Swan model, so we may use the same conclusions here. Therefore, Eq. (1) has a unique positive equilibrium, given by $k_* = \{[s_W + \alpha(s_c - s_W)]/n\}^{1/(1-\alpha)}$, which is globally asymptotically stable. All solutions starting near the steady state remain near the steady state for all the time and furthermore they tend towards k_* as t grows to infinity.

3. The Model with Kaldor-Pasinetti Saving and Time Delay

Here, we consider the delayed variant of our model formulated in terms of a time lag $T \geq 0$ in the production technology. The resulting capital accumulation dynamic equation is now a differential equation with time delay

$$\dot{k}_t = [s_W + \alpha(s_c - s_W)]k_{t-T}^{\alpha} - nk_{t-T}, \tag{2}$$

for some initial function $k_t = \phi_t$, $t \in [-T, 0]$. It is immediate that Eqs. (1) and (2) have the same equilibria. By the translation $x_t = k_t - k_*$ and considering the Taylor expansion of the right members from (2) until the first order, we obtain

$$\dot{x}_t = (\alpha - 1)nx_{t-T}. \tag{3}$$

The characteristic equation of Eq. (3) is

$$\lambda + (1-\alpha)ne^{-\lambda T} = 0. \tag{4}$$

We recall the sufficient condition for local asymptotical stability of the zero solution to be that the real parts of the eigenvalues are negative. Now, Eq. (4) is a function of λ and so are its roots. Since the zero solution is asymptotically stable for $T = 0$, by continuity we expect it to remain asymptotically stable with small values of T. All the eigenvalues λ of the above equation should lie in the left half of the complex λ-plane for the equilibrium point to be stable. That is, if for all the eigenvalues $Re\lambda < 0$, the corresponding solution is stable. On the other hand, even if one of the eigenvalues λ has a positive real part then the solution is unstable.

Proposition 1. *The equilibrium k_* is locally asymptotically stable if $0 < T < T_{bi} = \pi/2(1-\alpha)n$ and loses stability at $T = T_{bi}$, where it undergoes a Hopf bifurcation.*

Proof. An application of the Mikhailov criterion [5] shows that $x_t = 0$ is locally asymptotically stable for $T < T_{bi} = \pi/2(1-\alpha)n$ (see e.g. Hale [14]). Moreover, for $T = T_{bi}$ Eq. (4) has a pair of pure imaginary roots $\pm\omega = \pm\pi/2T_{bi}$. Calculating the first derivative of the characteristic function with respect to λ at an eigenvalue equal to the purely imaginary roots of (4) we easily find that the roots $\pm\omega$ are simple. Substituting $\lambda = i\omega$ into Eq. (4) and differentiating with respect to T, we have

$$\frac{d\lambda}{dT} - (1-\alpha)ne^{-\lambda T}\left(T\frac{d\lambda}{dT} + \lambda\right) = 0.$$

Hence,

$$\left(\frac{d\lambda}{dT}\right)^{-1} = \frac{1}{\lambda(1-\alpha)ne^{-\lambda T}} - \frac{T}{\lambda}. \tag{5}$$

Using (4) into Eq. (5), we derive

$$Re\left[\left(\frac{d\lambda}{dT}\right)^{-1}_{\lambda=i\omega}\right] = \frac{1}{\omega^2} > 0, \text{ i.e. } \left.\frac{d\,(Re\lambda)}{dT}\right|_{\lambda=i\omega} > 0.$$

All roots crossing the imaginary axis at $i\omega$ cross from left to right as T increases. We can conclude that for $T = T_{bi}$ Hopf bifurcation occurs. □

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