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CHARACTERIZATIONS OF n-INNER PRODUCT SPACE

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Abstract: In this paper, Dunkl-Williams inequality is established for n-normed space. Some necessary and sufficient conditions for an n-normed space to be an n-inner product space are also given.

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1. Introduction and Preliminaries

Let X be a real vector space with $\dim(X) \geq n$, where n is a positive integer. We allow $\dim(X)$ to be infinite. A real-valued function $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle : X^{n+1} \to \mathbb{R}$ is called an *n-inner product* on X if the following conditions hold:

1. $\langle x, x | a_2, \ldots, a_n \rangle \geq 0$, with equality if and only if $x, a_2, \ldots, a_n$ are linearly dependent;
2. $\langle x, y | a_2, \ldots, a_n \rangle = \langle y, x | a_2, \ldots, a_n \rangle$;
3. $\langle x, y | a_2, \ldots, a_n \rangle = \langle x, y | a_{i_1}, \ldots, a_{i_n} \rangle$ for every permutation $(i_1, \ldots, i_n)$ of $(2, \ldots, n)$;
4. $\langle x, x | a_2, a_3, \ldots, a_n \rangle = \langle a_2, a_2 | x, a_3, \ldots, a_n \rangle$;
5. $\langle \alpha x, y | a_2, \ldots, a_n \rangle = \alpha \langle x, y | a_2, \ldots, a_n \rangle$ for every $\alpha \in \mathbb{R}$;
6. $\langle x + x', y | a_2, \ldots, a_n \rangle = \langle x, y | a_2, \ldots, a_n \rangle + \langle x', y | a_2, \ldots, a_n \rangle$.

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The pair $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ is then called an *n-inner product space.*

Under the same conditions, a real-valued function $\|\cdot, \ldots, \cdot\| : X^n \to \mathbb{R}$ is called an *n-norm* on X^n if the following conditions hold:

1. $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;
2. $\|x_1, \ldots, x_n\|$ is invariant under permutations of $x_1, \ldots, x_n$;
3. $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for all $\alpha \in \mathbb{R}$ and $x_1, \ldots, x_n \in X$;
4. $\|x_0 + x_1, x_2, \ldots, x_n\| \leq \|x_0, x_2, \ldots, x_n\| + \|x_1, x_2, \ldots, x_n\|$, for all $x_0, x_1, \ldots, x_n \in X$.

The pair $(X, \|\cdot, \ldots, \cdot\|)$ is then called an *n-normed space.*

Note that by the above definition, an n-norm is always non-negative. We also have $\|x_1, \ldots, x_{n-1}, x_n\| = \|x_1, \ldots, x_{n-1}, x_n + \alpha_1 x_1 + \ldots + \alpha_{n-1} x_{n-1}\|$ for all $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$.

Using the definitions above, it is easy to see that in an n-inner product space X, the formula

$$\|x_1, x_2, \ldots, x_n\| = \sqrt{\langle x_1, x_1 | x_2, \ldots, x_n \rangle} \tag{1.1}$$

defines an n-norm on X.

Various aspects of the theory of n-normed space and n-inner product space has been investigated more recently, for instance in [1, 4, 7]. A survey of the theory of n-inner product space can be found in [3].

In [2], Cho et al. shows that a necessary and sufficient condition for an n-normed space to be an n-inner product space is that the following extended parallelogram law:

$$\begin{aligned} &\|x + y, a_2, \ldots, a_n\|^2 + \|x - y, a_2, \ldots, a_n\|^2 \\ &\qquad = 2\left(\|x, a_2, \ldots, a_n\|^2 + \|y, a_2, \ldots, a_n\|^2\right) \end{aligned} \tag{1.2}$$

holds, in which case the n-inner product is given by

$$\langle x, y | a_2, \ldots, a_n \rangle = \frac{1}{4}\left(\|x + y, a_2, \ldots, a_n\|^2 - \|x - y, a_2, \ldots, a_n\|^2\right) \tag{1.3}$$

We call this a *characterization* of n-inner product space.

In [5], Dunkl and Williams introduced the following Dunkl-Williams inequality in an inner product space:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x - y\|}{\|x\| + \|y\|} \tag{1.4}$$

It has since been improved by various authors ([9, 10]).

In this paper, we will extend the above and related inequalities to n-normed space and n-inner product space. We also derive several necessary and sufficient conditions for an n-normed space to be an n-inner product space.

2. Main Results

For ease of exposition, throughout this section let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. All the results hold in the more general case of n-normed space with minor modifications.

Proposition 2.1. *The following Dunkl-Williams inequality*

$$\left\| \frac{x}{\|x,a\|} - \frac{y}{\|y,a\|}, a \right\| \le \frac{4\|x-y,a\|}{\|x,a\| + \|y,a\|} \tag{2.1}$$

holds for any $x, y, a \in X$.

Proof. We have

$$\begin{aligned} \|x,a\| \left\| \frac{x}{\|x,a\|} - \frac{y}{\|y,a\|}, a \right\| &\le \|x,a\| \left\| \frac{x}{\|x,a\|} - \frac{y}{\|x,a\|}, a \right\| \\ &\quad + \|x,a\| \left\| \frac{y}{\|x,a\|} - \frac{y}{\|y,a\|}, a \right\| \\ &= \|x-y,a\| + \big| \|y,a\| - \|x,a\| \big| \\ &\le 2\|x-y,a\| \end{aligned}$$

Similarly, we have

$$\|y,a\| \left\| \frac{x}{\|x,a\|} - \frac{y}{\|y,a\|}, a \right\| \le 2\|x-y,a\|$$

The result follows by adding the two inequalities above. □

Proposition 2.2. *Let Y be a 2-inner product space. The following Dunkl-Williams inequality*

$$\left\| \frac{x}{\|x,a\|} - \frac{y}{\|y,a\|}, a \right\| \le \frac{2\|x-y,a\|}{\|x,a\| + \|y,a\|} \tag{2.2}$$

holds for any $x, y, a \in Y$.

Proof. We have

$$\begin{aligned}\left\|\frac{x}{\|x,a\|}-\frac{y}{\|y,a\|},a\right\|^2 &= \left\langle \frac{x}{\|x,a\|}-\frac{y}{\|y,a\|},\frac{x}{\|x,a\|}-\frac{y}{\|y,a\|}\,\middle|\, a\right\rangle \\ &= 2-2\left\langle \frac{x}{\|x,a\|},\frac{y}{\|y,a\|}\,\middle|\, a\right\rangle \\ &= \frac{1}{\|x,a\|\|y,a\|}(2\|x,a\|\|y,a\|-2\langle x,y|a\rangle) \\ &= \frac{1}{\|x,a\|\|y,a\|}[2\|x,a\|\|y,a\|-(\|x,a\|^2+\|y,a\|^2 \\ &\quad -\|x-y,a\|^2)] \\ &= \frac{\|x-y,a\|^2-(\|x,a\|-\|y,a\|)^2}{\|x,a\|\|y,a\|}\end{aligned}$$

Therefore

$$\begin{aligned}&\|x-y,a\|^2-\left(\frac{\|x,a\|+\|y,a\|}{2}\right)^2\left\|\frac{x}{\|x,a\|}-\frac{y}{\|y,a\|}\right\|^2 \\ &\quad = \frac{(\|x,a\|-\|y,a\|)^2}{4\|x,a\|\|y,a\|}[(\|x,a\|+\|y,a\|)^2-\|x-y,a\|^2]\geq 0\end{aligned}$$

The result then follows. □

Later we will prove that inequality (2.2) is actually a necessary and sufficient condition for a 2-normed space to be a 2-inner product space. Again, this is also true for n-normed space and n-inner product space with obvious modifications. To prove that, we need several lemmas.

Lemma 2.3. *Let $x,y,a\in X$. Suppose the following statement is true*

$$\|x,a\|=\|y,a\| \text{ implies } \|\alpha x+\alpha^{-1}y,a\|\geq\|x+y,a\| \text{ for all } \alpha\neq 0 \tag{2.3}$$

Then we have the following

$$\|\alpha x+\alpha^{-1}y,a\|\geq\|x+y,a\| \text{ for all } \alpha\neq 0 \text{ implies } \|x,a\|=\|y,a\| \tag{2.4}$$

Proof. Suppose (2.3) is true. Suppose $\|\alpha x+\alpha^{-1}y,a\|\geq\|x+y,a\|$. If $\{x+y,a\}$ are linearly dependent, then we have $\|x,a\|=\|y,a\|$. Now suppose $\{x+y,a\}$ are linearly independent, then we only need to consider when each of $\{x,a\}$ and $\{y,a\}$ are linearly independent, otherwise (2.4) is vacuously true.

Let $\beta = \frac{\|y,a\|}{\|x,a\|}$ and $y = \beta y'$. Then $\|y', a\| = \|x, a\|$. Using triangle inequality, we have

$$\begin{aligned}\|x+y,a\| = \|x+\beta y',a\| &= \beta^{1/2}\|\beta^{-1/2}x+\beta^{1/2}y',a\| \\ &\geq \beta^{1/2}\|x+y',a\| = \|\beta^{1/2}x+\beta^{-1/2}y,a\| \geq \|x+y,a\|\end{aligned}$$

whence equality is only attained when $\beta^{1/2} = 1$, i.e. $\|x, a\| = \|y, a\|$ as required. □

Lemma 2.4. *For all $u, v, a \in X$ and $r > 0$, there exist no more than 2 distinct real numbers t such that $\|u + t(v - u), a\| = r$.*

Proof. If $\{u, v, a\}$ are linearly dependent, then the result follows. Suppose $\{u, v, a\}$ are linearly independent. Now suppose the assertion is false. Then we can find among the vectors $u+t(v-u)$, two linearly independent vectors p and q, and a vector $w = sp+(1-s)q$, $0 < s < 1$, such that $\|p, a\| = \|q, a\| = \|w, a\|$. By our assumption, we have $\{p, q, a\}$ are linearly independent, and therefore we have strict inequality in the following triangle inequality $\|w, a\| < s\|p, a\| + (1 - s)\|q, a\| < \|p, a\|$, a contradiction. Hence, the lemma is proven. □

We need the following result. In [6], Ficken proved it for the case of normed space, and we will extend his result. This is one characterization of an n-inner product space.

Theorem 2.5. *Let $x, y, a \in X$. The following statements are equivalent:*

1. *X is a 2-inner product space.*
2. *$\|x, a\| = \|y, a\|$ implies $\|\alpha x + \beta y, a\| = \|\beta x + \alpha y, a\|$ for all $\alpha, \beta \in \mathbb{R}$.*

Proof. (1) ⇒ (2): Suppose $\|x, a\| = \|y, a\|$. Then we have

$$\begin{aligned}\|\alpha x+\beta y,a\|^2 &= \alpha^2\|x,a\|^2 + 2\alpha\beta\langle x,y|a\rangle + \beta^2\|y,a\|^2 \\ &= \langle \beta x+\alpha y, \beta x+\alpha y|a\rangle = \|\beta x+\alpha y,a\|^2\end{aligned}$$

from which the conclusion follows.

(2) ⇒ (1): We will show that X is a 2-inner product space by showing that the extended parallelogram law (1.2) holds. Suppose we have (2). The result is immediate if $\{x, y, a\}$ are linearly dependent. Now suppose $\{x, y, a\}$ are linearly independent. Using the assumption,

$$\|x+y,a\| = \|x-y+2y,a\| = \left\|\frac{2\|y,a\|}{\|x-y,a\|}(x-y) + \frac{\|x-y,a\|}{\|y,a\|}y, a\right\|$$

$$= \left\| \frac{2\|y,a\|}{\|x-y,a\|}x + \frac{\|x-y,a\|^2 - 2\|y,a\|^2}{\|x-y,a\|\|y,a\|}y, a \right\|$$

i.e.

$$\|y,a\|\|x+y,a\|\|x-y,a\| = \left\|2\|y,a\|^2 x + (\|x-y,a\|^2 - 2\|y,a\|^2)y, a\right\| \quad (2.5)$$

Similarly, considering $\|y-x+2x,a\|$, $\|x+y-2y,a\|$ and $\|x+y-2x,a\|$, we have

$$\begin{aligned} \|y,a\|\|x+y,a\|\|x-y,a\| &= \left\|2\|y,a\|^2 x + (\|x-y,a\|^2 - 2\|x,a\|^2)y, a\right\| \\ &= \left\|2\|y,a\|^2 x + (2\|y,a\|^2 - \|x+y,a\|^2)y, a\right\| \\ &= \left\|2\|y,a\|^2 x + (2\|x,a\|^2 - \|x+y,a\|^2)y, a\right\| \end{aligned}$$

Applying Lemma 2.4 with $u = 2\|x,a\|^2 y$, $v-u = y$ and $r = \|y,a\|\|x+y,a\|\|x-y,a\|$, we find that among the four numbers

$$\begin{aligned} m &= \|x-y,a\|^2 - 2\|y,a\|^2 \\ n &= \|x-y,a\|^2 - 2\|x,a\|^2 \\ p &= 2\|y,a\|^2 - \|x+y,a\|^2 \\ q &= 2\|x,a\|^2 - \|x+y,a\|^2 \end{aligned}$$

there are at most two distinct numbers. It remains to check case by case. This is routine and parallelogram law follows immediately except for the case where $m = n$ and $p = q$. Then for this last case, without loss of generality, we may assume $\|x,a\| = \|y,a\|$ and $\|x+y,a\| = \|x-y,a\|$. The details are similar to that in [6].

In this last case, suppose $m > 0$, then applying triangle inequality to (2.5), we have the strict inequality $\|x+y,a\| < \|x-y,a\|$ since $\{x,y,a\}$ are linearly independent, a contradiction. Similarly if $m < 0$, we obtain $\|x+y,a\| > \|x-y,a\|$, also a contradiction. Hence $m = 0 = n$. Similarly $p = 0 = q$. Extended parallelogram law follows in all case, hence X is a 2-inner product space. □

The following result is needed to prove our main theorem. The case when X is a normed space is due to Lorch ([8]). We will extend the result. This also provides another characterization of n-inner product space.

Theorem 2.6. *Let $x, y, a \in X$ and $\alpha \in \mathbb{R}$. The following statements are equivalent:*

1. *X is a 2-inner product space.*

2. *$\|x,a\| = \|y,a\|$ implies $\|\alpha x + \alpha^{-1}y, a\| \geq \|x+y,a\|$ for all $\alpha \neq 0$.*

Proof. (1) $\Rightarrow$ (2): Suppose X is a 2-inner product space. Suppose $\|x,a\| = \|y,a\|$. Then we have

$$\begin{aligned}\|\alpha x + \alpha^{-1}y, a\|^2 &= \langle \alpha x + \alpha^{-1}y, \alpha x + \alpha^{-1}y|a\rangle \\ &= \alpha^2\|x,a\|^2 + 2\langle x,y|a\rangle + \alpha^{-2}\|y,a\|^2 \\ &= (\alpha^2+\alpha^{-2})\left(\frac{\|x,a\|^2+\|y,a\|^2}{2}\right) + 2\langle x,y|a\rangle \\ &\geq \|x,a\|^2 + \|y,a\|^2 + 2\langle x,y|a\rangle \\ &= \|x+y,a\|^2\end{aligned}$$

(2) $\Rightarrow$ (1): Suppose (2) is true. We will use Theorem 2.5. Suppose $\|x,a\| = \|y,a\|$. It suffices to show that for any $\theta \in \mathbb{R}$, $\|e^\theta x + e^{-\theta}y, a\| = \|e^{-\theta}x + e^\theta y, a\|$. Note that for any $\phi \in \mathbb{R}$,

$$\begin{aligned}&\|e^\phi(e^\theta x + e^{-\theta}y) + e^{-\phi}(e^{-\theta}x + e^\theta y), a\| \\ &\qquad = \|(e^{\theta+\phi} + e^{-\theta-\phi})x + (e^{\theta-\phi} + e^{-\theta+\phi})y, a\|\end{aligned}$$

We denote $\alpha = e^{\theta+\phi} + e^{-\theta-\phi}$ and $\beta = e^{\theta-\phi} + e^{-\theta+\phi}$. Then $\alpha\beta \geq (e^\theta + e^{-\theta})^2$.

Now, writing $\gamma^2 = \alpha\beta$ and using the hypothesis, we have

$$\begin{aligned}\|\alpha x + \beta y, a\| &= \gamma\|\alpha\gamma^{-1}x + \beta\gamma^{-1}y, a\| \geq \gamma\|x+y,a\| \geq (e^\theta + e^{-\theta})\|x+y,a\| \\ &= \|(e^\theta x + e^{-\theta}y) + (e^{-\theta}x + e^\theta y), a\|\end{aligned}$$

The desired result then follows by applying Lemma 2.3. □

Now we are ready to prove the main theorem. This is our final characterization of n-inner product space.

Theorem 2.7. *Let $x,y,a \in X$. The following statements are equivalent:*

1. *X is a 2-inner product space.*

2. *The following Dunkl-Williams inequality*

$$\left\|\frac{x}{\|x,a\|} - \frac{y}{\|y,a\|}, a\right\| \leq \frac{2\|x-y,a\|}{\|x,a\| + \|y,a\|} \tag{2.6}$$

holds for any $x,y,a \in X$.

Proof. (1) $\Rightarrow$ (2): This is Proposition 2.2.

(2) $\Rightarrow$ (1): Suppose (2.6) holds. We will use Theorem 2.6. Suppose $\|x, a\| = \|y, a\|$. Now, replacing x by αx, and y by $-\alpha^{-1}y$ in inequality (2.6), it easily follows that $\|\alpha x + \alpha^{-1}y, a\| \geq \|x + y, a\|$. Therefore by Theorem 2.6, X is a 2-inner product space. □

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