International Journal of Pure and Applied Mathematics
Volume 78 No. 8 2012, 1111-1120
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

BUILDING BLOCKS AND HARMONIC SERIES

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Abstract: This chapter presents an explanation of the divergence and convergence of infinite series through the building block problem. The chapter simultaneously expresses the fact that mathematics is not just about manipulating complicated numerical formulae but is also a field in which logical ways of thought are acquired. It is emphasized that in order to overcome university students' aversions to mathematics, lecturers must pour their energies into developing study materials taken from topics relevant to students.

AMS Subject Classification: 00A09, 40A02, 97A20
Key Words: harmonic series, center of gravity, convergence and divergence, logarithmic functions, mathematical education

1. Is it Possible to Stagger Building Blocks by More than the Width of One Block?

There has been a lot of publicity about how young people avoid and are "allergic" to mathematics. The goal of mathematics is not difficult numerical formulas but a mathematical way of looking at and thinking about things and I would like to present one example of this. Let us think about the building blocks problem in Figure 1. There are a few building blocks stacked up, and

Received: March 5, 2012

the problem is whether or not it is possible to stack them in such a way that the positions of the bottom block and the top block are horizontally separated by more than the width of one block.

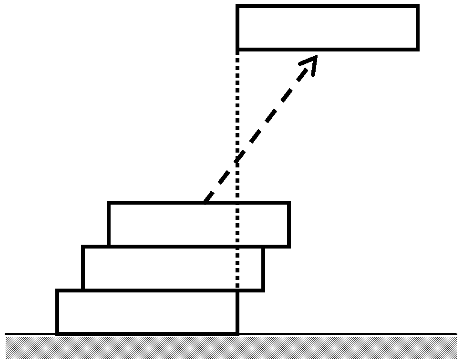

Figure 1: Is it possible to stagger building blocks by more than the width of one block?

Most people asked this question would immediately answer that it is not possible. I wonder if this tendency to come to a conclusion before even attempting to think about whether something is possible or not is a reflection of the digital age. Sometimes it is possible, sometimes it is not possible, additionally sometimes we do not know. But they hate vague answers very much. This is not magic or a trick, I promise that a solution certainly exists. If a person is told to stack building blocks in a staggered way, he or she will stagger them uniformly. But if they are staggered uniformly they will fall down every time. I wonder if this tendency to stack the blocks uniformly is also a manifestation of digital thinking.

We will not obtain a solution immediately. Let us start by looking at the case of two blocks. It is intuitively obvious that the distance they can be staggered is 1/2 of the width of the blocks. So the problem is the third block. Let us hold the third block in our right hand and think about this problem. Most people would try to stack this block on top of the other two but then they always fall down. If your approach does not work it is important to abandon it, and you should search for an alternative approach. To be able to do this, it is necessary to change your way of thinking about the problem.

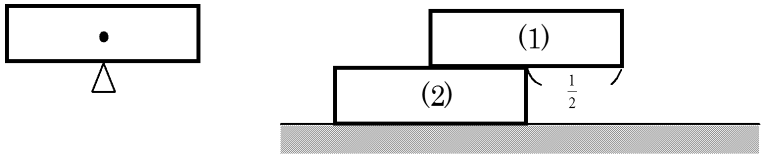

Figure 2: 1/2 Stagger

2. Calculating the Center of Gravity

This is the problem of calculating the center of gravity. Rather than thinking about this problem with a pen and paper, it is surprisingly fast to use building blocks and look for the answer through trial and error.

Here I will give you a hint. Are building blocks best stacked on top of each other? You will probably be perplexed by this hint. That is because of the fixed preconception that it is because we stack them on top of each other that they are building blocks. But building blocks should not be stacked on top of each other; they should be slid under each other. If the third building block is placed at the bottom, and we gradually stagger the first two building blocks on top of the third building block while maintaining the relationship between the first two building blocks as it was, we find that we can stagger the top two blocks by 1/4. In the same way, the fourth block can be placed under the other three and staggered by 1/6, and the fifth block can be placed under the other four and staggered by 1/8. If we add 1/2, 1/4, 1/6 and 1/8 the sum is greater than 1. In other words, we have stacked the blocks in such a way that the position of the top block is horizontally separated from that of the bottom block by more than the width of one block.

While referring to Figures 2, 3, and 4, let us confirm the above approach as a center of gravity calculation using numerical formulas. First of all let us think about building block (1) and building block (2). It is clear that we can only stagger them by 1/2 of the width of a block (Figure 2).

Next we are going to put building block (3) under the first two blocks so let us think about the center of gravity of building blocks (1) and (2) together (Figure 3, left). Because building block (3) can be staggered up to the center of gravity, I will obtain the moment, calling the stagger distance . Moment is the product of weight and arm length so the moment of building block (1) (rotated clockwise) is $1 \times x$, the moment of building block (2) (rotated anti-clockwise) is $1 \times (\frac{1}{2} - x)$ and because these two values are equal,

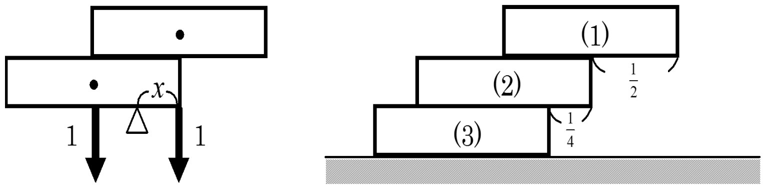

Figure 3: 1/4 Stagger

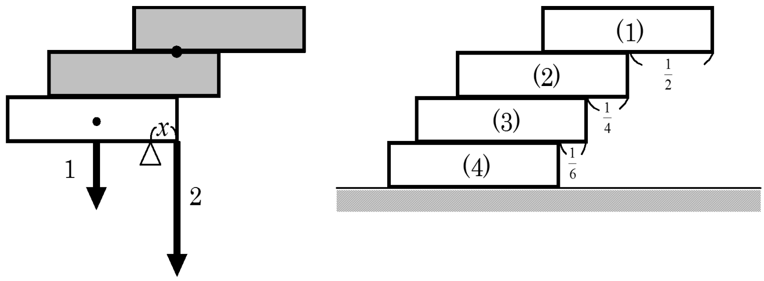

Figure 4: 1/6 Stagger

$$1 \times x = 1 \times (\frac{1}{2} - x)$$

Solving this equation, we show that $x = 1/4$. In other words, the stagger distance for building block (3) is 1/4 (Figure 3, right).

Next we are going to place building block (4) so let us think about the center of gravity of building blocks (1), (2) and (3) together. Let us obtain this center of gravity from the combination of the center of gravity of building blocks (1) and (2) together and the center of gravity of building block (3) (Figure 4, left). Taking building blocks (1) and (2) together gives a weight of 2. The moment of building blocks (1) and (2) (rotated clockwise) is , and the moment of building block (3) (rotated anti-clockwise) is $1 \times (\frac{1}{2} - x)$ and because these two values are equal,

$$2 \times x = 1 \times (\frac{1}{2} - x)$$

Solving this equation, we show that $x = 1/6$. In other words, the stagger distance for building block (4) is 1/6 (Figure 4, right).

Let us obtain the general result for the center of gravity of n building blocks. As this is determined by the center of gravity of $(n-1)$ building blocks plus the center of gravity of one building block,$(n-1) \times x = 1 \times (\frac{1}{2} - x)$, therefore $x = \frac{1}{2n}$.

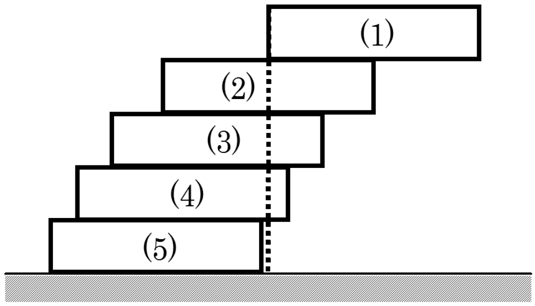

Figure 5: 1-Block Stagger

Rearranging this equation we can see that if the stagger position is as follows

$$\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2n}, \cdots$$

then the building blocks can be stacked so that they will not fall down. When the progression produced by reciprocal numbers is an arithmetic progression, it is called a harmonic progression. For example, $1, 1/2, 1/3, \cdots$ and $1, 1/3, 1/5, \cdots$ are harmonic progressions. Harmonic progressions are said to have been used in the study of harmonies theory by the Pythagorean School in ancient Greece and the name of harmonic progressions is derived from it. Harmonic series are the totals of harmonic progressions so we can also write:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n} = \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n})$$

So now let us calculate the value of this series.

$$\frac{1}{2} = 0.5, \ \frac{1}{4} = 0.25, \ \frac{1}{6} \approx 0.167, \ \frac{1}{8} = 0.125$$

therefore

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} \approx 1.042 > 1$$

So we now know that if we have 5 building blocks we can stagger them by more than the width of one block (Figure 5).

3. Convergence and Divergence

In high school and university differential and integral calculus textbooks there are chapters on progressions and series. In those chapters the following exercise invariably appears:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is divergent, and

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

is convergent.

When n goes to infinity, there are interesting exercises in which sometimes even if the general term of the progression converges to 0 the infinite series diverges. Convergence and divergence can be approximately known by performing integration as follows:

$$\sum \frac{1}{n} \approx \int \frac{dx}{x} = [\log x]$$

therefore

$$\sum \frac{1}{n^2} \approx \int \frac{dx}{x^2} = [-\frac{1}{x}].$$

The first of these two equations is in log order and diverges (Figure 6), and the second of these two equations converges (Figure 7). Generally, infinite series of the form $\sum \frac{1}{n^p}$ $(p > 0)$ diverge if $p \leq 1$ and converge if $p > 1$. Furthermore, it is known that $\sum \frac{1}{n^2}$ converges to $\frac{\pi^2}{6}$. Furthermore, whether or not $\sum \frac{1}{n}$ converges is determined by the Cauchy convergence criteria for the progression.

The sum of the first n terms of the progression $a_1, a_2, \cdots, a_n, \cdots$ is defined as

$$S_n = a_1 + a_2 + \cdots + a_n.$$

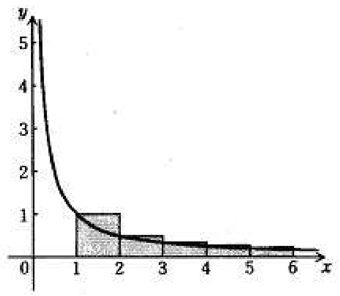

Figure 6: $y = \dfrac{1}{x}$

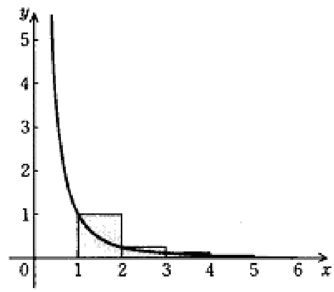

Figure 7: $y = \dfrac{1}{x^2}$

As for the necessary and sufficient condition for the series $\sum a_n$ to be convergent, if we make N sufficiently large compared to any given positive number ϵ, for all n and m where $m > n > N$ it can be shown that:

$$|S_m - S_n| = |a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon.$$

Assuming that

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

then no matter how big we make n,

$$\begin{aligned} |S_{2n} - S_n| &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+m} \\ &> \underbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}}_{n\ terms} \\ &= \frac{1}{2} \end{aligned}$$

So the *Cauchy* convergence criteria are not met. Therefore $\sum \dfrac{1}{n}$ is divergent. Let us look at this more closely. If we take the number of terms $2n$ as powers of 2 like this: $2, 4, 8, \cdots$, then

$|S_2 - S_1| = \dfrac{1}{2}$

$|S_4 - S_2| = \dfrac{1}{3} + \dfrac{1}{4} > \dfrac{2}{4} = \dfrac{1}{2}$

$$|S_8 - S_4| = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$$

$$|S_{2n} - S_1| = |S_{2n} - S_n| + \cdots + |S_8 - S_4| + |S_4 - S_2| + |S_2 - S_1|$$
$$> \frac{1}{2} + \cdots + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

So we can see that the series diverges.

4. Divergence in log *n* Order

I have explained that the harmonic series $\sum \frac{1}{n}$ diverges to infinity but let us look closely at how quickly $\sum \frac{1}{2n}$ diverge. I used a personal computer to calculate the value of $\sum \frac{1}{2n}$, the total stagger distance. The results were as follows:

When $n = 4$ $\quad \sum \frac{1}{2n} = 1.0417 > 1$

When $n = 31$ $\quad \sum \frac{1}{2n} = 2.0136 > 2$

When $n = 227$ $\quad \sum \frac{1}{2n} = 3.0022 > 3$

So the series does diverge to infinity but at an extremely slow speed.If we now compare $\sum \frac{1}{n}$ with the integration of the function $y = \frac{1}{x}$ we can establish an inequality as follows:

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^n \frac{dx}{x} < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$$

From the fact that $\int_1^n \frac{dx}{x} = [\log x]_1^n = \log n$, we can show that

$$\frac{1}{2}(\log n + \frac{1}{n}) < \sum \frac{1}{2n} < \frac{1}{2}(\log n + 1).$$

So we know that when $n \to \infty$, $\sum \frac{1}{2n}$ diverges in $\frac{1}{2} \log n$ order.

Because one more extra building block is necessary at the bottom, the number of building blocks necessary is actually $n+1$. Only five building blocks (4+1) are sufficient to stagger the pile of building blocks by the width of one building block, but 32 building blocks (31+1) are necessary to stagger the

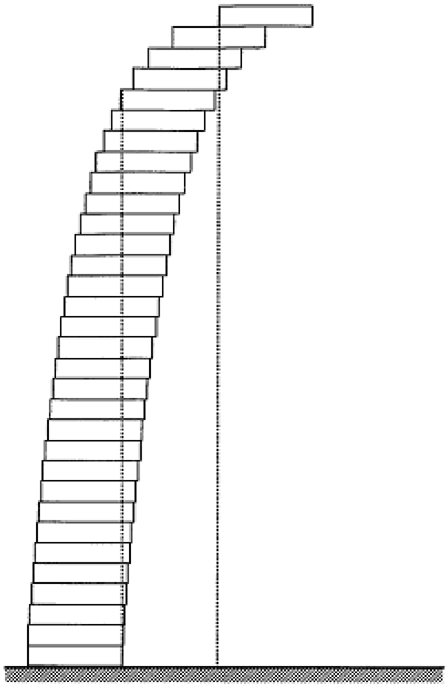

Figure 8: 2-Block Stagger

pile by the width of two building blocks and 228 building blocks (227+1) are necessary to stagger the pile by the width of three building blocks. Figure 8 shows a stack of 32 building blocks but in practice it is impossible to stack up 32 blocks accurately staggered in this way. This is just a theoretical discussion. Figure 9 is a graph showing the function $y = \frac{1}{x}$ and $y = \log x$, the function resulting from the integration of $y = \frac{1}{x}$. If we rotate the log function 90 degrees clockwise and reverse it horizontally it becomes the building block stacking problem in Figure 8. I will leave it to you to confirm this.

That completes the proof. I have shown that the harmonic series $\sum \frac{1}{n}$ describes the solution to the building blocks problem. If we solve these kinds of problems, mathematics should be more enjoyable I think. Mathematics in high school and university progressively becomes more distant from reality and sometimes students come close to losing sight of them. At times like that the student must not forget to apply the problems to reality. The building blocks

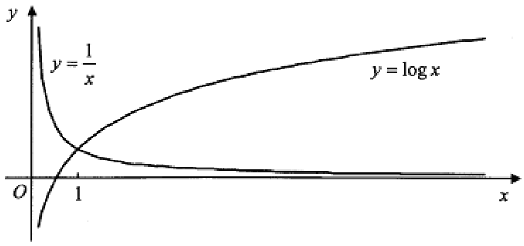

Figure 9: log function

problem is the problem of the calculation of the center of gravity; it also involves harmonic series and is extremely mathematical. If we limit ourselves to just solving the problem, we do not need to use complicated numerical formulas. The important things are to employ logical ways of thought and to have the ability to change your way of thinking. Ironically, university students doing science subjects cannot solve this building blocks problem. They can prove with numerical formulas that harmonic series diverge to infinity, but they cannot solve the real world problem of the building blocks. This is a blind spot in modern education.

I leaned about the building blocks problem from a 1958 work by George Gamow (see Gamow, Stern, 1958) [1]. He was both a researcher and educator and it appears that he was of the opinion that the students will not get excited about mathematics if the teacher is not excited about it. If you would like to confirm the solution to the building blocks problem but you do not have any building blocks at hand, you could try doing it with ten volumes of an encyclopedia or ten video tapes.

References

[1] G. Gamow, M. Stern, *Puzzle-Math*, The Viking Press Inc., USA, (1958).