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PLAYING WITH MÖBIUS STRIPS

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Abstract: Möbius strips are well known in topology as curved surfaces with no inside or outside. The degree of twist in the loop is particularly important, that is whether it is an odd or even multiple of 180 degrees. This article expresses how interesting mathematics can be, by splitting Möbius strips into halves or thirds using a pair of scissors, and contains a hint for unravelling the structure of space.

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1. A Single Sheet of B4 Paper

This time, let's talk about Möbius strips. Möbius strips, also known as Möbius loops, are discussed in topology but they also include very evocative aspects for the general public, so I always make a point of taking them up during my seminar time. I'd be pleased if those readers who are not familiar with Möbius strips would try them out.

To begin with, prepare yourself with a single sheet of B4 copy paper. From experience, a sheet with height 257 mm and width 364 mm is the best. Divide this into 6 equal parts as shown in Figure 1, making 6 long thin paper strips.

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These 6 strips will be used for experiments.

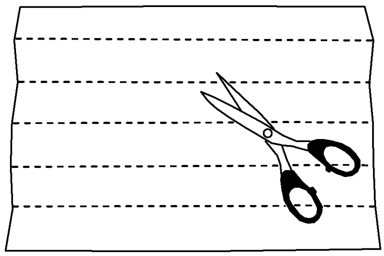


Figure 1: B4 sheet divided into 6 equal parts

So let's try out some experiments with Möbius strips. First, please make one Möbius strip. For those who don't know about Möbius strips, they can be made by gluing, just like a usual loop, except that Möbius strips have a 180 degree twist (Figure 2). There are two ways to twist by 180 degrees, clockwise and anticlockwise, but either direction is fine. Just by twisting the strip through 180 degrees and gluing it together, a weird world begins to open up.

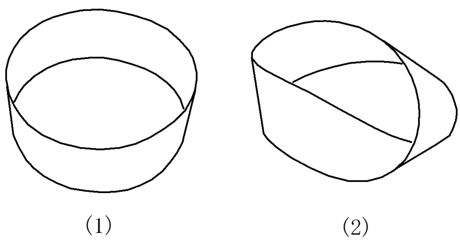


Figure 2: Normal loops and Möbius strips

Having checked that the Möbius strip is made correctly, let's think about what will happen if the strip is split in half. These days students seem to prefer cutting the loop with a pair of scissors before even thinking about the result, so students with scissors should be warned that they will lose points if they don't first try to imagine the result. Many students predict that cutting the loop will separate it into two pieces. Next comes the time to cut the loop and see. As an unexpected surprise, when the Möbius strip is cut, rather than separating into two pieces, one large loop is produced (Figure 3). Students who didn't know about the Möbius strip are surprised, and the experience is also useful for opening up the eyes of young people who have a dislike for mathematics.

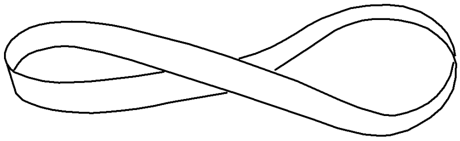
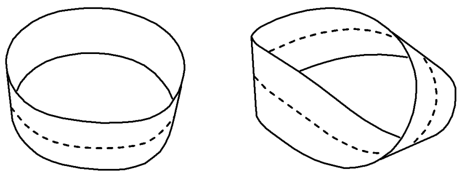


Figure 3: When a Möbius strip is split in half

The prediction that the strip would separate into two is way off. Let's think about why it forms a one big loop rather than separating into two. Since students know the difference between a normal loop and a Möbius strip with a 180 degree twist, have them make a normal loop and a Möbius strip, and try running a line along each using a pencil (Figure 4). With a normal loop, when a line is run along the outer surface it doesn't reach the inner surface, but with a Möbius strip the line runs along the inside and the outside. For students discovering the Möbius strip for the first time, the expectation that it would be the same as a normal loop makes this alone an impressive phenomenon.



(1) (2)

Figure 4: Running a line in pencil

2. Odd and Even Multiples of 180 Degrees

The work so far was round 1. Some students know that cutting a Möbius strip produces one long thin loop, but this story has a continuation. What happens when the loop produced so far (Figure 3) is again cut into two?

Have students make a prediction. At this point the prediction that one even longer loop will be produced is common. Is it perhaps a characteristic of the current younger generation that they are easily influenced by previous results. One person says that a large loop will be produced and then everyone responds that they think the same thing. Think about it carefully. Here's a hint. The thing we're trying to cut is a Möbius strip. Is it that the desire to see one big loop is stronger, or is it that there are few alternative predictions? The result is that two loops are produced as shown in Figure 5, and they are linked. Students' predictions are again mistaken.

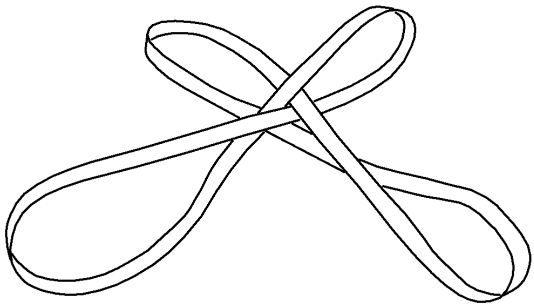


Figure 5: When the loop is cut again

Reviewing the results above yields the following. The degree of twist in the loop is particularly important. A normal loop without any twist has an inside and an outside, but the Möbius strip with its 180 degree twist is a 'curved surface with no inside or outside' since the inner and outer surfaces cannot be distinguished. Furthermore, it can be written on the board that a loop with an even number of 180 degree twists (0 degrees, 360 degrees, 720 degrees...) has an inside and an outside, while a loop with an odd number (180 degrees, 540 degrees, 900 degrees...) has no inside or outside.

If this were known, it should at least have been possible to predict that the loop in Figure 3 would separate into two pieces because it is twisted by 360 degrees and has an inside and an outside.

3. Splitting a Möbius Strip into Thirds

Let's make another Möbius strip, and this time let's think about what will happen if it is split into thirds. Running a pencil line around the strip reveals information significant for making a prediction, so have students do this. An approximate division into thirds is sufficient. When the pencil line running around the Möbius strip reaches the other side, just like the halfway line, there is no true reverse side. The line is offset, but ignoring this and continuing the line all the way round, it eventually reaches its initial point. Change the color of pencil and have them draw one more line. In this way the state divided into thirds can be confirmed. If it still seems to be difficult to make a prediction, applying color in the way shown in Figure 7, and thus dividing into different colored regions should help. It appears that the center strip (white) and the two outer strips (black and gray) are different pieces, although the black and gray parts are connected.

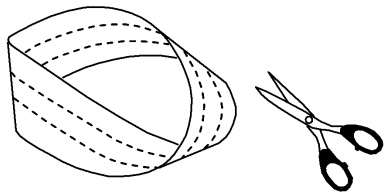


Figure 6: Möbius strip split into thirds

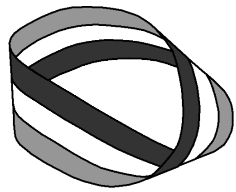


Figure 7: Applying color

Splitting the Möbius strip into thirds yields a small loop and a large loop, which are linked (Figure 8). The small loop is a Möbius strip. The large loop is a surface with an inside and outside, and has a 360 degree twist in it.

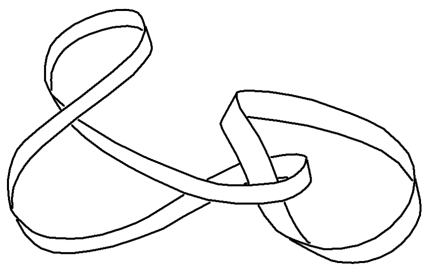


Figure 8: The large loop and short loop are linked.

4. A Hint for Unravelling the Structure of Space

Who devised the Möbius strip, and for what purpose? Möbius (A.F. Möbius, 1790-1868) was a German mathematician and astronomer active in the 19th century. He proposed it, so the Möbius strip is named after him. He was also an astronomer so he had an interest in the structure of space, and it is assumed that he devised the strip while exploring various notions about the boundaries of the universe.

Stepping back for a moment into an earlier age, it was thought that the world was flat. People thought that if you went to the edge where no one had been, there was a cliff that you would fall off. The fact that the world is round was demonstrated by Columbus, who discovered the 'new world' in 1492. Afterwards, Magellan (1480-1521) made the first circumnavigation of the world, thus proving the fact.

This also had a significant impact on geometry. Up until that time geometry was essentially Euclidean, and it was thought that all problems might be soluble with this well-integrated model. There was an approximate contemporary of Möbius known as Lobachevsky (1792-1856). He was a Russian mathematician who while pursuing research into parallel lines, validated a formulation of non-Euclidean geometry and presented his results.

Non-Euclidean geometry is born from a rejection of the axiom of parallel lines. In this geometry, the axiom of parallel lines is replaced by the notions that 'for a given straight line on a plane there are at least two straight lines no different from the given line that pass through a point which is not on the given line', 'the sum of the internal angles of a triangle is at least the sum of two right angles', and furthermore, 'straight lines are finitely closed, and divergences from Euclidean geometry can also be seen in the ordering of points on a straight line', *etc.*

For example, think about a triangle NAB formed by the North Pole and two points on the equator. In this case the sum of the internal angles is larger than the sum of two right angles. The sum of the internal angles of a triangle drawn on a notepad is the sum of two right angles, but triangles on the Earth's surface do not accord with Euclidean geometry. There is a relationship in the sense that non-Euclidean geometry is locally Euclidean.

When Columbus and Magellan proved that the Earth is round, the mistake regarding the edge of the world was resolved. The ancient mystery regarding the nature of the boundary of the universe on the other hand remains. Space is vast, and being unable to reach the boundary, humans cannot resolve this problem absolutely. There is a theory that space, like the Earth, is closed.

Until recently, it was thought that light traveled in straight lines, but the person who rejected this in an attempt to clarify the nature of space was Einstein (1879-1955). He established a new 4 dimensional model of the universe according to the theory of special relativity based on the principles of relativity and the constancy of the speed of light in an inertial reference frame, discarding the notions of absolute space, absolute time, the ether and so on occurring in Newtonian dynamics. The unity of mass and energy are derived from this theory. Beginning in 1907, it was attempted to expand the theory of relativity to gravitational fields, leading to the completion of general relativity theory in 1915.

Light bends according to gravity. The light released into space from the Earth experiences the effects of gravity and will at some point return. The theory that space does not have a boundary but is closed was therefore composed.

Regarding space with 3 dimensional coordinates, adding a further axis for time yields a 4 dimensional coordinate frame. On the Earth 3 dimensional coordinates are sufficient, but at the astronomical level 4 dimensional coordinates become necessary. Euclidean geometry and Newtonian dynamics are no good for understanding the boundaries of space, and non-Euclidean geometry and Einstein's theory of relativity become necessary.

5. Can We Go to a 4 Dimensional World?

It's somewhat metaphorical but let's try thinking about dimensions. Caterpillars and ants can only move in straight lines. They are 1 dimensional animals that only know a single route. Whirligig beetles, which float on water and can only glide on the water's surface are restricted to planar movement and are 2 dimensional animals. People are 3 dimensional animals capable of conceiving straight lines, planes and spaces. The only 4 dimensional animals capable of perceiving the speed of light as finite and grasping the structure of space are extraterrestrial.

We can understand that the animals living at each dimensionality are at a lower dimensional level than ourselves, but we cannot comprehend a higher level of dimensionality. For example, people are 3 dimensional animals, so we can comprehend ants, which are 1 dimensional animals and whirligig beetles, which are 2 dimensional animals, but we cannot imagine 4 dimensional space. This is the same relationship according to which 1 dimensional ants cannot comprehend planes and whirligig beetles cannot comprehend spaces.

Among the many ideas about how to express a 4 dimensional coordinate system in a 3 dimensional world, the Möbius strip is often cited. Even when living in a 2 dimensional world, mounted on a Möbius strip it is possible to reach the other side of the surface. If the relationship between the inside and the outside is taken as a concept going beyond the notion of a plane, then the Möbius strip is a bridge from 2 dimensions to a quasi- 3 dimensional world. This embodies the expectation that if a device like this could be manufactured, it might perhaps provide an interface from 3 dimensions to a 4 dimensional world.

6. Application to Endless Tapes

Möbius strips have been proposed as one method for clarifying the structure of space, but besides academia, their wonderful geometrical properties have also been applied in our daily lives, as endless tapes. Musical recording media are moving from CDs (compact discs) to hard disk devices such as iPods (Apple), but at the peak of the analogue generation magnetic tapes were in use.

Magnetic tape kept inside a plastic case is known as a cassette tape. A cassette tape has two sides, A and B. When each side had finished playing the tape had to be turned over or rewound. A way to save the time spent idling under these operations is to connect the tape with a twist of 180 degrees like a Möbius strip. These endless tapes were used in places like storefronts where it is necessary to play them repeatedly. Möbius strips were also applied to computer printer ribbons, although these devices are now to be found in museums.

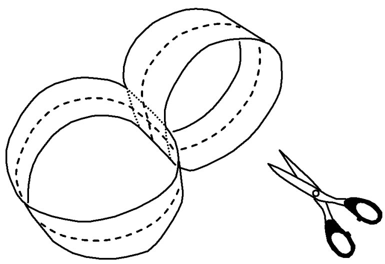


Figure 9: Cross shaped connection

After they have thoroughly played with the Möbius strip, I always make a point of presenting the following quiz to students. Two normal strips are prepared. They are firmly glued, and attached in a cross shape as shown in Figure 9. Well then, what happens when this is split into two along the dotted line?

This is also interesting to predict. Since the results so far have produced large loops, large loops linked with smaller loops and so on, predictions extending these results are common. After gathering all the students' predictions, have them cut with a pair of scissors and see. Lo and behold, a regular quadrilateral is produced (Figure 10). The fact that a square is produced from the two loops, thus yielding a 2 dimensional surface from a 3 dimensional solid, is an intriguing curiosity. No one expects the result to be planar.

The square thus produced can also be used to make two normal loops. It is helpful to imagine the images of the cutting process played in reverse. The

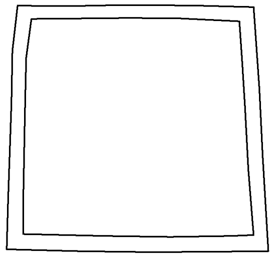


Figure 10: Square

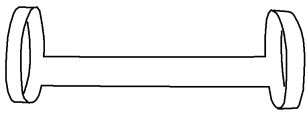


Figure 11: Intermediate form

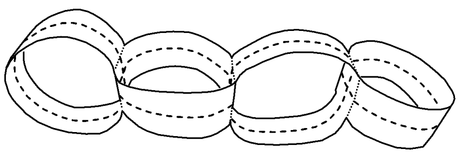


Figure 12: When normal loops in 4 crosses are cut

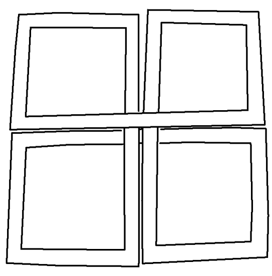


Figure 13: A boxed array is formed

square in Figure 10 passes through an intermediate form like that shown in Figure 11, and ends up in a state like that shown in Figure 9. Essentially, this can be understood by employing a flexible way of thinking.

When performing this quiz, one student asked the question “what happens when 4 loops are connected?” I had not tried this, so I attached normal loops in 4 crosses as shown in Figure 12 and used a pair of scissors to cut them and see. This time the result was a boxed array (Figure 13). What happens when 8 are connected together and cut? Such interesting topics arose one after the other, when students played with the Möbius strip for the first time.