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INVERSION OF THE TOEPLITZ-PLUS-HANKEL MATRICES VIA GENERALIZED INVERSION

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Abstract: The generalized inversion of the block Toeplitz-plus-Hankel matrix has been obtained. It allows to find the inverse (one-sided inverse) matrix of the block Toeplitz-plus-Hankel matrix provided that the this matrix is invertible (one-sided invertible).

AMS Subject Classification: 15A09
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1. Introduction

In many applications, e.g. digital signal processing, discrete inverse scattering, linear prediction etc., Toeplitz-plus-Hankel ($T + H$) matrices need to be inverted. (For further applications see [1] and references therein).

Firstly the $T + H$ matrix inversion problem has been solved in [2] where it was reduced to the inversion problem of the block Toeplitz matrix (the so-called mosaic matrix). The drawback of the method is that it does not work for any invertible $T + H$ matrix since it requires also invertibility of the corresponding $T - H$ matrix. Later on the drawback was put out [3], moreover, the inversion problem was solved for the block $T + H$ matrix [4], [5].

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Our goal is to restore the method of [2] in order to get the generalized inversion for the block $T+H$ matrix. To do it we will need the generalized inversion for the block Toeplitz matrix which has been already found in, e.g. [6]. It is shown in the present paper that there is no need for $T-H$ matrix to be inverted: if the $T+H$ matrix is invertible than the obtained generalized inverse matrix proves to be its inverse matrix.

2. The Basic Definitions and Notations

Let $T+H=||a_{i-j}+b_{i+j}||_{\substack{i=0,\dots,n,\\ j=0,\dots,m,}}$, $a_k,\ b_k\in\mathbb{C}^{p\times q}$, be the block Toeplitz-plus-Hankel matrix.

Denote $a_{-m}^{n}(z)=a_{-m}z^{-m}+\ldots+a_0+\ldots+a_nz^n$, $b_0^{n+m}(z)=b_0+b_1z+\ldots+b_{n+m}z^{n+m}$ and introduce an auxiliary matrix function

$$A(z)=\begin{pmatrix} z^nb_0^{n+m}(z^{-1}) & z^{n-m}a_{-m}^{n}(z^{-1})\\ a_{-m}^{n}(z) & z^{-m}b_0^{n+m}(z)\end{pmatrix}.$$

Obviously, $A(z)=\sum\limits_{j=-m}^{n}A_jz^j$, with $A_j\in\mathbb{C}^{2p\times 2q}$ and

$$A_j=\begin{pmatrix} b_{n-j} & a_{n-m-j}\\ a_j & b_{j+m}\end{pmatrix}. \tag{1}$$

Thus, $A(z)$ is the generating function for the sequence of matrices $A_{-m},\ldots,A_0,\ldots,A_n$.

Further we will need the generalized inversion for the block Toeplitz matrix $T_A=||A_{i-j}||_{\substack{i=0,\dots,n,\\ j=0,\dots,m,}}$ which has been found in [6]. In order to use this result we should introduce the definitions of essential indices and polynomials of the sequence $A_{-m},\ldots,A_0,\ldots,A_n$.

We include the matrix $T_A\equiv T_0$ into the family of the block Toeplitz matrices $T_k=||A_{i-j+k}||_{\substack{i=0,\dots,n-k,\\ j=0,\dots,m+k,}}$, $-m\le k\le n$. The matrices T_k are of the same structure and it is reasonable that they should be examined together.

We are interested in right kernels of T_k. For the sake of convenience let us pass from the spaces $\ker_{\mathrm{R}}T_k$ to the isomorphic spaces $\mathcal{N}_k^R$ of generating polynomials. To do this we define the operator σ_R acting from the space of rational matrix functions $R(z)=\sum_{j=-n}^{m}r_jz^j, r_j\in\mathbb{C}^{2q\times l}$ to the space $\mathbb{C}^{2p\times l}$ according to $\sigma_R\left\{R(z)\right\}=\sum_{j=-n}^{m}A_{-j}r_j$.

By $\mathcal{N}_k^R, k=-m,\ldots,n$, we denote the space of vector polynomials $R(z)=\sum_{j=0}^{k+m}r_jz^j, r_j\in\mathbb{C}^{2q\times 1}$, such that $\sigma_R\left\{z^{-i}R(z)\right\}=0$, $i=k,k+1,\ldots,n$. $\mathcal{N}_k^R$ is evident to be isomorphic to $\ker_{\mathrm{R}}T_k$.

It is convenient to put $\mathcal{N}^R_{-m-1}=0$ and denote by $\mathcal{N}^R_{n+1}$ the $2(n+m+2)q$-dimensional space of all vector polynomials in z with formal degree $n+m+1$.

Similarly, one may define spaces $\mathcal{N}^L_k$ which are isomorphic to $\ker_{\mathrm{L}} T_k$. We denote $\ker_L A=\{y \mid yA=0\}$.

Let us put also $\alpha=\dim \mathcal{N}^R_{-m}$ and $\omega=\dim \mathcal{N}^L_n$. The sequence $A_{-m},\dots,A_n$ is called *left (right) regular* if $\alpha=0 \quad (\omega=0)$. Otherwise, the sequence is not regular and $\alpha(\omega)$ is its *left (right) defect.* The sequence is called *regular* if $\alpha=\omega=0$. It is evident that $\alpha<2q, \quad \omega<2p$ for the nonzero sequence.

Denote $d^R_k=\dim \mathcal{N}^R_k$, $\Delta^R_k=d^R_k-d^R_{k-1}$, $k=-m,\dots,n+1$. For any sequence $A_{-m},\dots,A_n$ the following inequalities hold [6]:

$$\alpha=\Delta^R_{-m}\le\Delta^R_{-m+1}\le\dots\le\Delta^R_n\le\Delta^R_{n+1}=2(p+q)-\omega.$$

It means that there are $2(p+q)-\alpha-\omega$ integers $\mu_\alpha\le\mu_{\alpha+1}\le\dots\le\mu_{2(p+q)-\alpha-\omega}$, satisfying equations

$$\begin{array}{ccccccc}
\Delta^R_{-m} & = & \dots & = & \Delta^R_{\mu_{\alpha+1}} & = & \alpha, \\
 & & \dots & & & & \\
\Delta^R_{\mu_i+1} & = & \dots & = & \Delta^R_{\mu_{i+1}} & = & i, \\
 & & \dots & & & & \\
\Delta^R_{\mu_{2(p+q)-\omega}+1} & = & \dots & = & \Delta^R_{n+1} & = & 2(p+q)-\omega.
\end{array} \tag{2}$$

If the ith row in (2) is absent, we assume $\mu_i=\mu_{i+1}$. Let us put also $\mu_1=\dots=\mu_\alpha=-m-1$ if $\alpha\ne0$ and $\mu_{2(p+q)-\omega+1}=\dots=\mu_{2(p+q)}$ if $\omega\ne0$.

Thus, for any sequence $A_{-m},\dots,A_n$, there is a set of $2(p+q)$ integers, satisfying (2), which will be called *indices* of the sequence.

Let us define the right essential polynomials. It follows from the definition of $\mathcal{N}^R_k$ that $\mathcal{N}^R_k$ and $z\mathcal{N}^R_k$ are the subspaces of $\mathcal{N}^R_{k+1}$, $k=-m-1,\dots,n$, moreover, $\mathcal{N}^R_k\bigcap z\mathcal{N}^R_k=\mathcal{N}^R_{k-1}$. Then $\mathcal{N}^R_{k+1}=(\mathcal{N}^R_k+z\mathcal{N}^R_k)\oplus\mathcal{H}^R_{k+1}$, where $\mathcal{H}^R_{k+1}$ is the complement of $\mathcal{N}^R_k+z\mathcal{N}^R_k$ to the whole $\mathcal{N}^R_{k+1}$. Obviously, $\dim\mathcal{H}^R_{k+1}=\Delta^R_{k+1}-\Delta^R_k$. Hence $\dim\mathcal{H}^R_{k+1}\ne0$ iff $k=\mu_i$. In this case $\dim\mathcal{H}^R_{k+1}$ is equal to the multiplicity k_i of the index μ_i.

Definition 1. If $\alpha\ne0$ then any vector polynomials $R_1(z),\dots,\ R_\alpha(z)$ forming the basis of $\mathcal{N}^R_{-m}$ will be called right essential polynomials of the sequence $A_{-m},\dots,\ A_0,\ \dots,A_n$. They correspond to the index $\mu_1=-m-1$ with the multiplicity α.

Any vector polynomials $R_j(z),\dots,R_{j+k_j-1}(z)$ forming the basis for $\mathcal{H}^R_{\mu_j+1}$ will be called right essential polynomials of the sequence $A_{-m},\dots,\ A_0,\ \dots,A_n$. They correspond to the index μ_j with the multiplicity k_j, $\alpha+1\le j\le2(p+q)-\omega$.

Similarly, one may define the left essential polynomials.

There are $2(p+q)-\omega$ right and $2(p+q)-\alpha$ left essential polynomials of the sequence $A_{-m},\dots,A_n$. There is a lack of essential polynomials if $\alpha\neq 0$ or $\omega\neq 0$. But we can always complement the number of right (if $p\leq q$) or left (if $p\geq q$) essential polynomials to $2(p+q)$. (The complement procedure is described in [6]).

For definiteness sake, we will suppose that we have got the full set of $2(p+q)$ right essential polinomials, i.e. either $\omega=0$ or $p\leq q$.

The set of the left essential polynomials could always be recovered with the help of the so-called conformation procedure of the right and left essential polynomials. Let us describe how for the given set of the right essential polynomials $R_1(z),\ \dots, R_{2(p+q)}(z),\ R_j(z)\in\mathbb{C}^{2q\times 1}[z]$ one can construct the conforming left essential polynomials $L_1(z),\dots,L_{2(p+q)}(z),\ L_j(z)\in\mathbb{C}^{1\times 2p}[z]$.

We introduce the matrix $\mathcal{R}(z)=\big(R_1(z)\dots R_{2(p+q)}(z)\big)$ of the right essential polynomials and find the matrix polynomial $\alpha_-(z)$ from the decomposition $A(z)\mathcal{R}(z)=\alpha_-(z)d(z)-z^{n+1}\beta_+(z)$, where $d(z)=\operatorname{diag}\left[z^{\mu_1},\dots,z^{\mu_{2(p+q)}}\right]$, $\beta_+(z)$ ($\alpha_-(z)$) is the matrix polynomial in z (z^{-1}) of the size $2p\times 2(p+q)$.

Denote $\mathcal{R}_-(z)=z^{-m-1}\mathcal{R}(z)d^{-1}(z)$. Let $U_-(z)=\begin{pmatrix}\mathcal{R}_-(z)\\ \alpha_-(z)\end{pmatrix}$ be the matrix polynomial in z^{-1}. The polynomial $U_-(z)$ is shown in [6] to be unimodular, i.e. its determinant is equal to a constant. We pick the $2(p+q)\times 2p$ block $\mathcal{L}(z)$ out $U_-^{-1}(z)=\begin{pmatrix} * & \mathcal{L}(z)\end{pmatrix}$.

The matrix polynomial $\mathcal{L}(z)=\begin{pmatrix}L_1(z)\\ \vdots\\ L_{2(p+q)}(z)\end{pmatrix}$ turns out to be the matrix of the conforming left essential polynomials.

The case when $\alpha=0$ or $p\geq q$ may be considered in a similar manner with help of the left essential polynomials.

Now we may present the formula (5.13) from [6] for the generalized inverse of T_A:

$$T_A^{\dagger}=\begin{pmatrix}\mathcal{R}_0 & \dots & 0\\ \vdots & \ddots & \vdots\\ \mathcal{R}_m & \dots & \mathcal{R}_0\end{pmatrix}\Pi\begin{pmatrix}\mathcal{L}_0 & \dots & \mathcal{L}_{-n}\\ \vdots & \ddots & \vdots\\ 0 & \dots & \mathcal{L}_0\end{pmatrix}. \tag{3}$$

Here $\mathcal{R}_j\in\mathbb{C}^{2q\times 2(p+q)}$, $\ \mathcal{L}_j\in\mathbb{C}^{2(p+q)\times 2p}$ are the coefficients of the matrix polynomials $\mathcal{R}(z)$, $\mathcal{L}(z)$, respectively, and $R_j(z), L_j(z)$ are the conforming right and left essential polynomials of the sequence $A_{-m},\dots,A_0,\dots,A_n$. The generalized inversion for matrix A is meant to be the matrix $A^{\dagger}$ such that $AA^{\dagger}A=A$.

The matrix Π is constructed in a following way. Let $\lambda_1, \dots, \lambda_r$ be the distinct essential indices of the sequence $A_{-m}, \dots, A_0, \dots, A_n$ and let $\nu_1, \dots, \nu_r$ be their multiplicities ($\nu_1 + \dots + \nu_r = 2(p+q)$). Then $\Pi = ||\Pi_{i-j}||_{\substack{i=0,\dots,m,\\ j=0,\dots,n,}}$. Here $\Pi_k = 0$ for $-n \le k \le m,\ k \ne -\lambda_1, \dots, -\lambda_r$, $\Pi_{-\lambda_j} = \|\varepsilon_i^j \delta_{ik}\|_{i,k=1}^{2(p+q)}$,

$$\varepsilon_i^j = \begin{cases} 1, & i = \nu_1 + \dots + \nu_{j-1} + 1, \dots, \nu_1 + \dots + \nu_j, \\ 0, & \text{otherwise.} \end{cases}$$

For the generalized inversion of the $T+H$ matrix it will be useful to partition the right essential polynomials $R_j(z) = \begin{pmatrix} R_j^1(z) \\ R_j^2(z) \end{pmatrix}$. Here $R_j^{1,2} \in \mathbb{C}^{q\times 1}[z]$. In similar way we partition the left essential polynomials: $L_j(z) = \begin{pmatrix} L_j^1(z) & L_j^2(z) \end{pmatrix}$, with $L_j^{1,2} \in \mathbb{C}^{1\times p}[z]$.

Then the matrix of these essential polynomials may be represented as:

$$\mathcal{R}(z) = \begin{pmatrix} \mathcal{R}^1(z) \\ \mathcal{R}^2(z) \end{pmatrix}, \quad \mathcal{L}(z) = \begin{pmatrix} \mathcal{L}^1(z) & \mathcal{L}^2(z) \end{pmatrix}, \tag{4}$$

with $\mathcal{R}^{1,2}(z) \in \mathbb{C}^{q\times 2(p+q)}$, $\quad \mathcal{L}^{1,2}(z) \in \mathbb{C}^{2(p+q)\times p}$.

3. The Generalized Inversion

In the section we will present our main result. Let us denote

$$T_{\mathcal{R}_j} = \begin{pmatrix} \mathcal{R}_0^j & \dots & 0 \\ \vdots & \ddots & \vdots \\ \mathcal{R}_m^j & \dots & \mathcal{R}_0^j \end{pmatrix}, \quad T_{\mathcal{L}_j} = \begin{pmatrix} \mathcal{L}_0^j & \dots & \mathcal{L}_{-n}^j \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathcal{L}_0^j \end{pmatrix}, \quad j = 1, 2,$$

where $\mathcal{R}_k^j (\mathcal{L}_k^j)$ are the coefficients of the polynomials $\mathcal{R}^j (\mathcal{L}^j)$. We also put $H_{\mathcal{R}_2} = JT_{\mathcal{R}_2}$, $H_{\mathcal{L}_1} = T_{\mathcal{L}_1}J$.

Theorem 1. *The generalized inverses of the $T+H$ and $T-H$ matrices are found by the formulas:*

$$(T \pm H)^{\dagger} = \frac{1}{2} \left(T_{\mathcal{R}_1} \pm H_{\mathcal{R}_2} \right) \Pi \left(T_{\mathcal{L}_2} \pm H_{\mathcal{L}_1} \right). \tag{5}$$

If $T \pm H$ is invertible (one-sided invertible), then $(T \pm H)^{\dagger}$ is its inverse (one-sided inverse) matrix.

Proof. Let us construct the generalized inversion to $T_A \equiv T_0$ according to formula (3). We are going to pass from block Toeplitz matrix T_A to the mosaic matrix

$$M_A = \left(\begin{array}{cccc|cccc} b_n & \dots & b_{n+m} & & a_{n-m} & \dots & a_n \\ b_{n-1} & \dots & b_{n+m-1} & & a_{n-m-1} & \dots & a_{n-1} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ b_0 & \dots & b_m & & a_{-m} & \dots & a_0 \\ \hline a_0 & \dots & a_{-m} & & b_m & \dots & b_0 \\ a_1 & \dots & a_{-m+1} & & b_{m+1} & \dots & b_1 \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_n & \dots & a_{n-m} & & b_{n+m} & \dots & b_n \end{array}\right).$$

At first, according to the block structure of A_j (1), we partition each block column X_j of the matrix T_A into two block columns X_j^1, X_j^2 with sizes $2p(n+1)\times q$: $X_j = \begin{pmatrix} X_j^1 & X_j^2 \end{pmatrix}$. Then permute new block columns in T_A and construct the matrix

$$\begin{pmatrix} X_1^1 & \dots & X_m^1 & X_1^2 & \dots & X_m^2 \end{pmatrix}$$
$$= \left(\begin{array}{cccc|cccc} b_n & b_{n+1} & \dots & b_{n+m} & a_{n-m} & a_{n-m+1} & \dots & a_n \\ a_0 & a_{-1} & \dots & a_{-m} & b_m & b_{m-1} & \dots & b_0 \\ \hline b_{n-1} & b_n & \dots & b_{n+m-1} & a_{n-m-1} & a_{n-m} & \dots & a_{n-1} \\ a_1 & a_0 & \dots & a_{-m+1} & b_{m+1} & b_m & \dots & b_1 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline b_0 & b_1 & \dots & b_m & a_{-m} & a_{-m+1} & \dots & a_0 \\ a_n & a_{n-1} & \dots & a_{n-m} & b_{n+m} & b_{n+m-1} & \dots & b_n \end{array}\right).$$

This matrix is evident to be obtained by multiplying T_A on a permutation matrix P_2. Then we will do the analogous permutation with block rows in T_AP_2. As a result, we will get the matrix $P_1T_AP_2$, where P_1 is a permutation matrix. The matrix $P_1T_AP_2$ coincides with $M_A = P_1T_AP_2$. Thus we have passed from the block Toeplitz matrix T_A to the mosaic matrix M_A.

Since for a permutation matrix P the equality $P^{-1} = P^t$ holds, we get the generalized inversion for $M_A : M_A^\dagger = P_2^tT_A^\dagger P_1^t$. Let us specify the structure of factors in this product. The operations which P_2 has done with the block columns of T_A, the matrix P_2^t now will carry out with the block rows of the matrix $\begin{pmatrix} \mathcal{R}_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \mathcal{R}_m & \dots & \mathcal{R}_0 \end{pmatrix}$.

Thus

$$P_2^t \begin{pmatrix} \mathcal{R}_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \mathcal{R}_m & \dots & \mathcal{R}_0 \end{pmatrix} = \begin{pmatrix} \mathcal{R}_0^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \mathcal{R}_m^1 & \dots & \mathcal{R}_0^1 \\ \mathcal{R}_0^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \mathcal{R}_m^2 & \dots & \mathcal{R}_0^2 \end{pmatrix} \equiv \begin{pmatrix} T_{\mathcal{R}_1} \\ T_{\mathcal{R}_2} \end{pmatrix},$$

where $\mathcal{R}_j^{1,2}$ are the coefficients of the matrix polynomials $\mathcal{R}^{1,2}(z)$, presented in (4).

Similarly, we have

$$\begin{pmatrix} \mathcal{L}_0 & \dots & \mathcal{L}_{-n} \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathcal{L}_0 \end{pmatrix} P_1^t$$
$$= \begin{pmatrix} \mathcal{L}_0^1 & \dots & \mathcal{L}_{-n}^1 & \mathcal{L}_0^2 & \dots & \mathcal{L}_{-n}^2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \mathcal{L}_0^1 & 0 & \dots & \mathcal{L}_0^2 \end{pmatrix} \equiv \begin{pmatrix} T_{\mathcal{L}_1} & T_{\mathcal{L}_2} \end{pmatrix}.$$

Then

$$M_A^{\dagger} = \begin{pmatrix} T_{\mathcal{R}_1} \\ T_{\mathcal{R}_2} \end{pmatrix} \Pi \begin{pmatrix} T_{\mathcal{L}_1} & T_{\mathcal{L}_2} \end{pmatrix}.$$

Let us apply now the well-known method [2] of reducing the mosaic matrix M_A to the block-diagonal matrix formed from the Toeplitz-plus-Hankel and Toeplitz-minus-Hankel matrices:

$$M_A = \frac{1}{2} \begin{pmatrix} J & J \\ I & -I \end{pmatrix} \begin{pmatrix} T+H & 0 \\ 0 & T-H \end{pmatrix} \begin{pmatrix} I & J \\ -I & J \end{pmatrix}.$$

Then

$$G = \frac{1}{2} \begin{pmatrix} I & J \\ -I & J \end{pmatrix} M_A^{\dagger} \begin{pmatrix} J & J \\ I & -I \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} T_{\mathcal{R}_1} + JT_{\mathcal{R}_2} \\ -T_{\mathcal{R}_1} + JT_{\mathcal{R}_2} \end{pmatrix} \Pi \begin{pmatrix} T_{\mathcal{L}_1}J + T_{\mathcal{L}_2} & T_{\mathcal{L}_1}J - T_{\mathcal{L}_2} \end{pmatrix}$$

is the generalized inversion for the matrix

$$\begin{pmatrix} T+H & 0 \\ 0 & T-H \end{pmatrix}.$$

Let $G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$, where $G_{ij} \in \mathbb{C}^{(m+1)q\times(n+1)p}$. It is easy to get that $G_{11} = \frac{1}{2}\left(T_{\mathcal{R}_1} + H_{\mathcal{R}_2}\right)\Pi\left(T_{\mathcal{L}_2} + H_{\mathcal{L}_1}\right)$, is the generalized inverses to $T + H$ and $G_{22} = \frac{1}{2}\left(T_{\mathcal{R}_1} - H_{\mathcal{R}_2}\right)\Pi\left(T_{\mathcal{L}_2} - H_{\mathcal{L}_1}\right)$ is the generalized inverses to $T - H$.

The theorem statement concerning the invertibility (one-sided invertibility) is evident. The theorem has been proved. □

Given $T \pm H$ matrices are block matrices with the sizes of their blocks $p \times q$. The factors in the inverse formulas (5) have blocks with sizes $q \times 2(p+q)$, $2(p+q) \times p$. The compact form of the generalized inversion is in many respects because of such factors sizes. Sometimes it is convenient to have a formula for the generalized inversion where factors have blocks with sizes $q \times q, q \times p, p \times p$.

In order to obtain it we partition $\mathcal{R}(z)$ and $\mathcal{L}(z)$ into blocks:

$$\mathcal{R}(z) = \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \mathcal{R}_{13} & \mathcal{R}_{14} \\ \mathcal{R}_{21} & \mathcal{R}_{22} & \mathcal{R}_{23} & \mathcal{R}_{24} \end{pmatrix}, \quad \mathcal{L}(z) = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \\ \mathcal{L}_{31} & \mathcal{L}_{32} \\ \mathcal{L}_{41} & \mathcal{L}_{42} \end{pmatrix}.$$

Here $\mathcal{R}_{ij}$ have the sizes $q \times q$ for $i, j = 1, 2$, and $q \times p$ for $i = 1, 2,\ j = 3, 4$ and $\mathcal{L}_{ij}$ have the sizes $q \times p$ for $i, j = 1, 2$, and $p \times p$ for $i = 3, 4, j = 1, 2$. Let us also partition $D = \operatorname{diag}[z^{\mu_1} \dots z^{\mu_{2(p+q)}}] = (d_1\ d_2\ d_3\ d_4)$, where $d_{1,2}$ are diagonal matrices with the sizes $q \times q$ and $d_{3,4}$ are ones with the sizes $p \times p$. For $i, j = 1, \dots, 4$ denote

$$T_{\mathcal{R}_{ij}} = \begin{pmatrix} \mathcal{R}_0^{ij} & \dots & 0 \\ \vdots & \ddots & \vdots \\ \mathcal{R}_m^{ij} & \dots & \mathcal{R}_0^{j} \end{pmatrix}, \quad T_{\mathcal{L}_{ij}} = \begin{pmatrix} \mathcal{L}_0^{ij} & \dots & \mathcal{L}_{-n}^{ij} \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathcal{L}_0^{ij} \end{pmatrix}.$$

Then it is easy to see that

$$(T \pm H)^{\dagger} = \frac{1}{2}\left[\sum_{j=1}^{4} T_{\mathcal{R}_{1j}}\pi_j T_{\mathcal{L}_{j2}} + \sum_{j=1}^{4} H_{\mathcal{R}_{2j}}\pi_j H_{\mathcal{L}_{j1}} \right.$$
$$\left. \pm \left(\sum_{j=1}^{4} T_{\mathcal{R}_{1j}}\pi_j H_{\mathcal{L}_{j1}} + \sum_{j=1}^{4} H_{\mathcal{R}_{2j}}\pi_j T_{\mathcal{L}_{j2}}\right)\right],$$

where we denote $T_{\mathcal{L},\mathcal{R}}J = H_{\mathcal{L},\mathcal{R}}$ and π_j are the matrices constructed by d_j with the same manner as Π by d.

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