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INTEGRALS AND INVARIANTS FOR INVISCID, INCOMPRESSIBLE, TWO-DIMENSIONAL, IRROTATIONAL, HYDROMAGNETIC FLOWS UNDER GRAVITY

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Abstract: Conversation laws for two-dimensional, incompressible, irrotational hydromagnetic flows under gravity are investigated. It is observed that the conserved quantities presented by Longuet-Higgins [5] are also conserved in hydromagnetic case if the total pressure vanishes on the boundary of the fluid.

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1. Introduction

Longuet-Higgins [5] obtained using a direct and simplified method, the eight conservation laws in two-dimensional, incompressible, inviscid and irrotational flows. These results were obtained earlier by Benjamin and Olver [1] using a rather general analysis. These results have been generalized by Chakraborty, Khattar and Verma [2] for two-dimensional compressible flows. In these flows it

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is found that only seven conservation laws are valid. These results have been further extended to three dimensional compressible flows by Khattar, Chakraborty, Mittal and Kaushik [4]. In three-dimensional flows, eleven quantities are conserved. The aim of the present paper is to generalize these conservation laws to the two-dimensional, hydromagnetic and inviscid flows in which all physical quantities are independent of the z-coordinate and the magnetic field is applied along the z-axis. Some generalizations of the results for rotational flows are also obtained.

2. Definitions

We take the xy-plane as the plane of the two-dimensional fluid motion. All quantities are independent of z. Following Longuet-Higgins [5], following the quantities $M, M\bar{x}, M\bar{y}, I, J, A, B$ and T are defined as given below.

$$M = \int_D dxdy = \int_C ydx, \tag{2.1}$$

$$M\bar{x} = \int_D xdxdy = \int_C xydx, \tag{2.2}$$

$$M\bar{y} = \int_D ydxdy = \int_C \frac{1}{2}y^2dx, \tag{2.3}$$

$$I = \int_D \phi_x dxdy = \int_C (-\phi)dy, \tag{2.4}$$

$$J = \int_D \phi_y dxdy = \int_C \phi dx, \tag{2.5}$$

$$A = \int_D [(x\phi)_y - (y\phi)_x]dxdy = \int_C \phi(xdx + ydy), \tag{2.6}$$

$$B = \int_D [(x\phi)_x + (y\phi)_y]dxdy = \int_C \phi(ydx - xdy), \tag{2.7}$$

$$T = \int_D \frac{1}{2}(\phi_x^2 + \phi_y^2)dxdy = \frac{1}{2}\int_C \phi(\phi_y dx - \phi_x dy), \tag{2.8}$$

In defining (2.1)-(2.8), Green's identity (Eq. (2.1) in Longuet-Higgins [5]) has been used. C is taken to be simple closed contour bounding a domain D of the fluid. ϕ is the velocity potential in the incompressible, irrotational flow, so that the fluid velocity $\vec{v}$ is given as

$$\vec{v} = \nabla\phi \tag{2.9}$$

and

$$\nabla^2\phi = 0. \tag{2.10}$$

We also define

$$E = T + V, \quad \text{and} \quad V = Mg\bar{y}\,. \tag{2.11}$$

In the above equations, M is the total mass, density of the fluid being taken as unity. The centre of mass has coordinates $\bar{x}$ and $\bar{y}$. I and J are the two components of the momentum, A is the angular momentum and B is a quantity analogous to A. T, V and E are the kinetic, potential and total energy, respectively.

3. Some Theorems

Two-dimensional, incompressible, inviscid and magnetohydrodynamic flow is considered. Following Cowling [3], the equation of motion can be written as

$$\rho\frac{D\vec{v}}{Dt} = -\text{grad}p + \rho\vec{g} + \mu\vec{j}\times\vec{H}\,, \tag{3.1}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}.\text{grad} \tag{3.2}$$

is the material derivative and

$$\text{curl}\vec{H} = \vec{j} \tag{3.3}$$

is from Maxwell's equations.

Using the vector identity (Weatherburn [6])

$$\text{grad}\vec{u}^2/2 = \vec{u}.\nabla\vec{u} + \vec{u}\times\text{curl}\vec{u}, \tag{3.4}$$

the Lorentz force $\mu\vec{j}\times\vec{H}$ in (3.1) can be put, in view of (3.3), as

$$\mu\vec{j}\times\vec{H} = \mu\vec{H}.\nabla\vec{H} - \text{grad}(\mu\vec{H}^2)/2\,. \tag{3.5}$$

In two-dimensional flow the quantities are independent of the z-coordinate, therefore we take

$$\vec{H} = (0, 0, H),$$

Now, the equation (3.5) is reduced to

$$\mu\vec{j} \times \vec{H} = -\text{grad}(\mu\vec{H}^2/2)\,. \tag{3.6}$$

Taking $\vec{g} = (0, -g, 0)$ and using (3.2), (3.4) and (3.6) into the equation (3.1), we get

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + \text{grad}(\vec{v}^2/2)\right) = -\text{grad}(p + \mu\vec{H}^2/2 + \rho g y) \tag{3.7}$$

Using (2.9) for an irrotational flow and taking density as unity ($\rho = 1$), we obtain from (3.7), the equation (3.7) is reduced to

$$p + \mu\vec{H}^2/2 + gy + \frac{\partial\phi}{\partial t} + \frac{1}{2}(\phi_x^2 + \phi_y^2) = 0. \tag{3.8}$$

In (3.8), the arbitrary function of time has been absorbed in $\dfrac{\partial\phi}{\partial t}$.

The equation (3.8) is the modified Bernoulli's equation in the hydromagnetic flow being which differs from the classical Bernoulli's equation used by Longuet-Higgins [5] in that the hydrodynamic pressure p in the nonmagnetic case is replaced by the total pressure P(hydrodynamic plus magnetic pressure) given by

$$P = p + \mu\vec{H}^2/2 \tag{3.9}$$

in the present hydromagnetic case (eq. (3.8)).

Let the contour C move with the fluid, then proofs of the following results

$$\frac{dM}{dt} = 0, \tag{3.10}$$

$$M\frac{d\bar{x}}{dt} = I, \tag{3.11}$$

$$M\frac{d\bar{y}}{dt} = J, \tag{3.12}$$

$$\frac{dI}{dt} = \int_C P\,dy, \tag{3.13}$$

$$\frac{dJ}{dt} = -\int_C P\,dx - Mg, \tag{3.14}$$

$$\frac{dA}{dt} = -\int_C P(x\,dx + y\,dy) - Mg\bar{x}, \tag{3.15}$$

$$\frac{dB}{dt} = -\int_C P(ydx - xdy) + 4T - 3V, \tag{3.16}$$
$$\frac{dE}{dt} = -\int_C P(\phi_y dx - \phi_x dy) \tag{3.17}$$

are exactly the same as given in Longuet-Higgins [1] except that everywhere in their derivations the hydrodynamic pressure p should be replaced by the total pressure P.

4. Conserved Quantities

If the total pressure P vanishes everywhere on the contour C, then the equations (3.10)-(3.17) show that eight quantities are conserved. These eight quantities are given by Longuet-Higgins [5].

In the absence of gravity, these conserved quantities take simpler forms and reduce to $M, I, J, M\bar{x} - It$, $M\bar{y} - Jt, E, A$ and $B - 4Et$, respectively.

5. Motion with Vorticity

For a rotational flow there is no velocity potential ϕ. However, $M, \bar{x}$ and $\bar{y}$ have been defined through equations (2.1)-(2.3). I, J, A, E are defined in terms of u and v, the components of velocity, and are given through equations(6.1)-(6.4) bu Longuet-Higgins [5]. Following Longuet-Higgins [5], it can be shown that the result (3.10) to (3.15) are also true for rotational flows. In rotational flows the equation of motion (3.7) is replaced by

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + \text{grad}(\vec{v}^2/2) - \vec{v}\times curl\vec{v}\right) = -\text{grad}(p + \rho g y) \tag{5.1}$$

Taking curl of this equation, we find that for a two-dimensional flow ($\vec{v} = (u, v, 0)$) the relation

$$\frac{d}{dt}(v_x - u_y) = 0 \tag{5.2}$$

holds, where suffix denotes partial differentiation. Using (5.2), we can derive the circulation theorem

$$\frac{dC}{dt} = 0, \tag{5.3}$$

where the circulation C is given by the line integral $\int(udx - vdy)$ taken along the contour bounding the fluid.

6. Conclusions

It is observed that all the results obtained by Longuet-Higgins [5] can be extended to the two-dimensional hydromagnetic flow. The conserved quantities, given by Longuet-Higgins, are also conserved in the hydromagnetic case, if the total pressure P vanishes on the boundary C of the fluid motion. P will vanish on the boundary of the fluid if outside the boundary there is vacuum (so that pressure p vanishes) and the magnetic field H vanishes there.

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