

SOME EULER-TYPE FORMULAS FOR PLANAR GRAPHS

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Abstract: It is the aim of this note to derive several analytical relations which hold for certain classes of planar graphs.

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1. Introduction and Definitions

It is known that for every connected simple planar graph there holds the Euler's characteristic χ — a topological invariant, originally defined for polyhedra by the formula

$$\chi = V - E + F = 2, \quad (1)$$

where V is the number of vertices, E is the number of edges, and F is the number of faces in the given graph, including the exterior face. This formula corresponds to the special case $g = 0$ (simple connectedness) of the more general Poincaré formula for genus g surfaces, in which $\chi \equiv \chi(g) = 2 - 2g$.

In this note we derive several analytical relations, similar to Euler's formula, which hold for some classes of planar graphs that we introduce below. A topological graph is a graph drawn in the plane such that its vertices are represented by points and its edges are represented by arcs connecting the corresponding points such that no two arcs intersect except at a common endpoint. The classes of graphs considered here are defined as follows.

Definition 1. Let $G_1(V, E)$ be a topological graph. We say that the graph G_1 belongs to the class Γ_1 of planar graphs if the following conditions hold

- any two vertices in the graph G_1 are connected by at least two simple paths (chains) without a common vertex, i.e. the graph is 2-connected;
- all the interior faces have the same η -gonality;
- any vertex of the graph $v \in V$ has the degree $d_{G_1}(v) = 2, 3, \dots, r$; for simplicity, below we use the symbol d to denote the vertex' degree.

The other two classes of graphs are derivative classes of Γ_1 .

Definition 2. A graph G_2 belongs to the class Γ_2 of planar graphs if $G_2 \in \Gamma_1$ and all the interior faces of G_2 are 4-gons and at least one interior vertex $v \in G_2$ has the degree $d(v) \neq 4$.

Definition 3. A graph G_3 belongs to the class Γ_3 of planar graphs if $G_3 \in \Gamma_1$ and G_3 contains at least one interior face with $\eta = 3$.

2. Main Results

Let V^{ext} be the set of all exterior vertices of the graph, that is, those vertices which are incident to the exterior face, and let V^{int} be the set of all the rest vertices – interior ones, and

$$\begin{aligned}
 v' \in V^{int}, \quad v'' \in V^{ext}, \quad V^{int} \cup V^{ext} = V, \quad V^{int} \cap V^{ext} = \emptyset, \\
 \bigcup_{d=2}^r V_d^{int} = V^{int}, \quad \bigcup_{d=2}^r V_d^{ext} = V^{ext}, \\
 v'_d \in V_d^{int} \subseteq V^{int} \subset V, \quad v''_d \in V_d^{ext} \subseteq V^{ext} \subseteq V.
 \end{aligned}$$

Then we have

$$|V| = |V^{int}| + |V^{ext}| = \sum_{d=2}^r |V_d^{int}| + \sum_{d=2}^r |V_d^{ext}|, \tag{2}$$

where $|\cdot|$ denotes the cardinality of a set.

Constructing the simple graph of incidence “vertices–edges”, one finds that the vertex v_d forms $d/2$ edges. Hence, we get

$$|E| = \frac{1}{2} \sum_{d=2}^r d (|V_d^{int}| + |V_d^{ext}|). \tag{3}$$

Constructing the simple graph of incidence “vertices–faces”, we get that the vertex v'_d forms d/η interior faces of the graph, since the vertex is incident to d faces and each interior face is incident to η vertices. Analogously, the number of interior faces formed by the vertex v''_d equals to $(d - 1)/\eta$.

From the above it follows that

$$|F| - 1 = \frac{1}{\eta} \left[\sum_{d=2}^r r |V_d^{int}| + \sum_{d=2}^r (d - 1) |V_d^{ext}| \right]. \tag{4}$$

Now substituting (2), (3) and (4) into (1), we obtain the formula

$$\left[\sum_{d=2}^r |V_d^{int}| + \sum_{d=2}^r |V_d^{ext}| \right] - \left[\frac{1}{2} \sum_{d=2}^r d (|V_d^{int}| \cdot |V_d^{ext}|) \right] + \frac{1}{\eta} \left[\sum_{d=2}^r d |V_d^{int}| + \sum_{d=2}^r (d - 1) |V_d^{ext}| \right] = 1. \tag{5}$$

For planar graphs with vertices v_d , ($d = 2, 3, 4$), and η -gon faces, we get from the formula (5)

$$- \left(|V_4^{int}| + |V_4^{ext}| + \frac{1}{2} |V_3^{int}| + \frac{1}{2} |V_3^{ext}| \right) + \frac{1}{\eta} (4|V_4^{int}| + 3|V_3^{int}| + 2|V_2^{int}| + 3|V_4^{ext}| + 2|V_3^{ext}| + |V_2^{ext}|) = 1, \tag{6}$$

where

$$|V| = |V_4^{int}| + |V_3^{int}| + |V_2^{int}| + |V_4^{ext}| + |V_3^{ext}| + |V_2^{ext}|,$$

$$|E| = 2 (|V_4^{int}| + |V_4^{ext}|) + \frac{3}{2} (|V_3^{int}| + |V_3^{ext}|) + (|V_2^{int}| + |V_2^{ext}|),$$

and

$$|F| - 1 = \frac{1}{\eta} (4|V_4^{int}| + 3|V_3^{int}| + 2|V_2^{int}| + 3|V_4^{ext}| + 2|V_3^{ext}| + |V_2^{ext}|)$$

or

$$|F| - 1 = |V_4^{int}| + |V_4^{ext}| + \frac{1}{2} |V_3^{int}| + \frac{1}{2} |V_3^{ext}|.$$

From the formula (6) we find

$$|V_4^{int}|(8 - 2\eta) + |V_4^{ext}|(6 - 2\eta) + |V_3^{int}|(6 - \eta) + |V_3^{ext}|(4 - \eta)$$

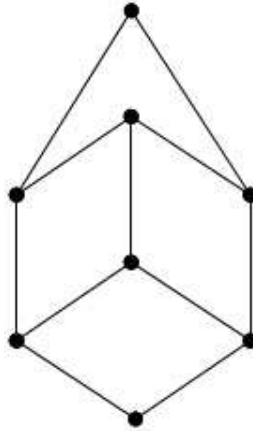


Figure 1: An example of graph from the class Γ_2

$$+ 4|V_2^{int}| + 2|V_2^{ext}| = 2\eta. \quad (7)$$

Putting $\eta = 4$ in the formula (7), we get the relation for the class Γ_2 of planar graphs (Fig. 1)

$$|V_3^{int}| + 2|V_2^{int}| = 4 + |V_4^{ext}| + |V_3^{ext}| - |V_2^{ext}|. \quad (8)$$

Now assume that some given planar graph contains exactly $|F_\eta|$ η -gon interior faces, $|F_{\eta-1}|$ $(\eta - 1)$ -gon interior faces, etc., and $|F_3|$ triangular interior faces. For such planar graphs we obtain from the formula (5) the following system of equations

$$\sum_{d=2}^r d|V_d^{int}| + \sum_{d=2}^r (d-1)|V_d^{ext}| = \sum_{\eta=3}^{\eta} \eta|F_\eta|, \quad (9)$$

$$-1 - \left(\sum_{d=2}^r |V_d^{int}| + \sum_{d=2}^r |V_d^{ext}| \right) + \frac{1}{2} \sum_{d=2}^r d(|V_d^{int}| + |V_d^{ext}|) = \sum_{\eta=3}^{\eta} |F_\eta|.$$

From the formula (9) for the class Γ_3 of planar graphs (Fig. 2) we get the following relation

$$4|V_4^{int}| + 3|V_3^{int}| + 2|V_2^{int}| + 3|V_4^{ext}| + 2|V_3^{ext}| + |V_2^{ext}| = 3|F_3| + 4|F_4|,$$

$$1 + |V_4^{int}| + |V_4^{ext}| + \frac{1}{2}|V_3^{int}| + \frac{1}{2}|V_3^{ext}| = |F_3| + |F_4|. \quad (10)$$

Solving the system of equations (10) with respect to $|F_3|$, we obtain

$$4 + |V_4^{ext}| - |V_3^{int}| - 2|V_2^{int}| - |V_2^{ext}| = |F_3|. \quad (11)$$

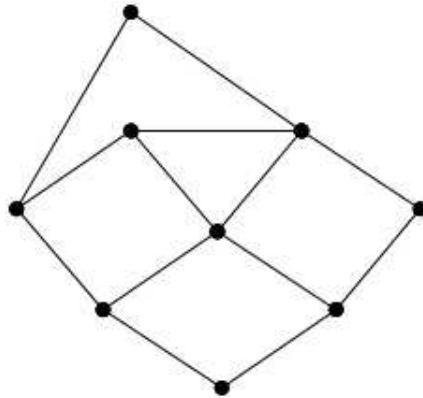


Figure 2: An example of graph from the class Γ_3

In the same way, using similar reasonings, one can derive analytical relations for other classes of planar graphs.

The formulas obtained in this note can find various applications, such as deriving quantitative and qualitative estimates for algorithms of mappings of different classes of planar graphs. For instance, the relation (8) allows one to recognize if the graph G_1 belongs to the class Γ_2 in $|V_{G_1}|$ steps, that is, by considering the $|V_{G_1}|$ rows of adjacency matrix of the graph G_1 vertices. In a subsequent paper we plan to deal with applications of the above formulas.

References

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